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On isometries of the Kobayashi and Carathéodory metrics

by Prachi Mahajan (Bangalore)

Abstract. This article considers C^1-smooth isometries of the Kobayashi and Carathéodory metrics on domains in $\mathbb{C}^n$ and the extent to which they behave like holomorphic mappings. First we provide an example which *suggests* that $\mathbb{B}^n$ cannot be mapped isometrically onto a product domain. In addition, we prove several results on continuous extension of C^0-isometries $f : D_1 \to D_2$ to the closures under purely local assumptions on the boundaries. As an application, we show that there is no C^0-isometry between a strongly pseudoconvex domain in $\mathbb{C}^2$ and certain classes of weakly pseudoconvex finite type domains in $\mathbb{C}^2$.

1. Introduction. The principal aim of this article is to explore the phenomenon of the rigidity of continuous isometries of the Kobayashi and the Carathéodory metrics. More precisely, if D, D' are two domains in $\mathbb{C}^n$ and $f : D \to D'$ is a continuous isometry of the Kobayashi metrics on D, D', it is not known whether f must necessarily be holomorphic or conjugate holomorphic. The same question can be asked about the Carathéodory metric or for that matter about any invariant metric as well. An affirmative answer for the Bergman metric was given in [GK] in the case when D and D' are both C^2-smooth strongly pseudoconvex domains in $\mathbb{C}^n$, and this required knowledge of the limiting behaviour of the holomorphic sectional curvatures of the Bergman metric near strongly pseudoconvex points. In general, the Kobayashi metric is just upper semicontinuous and therefore a different approach will be needed for this question. The case of isometries when D is smooth strongly convex and D' is the unit ball was dealt with in [SV2] and this was improved upon in [KK] to handle the case when D is a $C^{2,\epsilon}$-smooth strongly pseudoconvex domain; a common ingredient in both proofs was the use of Lempert discs. On the other hand, it was remarked in [F] that the localization of a biholomorphic mapping between bounded domains near a given boundary point should follow from general principles of Gromov

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hyperbolicity—an example of this localization can be found in [BB]. Motivated by such considerations it seemed natural to determine the extent to which isometries behave like holomorphic mappings. An example in this context is provided by the following result.

THEOREM 1.1. *There is no C^1-Kobayashi or Carathéodory isometry between $\mathbb{B}^n$ and the product of m domains $D_1 \times \cdots \times D_m$ for any $2 \le m \le n$ where each D_i is a bounded strongly convex domain in $\mathbb{C}^{n_i}$ with C^6-smooth boundary and $n = n_1 + \cdots + n_m$.*

Several remarks are in order here. Firstly, by a C^0-*Kobayashi* (*Carathéodory* or *inner Carathéodory*) *isometry* we mean a distance preserving bijection between the metric spaces (D_1, d_{D_1}) and (D_2, d_{D_2}) $((D_1, c_{D_1})$ and (D_2, c_{D_2}); $(D_1, c^i_{D_1})$ and $(D_2, c^i_{D_2})$ respectively). Here d_D, c_D and c^i_D denote the Kobayashi, Carathéodory and inner Carathéodory metrics respectively on the domain D. For $k \ge 1$, a C^k-*Kobayashi isometry* is a C^k-diffeomorphism f from D_1 onto D_2 with $f^*(F^K_{D_2}) = F^K_{D_1}$ where $F^K_{D_1}$ and $F^K_{D_2}$ denote the infinitesimal Kobayashi metrics on D_1 and D_2 respectively. Secondly, note that isometries are continuous when the domains are Kobayashi hyperbolic, for in this case the topology induced by the Kobayashi metric coincides with the intrinsic topology of the domain. As can be expected, the main step in proving Theorem 1.1 is to show that the C^1-smooth isometry, if it exists, is indeed a biholomorphic mapping to arrive at a contradiction. The proof of this is based on differential-geometric considerations, in particular the theorem of Myers–Steenrod as in [SV2]; the fact that the Kobayashi metric of the ball is a smooth Kähler metric of constant negative sectional curvature -4 plays a key role. The proof of Theorem 1.1 requires the existence of complex geodesics and a certain degree of smoothness of the Kobayashi metric, and hence we restrict to C^6-smooth strongly convex domains. It must be mentioned that the above result is motivated by the well known fact that there does not exist a proper holomorphic mapping from a product domain onto $\mathbb{B}^n$ for any $n > 1$. A different approach was used to get related results in [S].

This article also considers the question of continuous extendability up to the boundary of continuous isometries between domains in $\mathbb{C}^n$. Here is a prototype statement that can be proved.

THEOREM 1.2. *Let $f : D_1 \to D_2$ be a Kobayashi isometry between two bounded domains in $\mathbb{C}^2$. Let p^0 and q^0 be points on ∂D_1 and ∂D_2 respectively. Assume that ∂D_1 is C^∞-smooth weakly pseudoconvex of finite type near p^0 and that ∂D_2 is C^2-smooth strongly pseudoconvex in a neighbourhood U_2 of q^0. Suppose that q^0 belongs to the cluster set of p^0 under f. Then f extends as a continuous mapping to a neighbourhood of p^0 in $\overline{D}_1$.*

It should be noted that we make only purely local assumptions on D_1 and D_2; in particular, the domains are not assumed to be pseudoconvex away from p^0 and q^0 and there are no global smoothness assumptions on the boundaries. Theorem 1.2 is proved using the global estimates on the Kobayashi metric near weakly pseudoconvex boundary points of finite type from [HERB]. This is done in Propositions 4.1 and 4.2. It is worth mentioning that other relevant theorems of this nature for proper holomorphic mappings between strongly pseudoconvex domains were proved by Forstnerič and Rosay [FR] using global estimates on the Kobayashi metric. As an application of Theorem 1.2 we get:

THEOREM 1.3. *Let D_1 and D_2 be bounded domains in $\mathbb{C}^2$. Let $p^0 = (0,0)$ and q^0 be points on ∂D_1 and ∂D_2 respectively. Assume that ∂D_1 in a neighbourhood U_1 of the origin is weakly pseudoconvex of finite type and defined by $\{\rho^0(z) < 0\}$ with*

$$\rho^0(z_1, z_2) = 2\Re z_2 + |z_1|^{2m} + o(|z_1|^{2m} + \Im z_2)$$

where $m > 1$ is a positive integer, and that ∂D_2 is C^2-smooth strongly pseudoconvex in a neighbourhood U_2 of q^0. Then there cannot be a Kobayashi isometry f from D_1 onto D_2 such that q^0 belongs to the cluster set of p^0 under f.

Theorem 1.3 dispenses with the assumption of having a global biholomorphic mapping and replaces it with a global Kobayashi isometry at the expense of restricting to certain classes of weakly pseudoconvex finite type domains in $\mathbb{C}^2$. A particularly useful strategy to investigate this type of results in the holomorphic category has been Pinchuk's scaling technique (cf. [P2]). Scaling D_1 near p^0 with respect to a sequence of points that converges to p^0 along the inner normal yields a limit domain of the form

$$D_{1,\infty} = \{(z_1, z_2) \in \mathbb{C}^2 : 2\Re z_2 + |z_1|^{2m} < 0\}$$

for which the Kobayashi metric has some smoothness ([M2]). It is for this reason that we restrict attention to domains with a defining function as described in Theorem 1.3. In trying to adapt the scaling methods in our situation, the 'normality' of the scaled isometries has to be established. This requires the stability of the integrated Kobayashi distance under scaling of a given strongly pseudoconvex domain (this was done in [SV1]) and a weakly pseudoconvex finite type domain in $\mathbb{C}^2$—this was developed in [MV] for a different application and we intend to use it here as well. The conclusion then would be that the limit of scaled isometries exists and yields a Kobayashi isometry between the corresponding model domains, i.e., the ellipsoid $D_{1,\infty}$ and the ball $\mathbb{B}^2$. Another difficulty is that unlike the holomorphic case the restrictions of Kobayashi isometries to subdomains are not isometries with respect to the Kobayashi metric of the subdomain. The end game is to show that the isometry is holomorphic, and this is done using the techniques employed in the proof of Theorem 1.1.

Several other statements about the continuous extendability of isometries are possible—these relate to isometries between either a pair of strongly pseudoconvex domains in $\mathbb{C}^n$ or a pair of weakly pseudoconvex domains of finite type in $\mathbb{C}^2$. These are stated (cf. Theorems 5.9 and 5.10) and elaborated upon towards the end of Section 5. They are valid for isometries of the inner Carathéodory distance as well (cf. Theorem 5.11).

2. Notation and terminology. Let Δ denote the open unit disc in the complex plane and let $d_{\mathrm{hyp}}(a,b)$ denote the distance between $a,b \in \Delta$ with respect to the hyperbolic metric. For $r>0$, $\Delta(0,r) \subset \mathbb{C}$ will be the disc of radius r around the origin and $B(z,\delta) \subset \mathbb{C}^n$ will be the Euclidean ball of radius $\delta > 0$ around z. Let X be a complex manifold of dimension n. The Kobayashi and the Carathéodory distances on X, denoted by d_X and c_X respectively, are defined as follows:

Let $z \in X$ and fix a holomorphic tangent vector ξ at z. Define the associated infinitesimal Carathéodory and Kobayashi metrics as

$$F_X^C(z,\xi) = \sup\{|df(z)\xi| : f \in \mathcal{O}(X,\Delta)\},$$
$$F_X^K(z,\xi) = \inf\left\{\frac{1}{\alpha} : \alpha > 0,\ f \in \mathcal{O}(\Delta, X) \text{ with } f(0)=z,\ f'(0)=\alpha\xi\right\}$$

respectively. This induces a concept of length of a path. If $\gamma : [0,1] \to X$ is a piecewise smooth path, then the *Carathéodory length* is given by

$$L_X^C(\gamma) = \int_0^1 F_X^C(\gamma(t), \dot\gamma(t))\,dt,$$

and this in turn induces the associated *inner Carathéodory distance*

$$c_X^i(p,q) = \inf L_X^C(\gamma)$$

where the infimum is taken over all piecewise smooth curves γ in X joining p to q. Likewise, the *Kobayashi length* of a piecewise C^1-curve $\gamma : [0,1] \to X$ is given by

$$L_X^K(\gamma) = \int_0^1 F_X^K(\gamma(t), \dot\gamma(t))\,dt,$$

and finally the *Kobayashi distance* between $p,q \in X$ is defined as

$$d_X(p,q) = \inf L_X(\gamma)$$

where the infimum is taken over all piecewise differentiable curves γ in X joining p to q. Furthermore, $B_X(z,r)$ will denote the ball of radius $r>0$ around $z \in X$ with respect to the Kobayashi distance d_X. Recall that X is taut if $\mathcal{O}(\Delta, \mathcal{X})$ is a normal family and any Kobayashi complete domain is taut. The *Carathéodory distance* c_X between $p, q \in X$ is defined by setting

$$c_X(p,q) = \sup_f d_{\text{hyp}}(f(p), f(q))$$

where the supremum is taken over the family of all holomorphic mappings $f : X \to \Delta$.

A domain $D \subset \mathbb{C}^n$ with C^2-smooth boundary is said to be *strongly convex* if there is a defining function ρ for ∂D such that the real Hessian of ρ is positive definite as a bilinear form on $T_p(\partial D)$ for every $p \in \partial D$.

Let $D \subset \mathbb{C}^n$ be a bounded domain. A holomorphic mapping $\phi : \Delta \to D$ is said to be an *extremal disc* or a *complex geodesic* for the Kobayashi distance if it is distance preserving, i.e., $d_D\big(\phi(p), \phi(q)\big) = d_\Delta(p, q)$ for all p, q in Δ.

A sequence D^j of domains in $\mathbb{C}^n$ is said to converge to a domain $D_\infty \subset \mathbb{C}^n$ in the *Hausdorff sense* if two things happen: first, given any compact set $K \subset \mathbb{C}^n$ such that K is compactly contained in D^j for all j large, then K is a relatively compact subset of D_∞; second, any compact subset of D_∞ is contained in D^j for all j large.

The notion of finite type for a smooth hypersurface $M \subset \mathbb{C}^n$ will be in the sense of D'Angelo, i.e., the order of contact of one-dimensional varieties with M is finite.

Recall some properties of the infinitesimal Kobayashi metric on taut domains:

Proposition 2.1. *Let D be a taut domain in $\mathbb{C}^n$. Then*

(i) *for any $z \in D$ and $v \in \mathbb{C}^n$ there exists an extremal disc $g : \Delta \to D$, i.e., $g(0) = z$ and $F_D^K(z, v) g'(0) = v$,*
(ii) *the function $F_D^K(\cdot, \cdot)$ is jointly continuous on $D \times \mathbb{C}^n$.*

A good source of details on holomorphically invariant functions is [JP].

3. Isometries versus biholomorphisms

Proof of Theorem 1.1. Suppose for some $2 \le m \le n$ there exists a C^1-Kobayashi isometry $f : D_1 \times \cdots \times D_m \to \mathbb{B}^n$. Now, fix $a = (a_2, \ldots, a_m) \in D_2 \times \cdots \times D_m$ and consider $f_a : D_1 \to \mathbb{B}^n$ defined by $f_a(z) = f(z, a_2, \ldots, a_m)$ for z in D_1. Using the product formula for the Kobayashi metric, we see that $d_{\mathbb{B}^n}\big(f_a(z), f_a(w)\big) = d_{D_1}(z, w)$ for all z, w in D_1. Said differently, the mapping $f_a : D_1 \to \mathbb{B}^n$ is distance preserving.

Step I. By Lemma 3.3 of [SV2], d_{D_1} is Lipschitz equivalent to the Euclidean distance on compact convex subdomains of D_1. To see this, observe that $F_{D_1}^K$ is jointly continuous by the tautness of D_1. Hence, $F_{D_1}^K(\cdot, v) \approx |v|$ on any compact subset of D_1. Integrating the above estimate along straight line segments and complex geodesics joining any two points $p, q \in D_1$, we get the required result. Note that convexity of D_1 guarantees the existence of geodesics between any two points in D_1 and that the line segment joining these two points is contained in D_1. Therefore, from the classical theorem of Rademacher and Stepanov, we see that f_a is differentiable almost everywhere.

STEP II. Firstly, it follows from [L1] that $F^K_{D_1}$ is C^1-smooth on $D_1 \times \mathbb{C}^{n_1} \setminus \{0\}$. Secondly, an argument similar to that used in [SV2] shows that the infinitesimal metric $F^K_{D_1}$ is Riemannian. $F^K_{D_1}$ being Riemannian at $p \in D_1$ is equivalent to the 'parallelogram law' being satisfied on T_pD_1, i.e.,

$$(F^K_{D_1}(p, v+w))^2 + (F^K_{D_1}(p, v-w))^2 = 2((F^K_{D_1}(p, v))^2 + (F^K(p, w))^2) \tag{3.1}$$

for all $v, w \in T_pD_1$. This is verified by first showing that $F^K_{D_1} = f_a^*(F^K_{\mathbb{B}^n})$ at every point of differentiability of f_a, which in turn relies on [MS] and on existence of smooth geodesics in D_1. Once we know that $F^K_{D_1}$ is Riemannian at every point of differentiability of f_a, which is a dense subset of D_1, we fix v, w in (3.1) and use the continuity of $F^K_{D_1}$ in the domain variable to conclude.

STEP III. Since f_a is a continuous distance preserving mapping between two C^1 Riemannian manifolds $(D_1, F^K_{D_1})$ and $(\mathbb{B}^n, F^K_{\mathbb{B}^n})$, applying the theorem of Myers–Steenrod ([MS]) gives us that f_a is C^1.

STEP IV. f_a is holomorphic or antiholomorphic. Let J_0 and J denote the almost complex structures on $T\mathbb{B}^n$ and TD_1 respectively. It suffices to prove that $df_a \circ J = \pm J_0 \circ df_a$. To do this, fix $p \in D_1$ and let S_0 and S denote the set of complex lines, i.e. 2-planes invariant under J_0 and J respectively, in $T_{f_a(p)}\mathbb{B}^n$ and T_pD_1. The goal now is to show that J-invariant 2-planes go to J_0-invariant 2-planes under df_a. To see this, first note that since $F^K_{\mathbb{B}^n}$ has constant holomorphic sectional curvature -4, at any point the sectional curvature of a 2-plane P spanned by an orthonormal pair of tangent vectors X, Y is $-(1 + 3\langle X, J_0Y\rangle)$. In particular, a two-dimensional subspace Q of $T_{f_a(p)}\mathbb{B}^n$ is in S_0 if and only if the sectional curvature of Q is -4.

Next, we claim that if $P \in S$ then $df_a(P) \in S_0$. The following observation will be needed to establish the above claim: The image $f_a \circ \phi(\Delta)$ is a C^∞-submanifold of $\mathbb{B}^n$ for any complex geodesic $\phi : \Delta \to D_1$ with $\phi(0) = p$ and $d\phi(T_0\Delta) = P$ (note that the convexity of D_1 ensures that such a ϕ exists). For this, it suffices to show that $f_a \circ \phi(\Delta) = \exp_{f_a(p)} df_a(P)$. Indeed, since $f_a \circ \phi$ is distance preserving, it takes geodesics in Δ to geodesics in $\mathbb{B}^n$. Therefore, $f_a \circ \phi(\Delta)$ is the union of geodesics which originate at $f_a(p)$ in directions along $df_a(P)$.

Firstly, since $f_a \circ \phi : \Delta \to f_a \circ \phi(\Delta)$ is a C^1-smooth distance preserving map for the induced metric on the image, appealing to the Myers–Steenrod theorem, we infer that $f_a \circ \phi$ is C^2. Hence, the sectional curvature of $f_a \circ \phi(\Delta)$ at $f_a(p)$ with respect to the metric induced by the mapping $f_a \circ \phi$ is equal to that of Δ with respect to the hyperbolic metric, i.e., -4. Secondly, $f_a \circ \phi(\Delta)$ is a totally geodesic two-dimensional submanifold of $\mathbb{B}^n$ as $f_a \circ \phi$ is distance preserving. This implies that the sectional curvature of $f_a \circ \phi(\Delta)$ at $f_a(p)$ with respect to the metric induced from $\mathbb{B}^n$ is equal to the sectional curvature in the $F^K_{\mathbb{B}^n}$-metric. Since this can be realized only by holomorphic sections in the ball, we conclude that complex lines are taken to complex lines by df_a.

Now, use the fact that the metrics involved are invariant under the almost complex structures to get $df_a \circ J = \pm J_0 \circ df_a$ on any $P \in S$. Next, the connectedness of S as a subset of the Grassmann manifold of 2-planes in $T_p D_1$ implies that either $df_a \circ J = J_0 \circ df_a$ on every $P \in S$ or $df_a \circ J = -J_0 \circ df_a$ on every $P \in S$, i.e., $df_a \circ J = \pm J_0 \circ df_a$ on $T_p D_1$. By the connectedness of D_1, it follows that $df_a \circ J = \pm J_0 \circ df_a$ on TD_1. This completes Step IV.

Recall that $f \in C^1$ by assumption and consequently the mapping $a = (a_2, \ldots, a_m) \mapsto f_a$ is also C^1—this is the only point in the proof that uses the C^1-smoothness of f. Now, from the connectedness of $D_2 \times \cdots \times D_m$, we see that either

$$df_a \circ J = J_0 \circ df_a \quad \text{or} \quad df_a \circ J = -J_0 \circ df_a$$

for every $a \in D_2 \times \cdots \times D_m$. In other words, f_a is either holomorphic for every choice of $a \in D_2 \times \cdots \times D_m$ or antiholomorphic for every a. Replacing f_a by its complex conjugate, if necessary, we may assume that f_a is holomorphic for every $a \in D_2 \times \cdots \times D_m$. Likewise, it can be shown that the mappings f_b given by $f_b(z) = f(b_1, z, b_3, \ldots, b_m)$, $z \in D_2$, are holomorphic for every parameter $b = (b_1, b_3, \ldots, b_m) \in D_1 \times D_3 \times \cdots \times D_m$. Repeating this argument, we see that f is separately holomorphic with respect to a group of variables for any fixed value of the other ones. In this setting, a generalization due to Hervé of the classical Hartogs theorem (see Theorem 2 in Section II.2.1 of [HERV]) shows that f is holomorphic on $D_1 \times \cdots \times D_m$ and consequently $D_1 \times \cdots \times D_m$ is biholomorphic to $\mathbb{B}^n$. This contradicts the fact that there cannot be a biholomorphism from a product domain onto $\mathbb{B}^n$, and finishes the proof for the Kobayashi metric. Since the Kobayashi and the Carathéodory metrics coincide on bounded convex domains (cf. [L2]), the theorem is completely proven. ∎

We record two simple corollaries of Theorem 1.1.

Corollary 3.1. *There is no C^1-Kobayashi or Carathéodory isometry between $\mathbb{B}^n$, the unit ball in $\mathbb{C}^n$, and Δ^n, the unit polydisc in $\mathbb{C}^n$, for any $n > 1$.*

Corollary 3.2. *There is no C^1-Kobayashi or Carathéodory isometry between $\mathbb{B}^n$ and the product of m Euclidean balls $\mathbb{B}^{n_1} \times \cdots \times \mathbb{B}^{n_m}$ for any $2 \le m \le n$ where $n = n_1 + \cdots + n_m$.*

A similar situation to the above corollary was considered in Proposition 2.2.8 of [JP] (see also [KR]), the emphasis here being on a different approach which is valid in a more general context.

4. Continuous extendability of isometries—Proof of Theorem 1.2. Recall the special coordinates constructed for weakly pseudoconvex finite type domains in [C]: Let $D \subset \mathbb{C}^2$ be a domain whose boundary is smooth pseudoconvex and of finite type $2m$, $m \in \mathbb{N}$, near the origin. Let U be a tiny neighbourhood of the origin and ρ a smooth defining function on U such that $U \cap \partial D = \{\rho = 0\}$ and $\frac{\partial \rho}{\partial z_2}(0,0) \neq 0$. Then for each $\zeta \in U \cap D$, there is a unique automorphism ϕ^ζ of $\mathbb{C}^2$ defined by

$$\phi^\zeta(z_1, z_2) = \Big(z_1 - \zeta_1, \Big(z_2 - \zeta_2 - \sum_{l=1}^{2m} d^l(\zeta)(z_1 - \zeta_1)^l\Big)(d^0(\zeta))^{-1}\Big),$$

where $d^l(\zeta)$ are non-zero functions depending smoothly on ζ with the property that the function $\rho \circ (\phi^\zeta)^{-1}$ satisfies

$$\rho \circ (\phi^\zeta)^{-1}(w_1, w_2) = 2\Re w_2 + \sum_{l=2}^{2m} P_{l,\zeta}(w_1, \bar{w}_1) + o\big(|w_1|^{2m} + \Im w_2\big)$$

where $P_{l,\zeta}(w_1, \bar{w}_1)$ are real-valued homogeneous polynomials of degree l without any harmonic terms. Let $\|\cdot\|$ be a fixed norm on the finite-dimensional space of all real-valued polynomials on the complex plane with degree at most $2m$ that do not contain any harmonic terms. Define, for some small $\delta > 0$,

$$\tau(\zeta, \delta) = \min_{2 \le l \le 2m} (\delta / \|P_{l,\zeta}(w_1, \bar{w}_1)\|)^{1/l}.$$

Let $\Delta^\delta_\zeta : \mathbb{C}^2 \to \mathbb{C}^2$ be anisotropic dilations defined by

$$\Delta^\delta_\zeta(z_1, z_2) = \big((\tau(\zeta, \delta))^{-1} z_1, \delta^{-1} z_2\big).$$

A useful set for approximating the geometry of D near the origin is Catlin's bidisc $Q(\zeta, \delta)$ determined by the quantities $\tau(\zeta, \delta)$ where

$$Q(\zeta, \delta) = (\Delta^\delta_\zeta \circ \phi^\zeta)^{-1}(\Delta \times \Delta).$$

The proof of Theorem 1.2 requires the following estimates on the Kobayashi metric near a weakly pseudoconvex boundary point of finite type.

PROPOSITION 4.1. *Let D be a bounded domain in $\mathbb{C}^2$. Assume that ∂D is C^∞-smooth weakly pseudoconvex of finite type near a point $p^0 \in \partial D$. Given $\epsilon > 0$, there exist positive numbers $r_2 < r_1 < \epsilon$ and C such that*

$$d_D(a, b) \ge -\tfrac{1}{2} \log d(b, \partial D) - C, \qquad a \in D \setminus B(p^0, r_1),\ b \in B(p^0, r_2) \cap D.$$

Proof. By Theorem 1.1 of [BER] there exists a neighbourhood U of p^0 in $\mathbb{C}^2$ such that

$$F_D^K(z,v) \approx \frac{|v_T|}{\tau(z, d(z,\partial D))} + \frac{|v_N|}{d(z,\partial D)} \tag{4.1}$$

for all $z \in U \cap D$ and v a tangent vector at z. As usual the decomposition $v = v_T + v_N$ into the tangential and normal components is taken at $\pi(z) \in \partial D$, which is the closest point on ∂D to z, and $\tau(z, d(z,\partial D))$ is as described above. Let γ be an arbitrary piecewise C^1-smooth curve in D joining a and b, i.e., $\gamma(0) = a, \gamma(1) = b$. As we travel along γ starting from a, there is a last point α on the curve with $\alpha \in \partial U \cap D$. Let $\gamma(t) = \alpha$ and denote by σ the subcurve of γ with end-points b and α. Then σ is contained in a δ-neighbourhood of ∂D for some fixed uniform $\delta > 0$. Using (4.1) we get

$$\begin{aligned}\int_0^1 F_D^K(\gamma(t),\dot\gamma(t))\,dt &\ge \int_t^1 F_D^K(\sigma(t),\dot\sigma(t))\,dt\\ &\gtrsim \int_t^1 \frac{|\dot\sigma_T(t)|}{\tau(\sigma(t), d(\sigma(t),\partial D))}\,dt + \int_t^1 \frac{|\dot\sigma_N(t)|}{d(\sigma(t),\partial D)}\,dt\\ &\ge \int_t^1 \frac{|\dot\sigma_N(t)|}{d(\sigma(t),\partial D)}\,dt.\end{aligned}$$

The last integrand is seen to be at least

$$\frac{d}{dt}\log(d(\sigma(t),\partial D))^{1/2}$$

(see for example Lemma 4.1 of [BB]) and hence

$$\int_0^1 F_D^K(\gamma(t),\dot\gamma(t))\,dt \gtrsim -\tfrac12 \log d(b,\partial D) - C$$

for some uniform $C > 0$. Taking the infimum over all such γ yields

$$d_D(a,b) \gtrsim -\tfrac12 \log d(b,\partial D) - C. \ \blacksquare$$

PROPOSITION 4.2. *Let D be a bounded domain in $\mathbb{C}^2$. Assume that ∂D is C^∞-smooth weakly pseudoconvex of finite type near two distinct boundary points a^0 and b^0. Then for a suitable constant C,*

$$d_D(a,b) \ge -\tfrac12 \log d(a,\partial D) - \tfrac12 \log d(b,\partial D) - C$$

whenever $a, b \in D$, a is near a^0 and b is near b^0.

Proof. Each path in D joining a and b must exit from neighbourhoods of a^0 and b^0. Hence the result follows from Proposition 4.1. $\blacksquare$

The following lemma will be useful for our purposes.

Lemma 4.3. *Let $D \subset \mathbb{C}^n$ be a bounded domain and $p^0 \in \partial D$ be a local holomorphic peak point. Then for any fixed $R > 0$ and every neighbourhood U of p^0 there exists a neighbourhood $V \subset U$ of p^0 with V relatively compact in U such that for all $z \in V \cap D$ we have*

$$B_D(z, cR) \subset B_{U\cap D}(z, R) \subset B_D(z, R)$$

where $c > 0$ is a constant independent of $z \in V \cap D$.

Proof. Let U be a neighbourhood of p^0 and let $g \in \mathcal{A}(U\cap D)$, the algebra of continuous functions on the closure of $U\cap D$ that are holomorphic on $U\cap D$, be such that $g(p^0) = 1$ and $|g(p)| < 1$ for $p \in \overline{U \cap D} \setminus \{p^0\}$. Fix $\epsilon > 0$. Then there exists a neighbourhood $V_1 \subset U$ of p^0 such that

$$F_D^K(z, v) \le F_{U\cap D}^K(z, v) \le (1 + \epsilon)F_D^K(z, v)$$

for $z \in V_1 \cap D$ and v a tangent vector at z. This is possible by the localization property of the Kobayashi metric (see for example Lemma 2 in [RO], or [G]).

The first inequality evidently implies that $B_{U\cap D}(z, R) \subset B_D(z, R)$ for all $z \in V_1 \cap D$ and all $R > 0$. For the left inclusion in the lemma, the following observation will be needed. For every $R > 0$ there is a neighbourhood $V \subset V_1$ of p^0 with the property that if $z \in V \cap D$ then $B_{U\cap D}(z, R/2) \subset V_1 \cap D$. For this it suffices to show that

$$\lim_{z\to p^0} d_{U\cap D}(z, (U \cap D) \setminus \overline{V_1 \cap D}) = \infty.$$

Indeed, for every $p \in (U \cap D) \setminus \overline{V_1 \cap D}$,

$$d_{U\cap D}(z, p) \ge d_\Delta(g(z), g(p)) \to \infty \tag{4.2}$$

as $z \to p^0$ since $g(p^0) = 1$ and $|g| < 1$ on $(U \cap D) \setminus \overline{V_1 \cap D}$. The estimate (4.2) is uniform with respect to $p \in (U \cap D) \setminus \overline{V_1 \cap D}$ and hence proves the claim.

Now for a given $R > 0$ let V be a sufficiently small neighbourhood of p^0 so that

$$B_{U\cap D}(z, R/2) \subset V_1 \cap D$$

if $z \in V \cap D$. Pick $p \in D$ in the complement of the closure of $B_{U\cap D}(z, R/2)$ and let $\gamma : [0, 1] \to D$ be a piecewise C^1-path with $\gamma(0) = z$ and $\gamma(1) = p$. Then there is a $t_0 \in (0, 1)$ such that $\gamma([0, t_0)) \subset B_{U\cap D}(z, R/2)$ and $\gamma(t_0) \in \partial B_{U\cap D}(z, R/2)$. Hence

$$
\begin{aligned}
\int_0^1 F_D^K(\gamma(t), \dot\gamma(t))\, dt &\ge \int_0^{t_0} F_D^K(\gamma(t), \dot\gamma(t)) dt \\
&\ge \frac{1}{1+\epsilon} \int_0^{t_0} F_{U\cap D}^K(\gamma(t), \dot\gamma(t))\, dt \\
&\ge \frac{1}{1+\epsilon}\, d_{U\cap D}(z, \gamma(t_0)) = \frac{R}{2(1+\epsilon)},
\end{aligned}
$$

which implies that $d_D(z,p) \ge R/(2(1+\epsilon))$. In other words,

$$\overline{B_D(z, R/(2(1+\epsilon)))} \subset \overline{B_{U\cap D}(z, R/2)}$$

and consequently

$$B_D(z, R/(2(1+\epsilon))) \subset B_{U\cap D}(z, R)$$

if $z \in V \cap D$. Finally observe that

$$B_D(z, R/(2(1+\epsilon))) \subset B_{U\cap D}(z, R) \subset B_D(z, R)$$

for all $z \in V \cap D$. ∎

Proof of Theorem 1.2. We first claim that f extends to $D_1 \cup \{p^0\}$ as a continuous mapping. To establish this, suppose that the claim is false. Then there is a sequence of points s^j in D_1 converging to $p^0 \in \partial D_1$ such that $f(s^j)$ does not converge to $q^0 \in \partial D_2$. Moreover, there exists a sequence $p^j \in D_1$ with $p^j \to p^0$ such that $f(p^j) \to q^0 \in \partial D_2$.

Consider polygonal paths γ^j in D_1 joining p^j and s^j defined as follows: for each j, choose $p^{j0}, s^{j0} \in \partial D_1$ closest to p^j and s^j respectively. Set $p^{j'} = p^j - |p^j - s^j| n(p^{j0})$ and $s^{j'} = s^j - |p^j - s^j| n(s^{j0})$ where $n(z)$ denotes the outward unit normal to ∂D_1 at $z \in \partial D_1$. Let γ^j be the union of three segments: the first is the straight line path joining p^j and $p^{j'}$ along the inward normal to ∂D_1 at the point p^{j0}, the second is a straight line path joining $p^{j'}$ and $s^{j'}$, and finally the third is the straight line path joining $s^{j'}$ and s^j along the inward normal to the point s^{j0}. Then $f \circ \gamma^j$ is a continuous path in D_2 joining $f(p^j)$ and $f(s^j)$. Now, for each j, pick $u^j \in \partial B(q^0, \epsilon) \cap U_2$ on trace$(f \circ \gamma^j)$ for some $\epsilon > 0$ sufficiently small. Let $t^j \in D_1$ be such that $f(t^j) = u^j$. Then $t^j \in \mathrm{trace}(\gamma^j)$ and hence $t^j \to p^0$ by construction. Moreover, $f(t^j) = u^j \to u^0 \in U_2 \cap \partial D_2$ $(u^0 \ne q^0)$. It follows from [FR] that

$$
\begin{aligned}
(4.3) \qquad d_{D_1}(p^j, t^j) \le& -\tfrac{1}{2} \log d(p^j, \partial D_1) \\
&+ \tfrac{1}{2} \log(d(p^j, \partial D_1) + |p^j - t^j|) \\
&+ \tfrac{1}{2} \log(d(t^j, \partial D_1) + |p^j - t^j|) \\
&- \tfrac{1}{2} \log d(t^j, \partial D_1) + C_1
\end{aligned}
$$

and

$$d_{D_2} f(p^j), f(t^j)) \geq -\tfrac{1}{2} \log d(f(p^j), \partial D_2) - \tfrac{1}{2} \log d(f(t^j), \partial D_2) - C_2 \tag{4.4}$$

for all j large and uniform positive constants C_1 and C_2.

Assertion. $d(f(p^j), \partial D_2) \leq C_3 d(p^j, \partial D_2)$ and $d(f(t^j), \partial D_2) \leq C_3 d(t^j, \partial D_2)$ for some uniform positive constant C_3.

Grant this for now. Using the fact that $d_{D_1}(p^j, t^j) = d_{D_2}(f(p^j), f(t^j))$ and comparing the inequalities (4.3) and (4.4), it follows from the Assertion that for all j large

$$-(C_1 + C_2 + \log C_3) \leq \tfrac{1}{2} \log(d(p^j, \partial D_1) + |p^j - t^j|) + \tfrac{1}{2} \log(d(t^j, \partial D_1) + |p^j - t^j|),$$

which is impossible. This contradiction proves the claim.

It remains to establish the Assertion. For this, fix $a \in D_1$ and use Proposition 4.1 to infer that

$$d_{D_1}(p^j, a) \geq -\tfrac{1}{2} \log d(p^j, \partial D_1) - C_4 \tag{4.5}$$

for some uniform positive constant C_4. On the other hand,

$$d_{D_2}(f(p^j), f(a)) \leq -\tfrac{1}{2} \log d(f(p^j), \partial D_2) + C_5 \tag{4.6}$$

for all j large and a uniform constant $C_5 > 0$. Fixing a in D_1, using $d_{D_1}(a, p^j) = d_{D_2}(f(p^j), f(a))$, and comparing (4.5) and (4.6), we get the required estimates. Hence the Assertion. This completes the proof of the claim.

Now, let z^j be a sequence of points in $U_1 \cap D_1$ with $z^j \to z^0 \in U_1 \cap \partial D_1$. The goal now is to show that f extends continuously to the point z^0. To see this, pick $z' \in U_1 \cap D_1$ such that $f(z') \in U_2 \cap D_2$. This can be achieved using the continuity of f. There are two cases to consider. After passing to a subsequence if needed, we have either

(i) $f(z^j) \to w^0 \in U_2 \cap \partial D_2$, or
(ii) $f(z^j) \to w^1 \in U_2 \cap D_2$ as $j \to \infty$.

In case (ii), observe that the quantity $d_{U_2 \cap D_2}(f(z^j), f(z'))$ is uniformly bounded (say by R) because of the completeness of $U_2 \cap D_2$. Therefore, for all j large,

$$d_{D_2}(f(z^j), f(z')) \leq d_{U_2 \cap D_2}(f(z^j), f(z')) < R.$$

Using the fact that $d_{D_1}(z^j, z') = d_{D_2}(f(z^j), f(z'))$, we get $z' \in B_{D_1}(z^j, R)$. Applying Lemma 4.3 forces that $z' \in B_{U_1 \cap D_1}(z^j, R/c)$ for some uniform constant c. This exactly means that

$$d_{U_1 \cap D_1}(z^j, z') < R/c.$$

This is however a contradiction as $d_{U_1 \cap D_1}(z^j, z')$ remains unbounded because of the completeness of $U_1 \cap D_1$. Hence, $f(z^j) \to w^0 \in U_2 \cap \partial D_2$ and consequently w^0 belongs to the cluster set of z^0 under f. From this point, proceeding exactly as in the first part of the proof we show that f extends continuously to the point z^0. Since $z^0 \in U_1 \cap \partial D_1$ was arbitrary, Theorem 1.2 is completely proven. ■

5. Non-existence of isometry—Proof of Theorem 1.3. The proof of Theorem 1.3 relies on the following lemma.

LEMMA 5.1. *Let D be a Kobayashi hyperbolic domain in $\mathbb{C}^n$ with a subdomain $D' \subset D$. Let $p, q \in D'$, $d_D(p,q) = a$ and $b > a$. If $B_D(q,b) \subset D'$, then*

$$d_{D'}(p,q) \le \frac{1}{\tanh(b-a)} d_D(p,q), \qquad F^K_{D'}(p,v) \le \frac{1}{\tanh(b-a)} F^K_D(p,v).$$

The reader is referred to [KK] (or [KM]) for a proof, but it should be noted that this statement emphasizes an upper bound for $d_{D'}$ in terms of d_D. An estimate with the inequality reversed is an immediate consequence of the definition of the Kobayashi metric.

The second ingredient is an estimate for the Kobayashi and the inner Carathéodory distance between two points in a weakly pseudoconvex finite type domain D in $\mathbb{C}^2$, due to Herbort ([HERB]). To state this, let $d(\cdot, \partial D)$ be the Euclidean distance to the boundary and ρ a smooth defining function for ∂D. For $a, b \in D$, define

$$\begin{aligned}
\rho^*(a,b) &= \log\left(1 + \frac{\tilde{d}(a,b)}{d(a,\partial D)} + \frac{|\langle L(a), a-b\rangle|}{\tau(a, d(a,\partial D))}\right),\\
L(a) &= \left(-\frac{\partial \rho}{\partial z_2}(a), \frac{\partial \rho}{\partial z_1}(a)\right),\\
d'(a,b) &= \inf\{\delta > 0 : a \in Q(b,\delta)\},\\
\tilde{d}(a,b) &= \min\{d'(a,b), |a-b|\},
\end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard hermitian inner product in $\mathbb{C}^2$. The main result of [HERB] that is needed is:

THEOREM 5.2. *Assume that $D = \{\rho < 0\} \subset \mathbb{C}^2$ be a bounded pseudoconvex domain with smooth boundary such that all boundary points are of finite type. Then there exists a positive constant C_* such that for any $a, b \in D$,*

$$C_*(\rho^*(a,b) + \rho^*(b,a)) \le c^i_D(a,b) \le d_D(a,b) \le \frac{1}{C_*}(\rho^*(a,b) + \rho^*(b,a)).$$

Proof of Theorem 1.3. Suppose that there exists an isometry $f : D_1 \to D_2$ with q^0 belonging to the cluster set of p^0. The proof is in several steps.

Step I. Considering the Levi form

$$\sum_{i,j=1}^{2} \frac{\partial^2 \rho^0}{\partial z_i \bar{z}_j}(p^0) v_i \bar{v}_j$$

for every complex tangent vector $v = (v_1, 0)$ at p^0, we see that ∂D_1 is weakly pseudoconvex near p^0. Using the explicit form of the defining function ρ^0, it is a straightforward calculation that ∂D_1 is of finite type $2m$ and smooth near p^0. Applying Theorem 1.2, we immediately see that f extends continuously to p^0.

Step II. Choose a sequence $a^j = (0, -\delta_j)$ in D_1 along the inner normal at the origin, where $\delta_j > 0$ and $\delta_j \to 0$. Step I shows that $b^j = f(a^j) \to q^0 \in \partial D_2$.

The scaling method applied to (D_1, D_2, f). Let Δ^j be the dilation defined by

$$\Delta^j(z_1, z_2) = (\delta_j^{-1/2m} z_1, \delta_j^{-1} z_2),$$

and note that $\Delta^j(a^j) = (0, -1)$ while the domains $D_1^j = \Delta^j(D_1)$ are defined in a neighbourhood of the origin by

$$\delta_j^{-1} \rho^0 \circ (\Delta^j)^{-1}(z) = 2\Re z_2 + |z_1|^{2m} + \delta_j^{-1} o(\delta_j |z_1|^{2m} + \delta_j \Im z_2).$$

On each compact set in $\mathbb{C}^2$ the error term converges to zero and hence the sequence of domains D_1^j converges in the Hausdorff sense to

$$D_{1,\infty} = \{z \in \mathbb{C}^2 : 2\Re z_2 + |z_1|^{2m} < 0\}.$$

To scale D_2 recall that by [P1], for each ζ near $q^0 \in \partial D_2$ there is a unique automorphism $h_\zeta : \mathbb{C}^2 \to \mathbb{C}^2$ with $h_\zeta(\zeta) = 0$ such that

$$h_\zeta(D_2) = \{z \in \mathbb{C}^2 : 2\Re(z_2 + K_\zeta(z)) + H_\zeta(z) + \alpha_\zeta(z) < 0\}$$

where $K_\zeta(z) = \sum_{i,j=1}^{2} a_{ij}(\zeta) z_i z_j$, $H_\zeta(z) = \sum_{i,j=1}^{2} b_{ij}(\zeta) z_i \bar{z}_j$ and $\alpha_\zeta(z) = o(|z|^2)$ with $K_\zeta(z_1, 0) \equiv 0$ and $H_\zeta(z_1, 0) \equiv |z_1|^2$. Furthermore, if $\zeta = (\zeta_1, \zeta_2) \in D_2$ as above, we consider the point $\tilde{\zeta} = (\zeta_1, \zeta_2 + \epsilon)$ where $\epsilon > 0$ is chosen to ensure that $\tilde{\zeta} \in \partial D_2$. Then the actual form of h_ζ shows that $h_{\tilde{\zeta}}(\zeta) = (0, -\epsilon)$.

Consider the sequence $b^j = f(a^j) \in D_2$ that converges to q^0 and denote by ζ^j the point on ∂D_2 chosen such that if $b^j = (b_1^j, b_2^j)$ then $\zeta^j = (b_1^j, b_2^j + \epsilon_j)$ for some $\epsilon_j > 0$. Note that $\epsilon_j \approx \operatorname{dist}(b^j, \partial D_2)$ for all j large. Let $h^j := h_{\zeta^j}$ be the biholomorphism corresponding to ζ^j as described above. Define a dilation of coordinates by

$$T^j(w_1, w_2) = (\epsilon_j^{-1/2} w_1, \epsilon_j^{-1} w_2).$$

Let $D_1^j = \Delta^j(D_1)$ and $D_2^j = T^j \circ h^j(D_2)$ be the scaled domains and let the scaled maps between them be

$$f^j = T^j \circ h^j \circ f \circ (\Delta^j)^{-1} : D_1^j \to D_2^j.$$

Note first that $T^j \circ h^j(b^j) = (0, -1)$, which implies that $f^j(0, -1) = (0, -1)$ and f^j is an isometry for the Kobayashi distances on D_1^j and D_2^j for each j. The defining function for D_2^j is given by

$$2\Re w_2 + |w_1|^2 + A^j(w)$$

where

$$|A^j(w)| \leq |w|^2(c\sqrt{\epsilon_j} + \eta(\epsilon_j|w|^2))$$

and $\eta(t)$ is a function of one real variable such that $\eta(t) = o(1)$ as $t \to 0$. Hence D_2^j converge to

$$D_{2,\infty} = \{w \in \mathbb{C}^2 : 2\Re w_2 + |w_1|^2 < 0\},$$

which is the unbounded realization of the unit ball.

Stability of the Kobayashi metric

PROPOSITION 5.3. $d_{D_1^j}((0, -1), \cdot) \to d_{D_{1,\infty}}((0, -1), \cdot)$ *uniformly on compact subsets of* $D_{1,\infty}$.

This was proved in [MV]; we include the proof for completeness. The proof requires several steps. First, it is natural to prove convergence at the infinitesimal level:

LEMMA 5.4. *For* $(s, v) \in D_{1,\infty} \times \mathbb{C}^2$,

$$\lim_{j\to\infty} F^K_{D_1^j}(s, v) = F^K_{D_{1,\infty}}(s, v).$$

Moreover, the convergence is uniform on compact subsets of $D_{1,\infty} \times \mathbb{C}^2$.

Proof. Let $S \subset D_{1,\infty}$ and $G \subset \mathbb{C}^2$ be compact and suppose that the desired convergence does not occur. Then there is an $\epsilon_0 > 0$ such that after passing to a subsequence if necessary, we may assume that there exists a sequence of points s^j in S which is relatively compact in D_1^j and a sequence $v^j \in G$ such that

$$|F^K_{D_1^j}(s^j, v^j) - F^K_{D_{1,\infty}}(s^j, v^j)| > \epsilon_0$$

for j large. Let $s^j \to s \in S$ and $v^j \to v \in G$. Since $F^K_{D_{1,\infty}}(s, \cdot)$ is homogeneous, we may assume that $|v^j| = 1$ for all j. Observe that $D_{1,\infty}$ is complete hyperbolic and hence taut. The tautness of $D_{1,\infty}$ implies via a normal family argument that $F^K_{D_{1,\infty}}(\cdot, \cdot)$ is jointly continuous, $0 < F^K_{D_{1,\infty}}(s, v) < \infty$ and there exists a holomorphic extremal disc $g : \Delta \to D_{1,\infty}$ that by definition satisfies $g(0) = s, g'(0) = \mu v$ where $\mu > 0$ and $F^K_{D_{1,\infty}}(s, v) = 1/\mu$. Hence

$$|F^K_{D_1^j}(s^j, v^j) - F^K_{D_{1,\infty}}(s, v)| > \epsilon_0/2 \tag{5.1}$$

for j sufficiently large. Fix $\delta \in (0,1)$ and define the holomorphic mappings $g^j : \Delta \to \mathbb{C}^2$ by

$$g^j(z) = g((1-\delta)z) + (s^j - s) + \mu(1-\delta)z(v^j - v).$$

The image $g((1-\delta)\Delta)$ is compactly contained in $D_{1,\infty}$, $s^j \to s$ and $v^j \to v$, therefore $g^j : \Delta \to D_1^j$ for j large. Also, $g^j(0) = s^j$ and $(g^j)'(0) = \mu(1-\delta)v^j$. By the definition of the infinitesimal metric it follows that

$$F^K_{D_1^j}(s^j, v^j) \le 1/(\mu(1-\delta)) = F^K_{D_{1,\infty}}(s,v)/(1-\delta).$$

Letting $\delta \to 0^+$ yields

$$\limsup_{j\to\infty} F^K_{D_1^j}(s^j, v^j) \le F^K_{D_{1,\infty}}(s,v). \tag{5.2}$$

Conversely, for $\epsilon > 0$ arbitrarily small, there exist holomorphic mappings $h^j : \Delta \to D_1^j$ satisfying $h^j(0) = s^j$ and $(h^j)'(0) = \mu^j v^j$ where $\mu^j > 0$ and

$$F^K_{D_1^j}(s^j, v^j) \ge 1/\mu^j - \epsilon. \tag{5.3}$$

The sequence h^j has a subsequence that converges to a holomorphic mapping $h : \Delta \to D_{1,\infty}$ uniformly on compact sets of Δ. To see this, consider $\Delta(0,r)$ for $r \in (0,1)$. Now, $\phi^{p^0} = \phi^{(0,0)} = \mathrm{id}_{\mathbb{C}^2}$ and $\tau(p^0, \delta_j) = \tau((0,0), \delta_j) \approx \delta_j^{1/2m}$. Further, we may assume that S is compactly contained in $\Delta(0, C_1^{1/2m}) \times \Delta(0, C_1)$ for some $C_1 > 1$. As a consequence,

$$(\Delta^j)^{-1}(s^j) \in Q(p^0, C_1\delta_j)$$

for all j. Also, note that

$$(\Delta^j)^{-1}(s^j) \to p^0 \in \partial D_1$$

as $j \to \infty$. Now, applying Proposition 1 of [BERC] to the mappings

$$(\Delta^j)^{-1} \circ h^j : \Delta \to D_1$$

shows that there exists a uniform positive constant $C_2 = C_2(r)$ with

$$(\Delta^j)^{-1} \circ h^j(\Delta(0,r)) \subset Q(p^0, C_2C_1\delta_j)$$

or equivalently

$$h^j(\Delta(0,r)) \subset \Delta(0, (C_1C_2)^{1/2m}) \times \Delta(0, C_1C_2).$$

Therefore, $\{h^j\}$ is uniformly bounded on each compact set in Δ and is therefore normal. Let $h : \Delta \to \mathbb{C}^2$ be a holomorphic limit of some subsequence of $\{h^j\}$. Since $h^j(0) = s^j$, it follows that $h(0) = s$. It remains to show that h maps Δ into $D_{1,\infty}$. For this, note that D_1^j are defined in a neighbourhood of the origin by

$$2\Re z_2 + |z_1|^{2m} + \frac{1}{\delta_j} o(\delta_j |z_1|^{2m} + \delta_j \Im z_2) < 0.$$

If $h^j(z) = (h_1^j(z), h_2^j(z))$ for each j, then

$$2\Re(h_2^j(z)) + |h_1^j(z)|^{2m} + \frac{1}{\delta_j} o(\delta_j |h_1^j(z)|^{2m} + \delta_j \Im(h_2^j(z))) < 0$$

whenever $z \in \Delta(0,r)$, $r \in (0,1)$. Letting $j \to \infty$ yields

$$2\Re(h_2(z)) + |h_1(z)|^{2m} \leq 0,$$

which exactly means that $h(\Delta(0,r)) \subset \overline{D}_{1,\infty}$. Since $r \in (0,1)$ was arbitrary, it follows that $h(\Delta) \subset \overline{D}_{1,\infty}$. Since $h(0) = s$, the maximum principle shows that $h(\Delta) \subset D_{1,\infty}$. Note that

$$h'(0) = \lim_{j\to\infty} (h^j)'(0) = \lim_{j\to\infty} \mu^j v^j = \mu v$$

for some μ. The inequalities (5.2), (5.3) and $0 < F^K_{D_{1,\infty}}(s,v) < \infty$ force that $\mu > 0$. Therefore,

$$F^K_{D_{1,\infty}}(s,v) \leq 1/\mu.$$

The above observation together with the inequality (5.3) yields

$$\liminf_{j\to\infty} F^K_{D_1^j}(s^j, v^j) \geq F^K_{D_{1,\infty}}(s,v). \tag{5.4}$$

Combining (5.2) and (5.4) shows that

$$\lim_{j\to\infty} F^K_{D_1^j}(s^j, v^j) = F^K_{D_{1,\infty}}(s,v),$$

which contradicts the assumption (5.1) and proves the lemma. ∎

To control the integrated Kobayashi distance on the domains D_1^j, we first note the following:

Lemma 5.5. *For any $R > 0$ and for all j large, $B_{D_1^j}\big((0,-1), R\big)$ is compactly contained in $D_{1,\infty}$.*

Proof. First note that

$$B_{D_1^j}((0,-1), R) = \Delta^j(B_{D_1}(a^j, R)).$$

Since $p^0 \in \partial D_1$ is a weakly pseudoconvex point, it is possible (see [BC] for details) to find a smaller domain $\Omega \subset D_1$ with the following three properties:

(i) Ω is a C^∞-smooth bounded weakly pseudoconvex domain contained in D_1.
(ii) $p^0 \in \partial\Omega$.
(iii) $\partial\Omega \cap \partial D_1$ contains a relatively open neighbourhood of p^0 in $\partial\Omega$.

Moreover, since $p^0 \in \partial D_1$ is a local holomorphic peak point, by Lemma 4.3 there exists a neighbourhood V of p^0 with $V \cap D_1 \subset \Omega$ and a uniform positive constant c such that for all $z \in V \cap D_1$,

$$B_{D_1}(z, cR) \subset B_\Omega(z, R) \subset B_{D_1}(z, R),$$

and therefore it will suffice to show that $\Delta^j(B_\Omega(a^j, R))$ is compactly contained in $D_{1,\infty}$. The proof now divides into two parts. In the first part we show that the sets $\Delta^j(B_\Omega(a^j, R))$ cannot accumulate at the point at infinity in $\partial D_{1,\infty}$ and in the second part we show that the sets $B_{D_1^j}((0,-1), R)$ do not cluster at any finite boundary point. Assume that $p \in B_\Omega(a^j, R)$. Using Herbort's lower estimate for the Kobayashi metric gives us

$$C_*(\rho^*(a^j, p) + \rho^*(p, a^j)) \leq d_\Omega(a^j, p) < R.$$

As a consequence,

$$\tilde{d}(a^j, p) < \exp(R/C_*) d(a^j, \partial\Omega) = \exp(R/C_*) d(a^j, \partial D_1),$$

which in turn implies that either

- $|a^j - p| < d(a^j, \partial D_1) \exp(R/C_*)$, or
- for each j, there exists a $\delta_j \in (0, d(a^j, \partial D_1) \exp(R/C_*))$ such that $a^j \in Q(p, \delta_j)$.

It follows from Proposition 1.7 in [C] that there exists a uniform positive constant C such that for each j, the following holds: if $a^j \in Q(p, \delta_j)$, then $p \in Q(a^j, C\delta_j)$. Hence, the second statement above can be rewritten as: there exists a positive constant C such that for each j, there exists a $\delta_j \in (0, d(a^j, \partial D_1) \exp(R/C_*))$ with

$$p \in (\phi^{a^j})^{-1}(\Delta(0, \tau(a^j, C\delta_j)) \times \Delta(0, C\delta_j)).$$

Said differently, $B_\Omega(a^j, R)$ is contained in the union

$$B(a^j, d(a^j, \partial D_1) \exp(R/C_*)) \cup (\phi^{a^j})^{-1}(\Delta(0, \tau(a^j, C\delta_j)) \times \Delta(0, C\delta_j))$$

with δ_j as described above. Now,

$$\begin{aligned} (5.5) \quad & \Delta^j\{(z_1, z_2) : |z_1 - a_1^j|^2 + |z_2 - a_2^j|^2 < (d(a^j, \partial D_1))^2 \exp(2R/C_*)\} \\ &= \{(w_1, w_2) : |w_1|^2 + \delta_j^{-1/m} \delta_j^2 |w_2 + 1|^2 < \delta_j^{-1/m} (d(a^j, \partial D_1))^2 \exp(2R/C_*)\}. \end{aligned}$$

If $w = (w_1, w_2)$ belongs to the set described above, then

$$(5.6) \qquad |w_1| \leq \delta_j^{-1/2m} d(a^j, \partial D_1) \exp(R/C_*) = \delta_j^{-1/2m} \delta_j \exp(R/C_*),$$

$$(5.7) \qquad |w_2 + 1| \leq \delta_j^{-1} d(a^j, \partial D_1) \exp(R/C_*) = \exp(R/C_*).$$

Moreover, for $\delta_j \in (0, d(a^j, \partial D_1) \exp(R/C_*))$,

$$\begin{aligned} &(\phi^{a^j})^{-1}\{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < \tau(a^j, C\delta_j), |z_2| < C\delta_j\} \\ &= \Big\{w : |w_1 - a_1^j| < \tau(a^j, C\delta_j), \Big|w_2 - a_2^j - \sum_{l=1}^{2m} d^l(a^j)(w_1 - a_1^j)^l\Big| < C\delta_j d^0(a^j)\Big\}, \end{aligned}$$

so that

$$\begin{aligned}(5.8)\quad &\Delta^j \circ (\phi^{a^j})^{-1}(\Delta(0,\tau(a^j,C\delta_j)) \times \Delta(0,C\delta_j)) \\ &= \Big\{ w : |w_1| < \delta_j^{-1/2m}\tau(a^j,C\delta_j),\ \Big| w_2 + 1 + \delta_j{}^{-1}\Big(\sum_{l=1}^{2m} \alpha^{j,l} w_1^l\Big)\Big| < Cd^0(a^j)\Big\}\end{aligned}$$

where

$$\alpha^{j,l} = d^l(a^j)\delta_j^{l/2m}.$$

If $w = (w_1, w_2)$ belongs to the set given by (5.8), then

$$|w_1| < \delta_j^{-1/2m}\tau(a^j, C\delta_j), \tag{5.9}$$

$$\Big| w_2 + 1 + \delta_j{}^{-1}\Big(\sum_{l=1}^{2m} d^l(a^j)\delta_j^{l/2m} w_1^l\Big)\Big| < Cd^0(a^j). \tag{5.10}$$

Among other things, it was shown in [C] that

- $\delta_j^{1/2} \lesssim \tau(a^j, \delta_j) \lesssim \delta_j^{1/2m}$,
- $|d^l(a^j)| \lesssim \delta_j(\tau(a^j,\delta_j))^{-l}$ for all $1 \le l \le 2m$,
- $d^0(a^j) \approx 1$.

These estimates together with (5.6), (5.7), (5.9) and (5.10) show that if $w = (w_1, w_2)$ belongs to either (5.5) or (5.8), then $|w|$ is uniformly bounded. In other words, the sets

$$\begin{aligned}&\Delta^j(B(a^j, d(a^j,\partial D_1)\exp(R/C_*))) \\ &\qquad\qquad \cup \Delta^j((\phi^{a^j})^{-1}(\Delta(0,\tau(a^j,C\delta_j)) \times \Delta(0,C\delta_j)))\end{aligned}$$

are uniformly bounded. Therefore, $\Delta^j(B_\Omega(a^j,R))$ and hence $B_{D_1^j}((0,-1),R)$ as a set cannot cluster at the point at infinity on $\partial D_{1,\infty}$.

It remains to show that the sets $B_{D_1^j}((0,-1),R)$ do not cluster at any finite point of $\partial D_{1,\infty}$. Suppose there is a sequence of points $z^j \in B_{D_1^j}((0,-1),R)$ such that $z^j \to z^0$ where z^0 is a finite point on $\partial D_{1,\infty}$. Applying Theorem 1.1 of [BER], we see that there exists a neighbourhood U of z^0 in $\mathbb{C}^2$ such that

$$F^K_{D_1^j}(z,v) \approx \frac{|v_T|}{\tau(z, d(z,\partial D_1^j))} + \frac{|v_N|}{d(z,\partial D_1^j)} \tag{5.11}$$

uniformly for all j large, $z \in U \cap D_{1,\infty}$ and v a tangent vector at z; this stable version holds since the defining functions for D_1^j converge to that of $D_{1,\infty}$ in the C^∞-topology on a given compact set. Here the decomposition $v = v_T + v_N$ into the tangential and normal components is taken at $\pi^j(z) \in \partial D_1^j$, which is the closest point on ∂D_1^j to z. Note that

$$d(z,\partial D_1^j) \approx d(z,\partial D_{1,\infty})$$

for $z \in U \cap D_{1,\infty}$ and that $\pi^j(z) \to \pi(z) \in \partial D_{1,\infty}$ where $|\pi(z) - z| = d(z, \partial D_{1,\infty})$. Let γ^j be an arbitrary piecewise C^1-smooth curve in D_1^j joining z^j and $(0,-1)$, that is, $\gamma^j(0) = (0,-1)$, $\gamma^j(1) = z^j$. As we travel along γ^j starting from $(0,-1)$, there is a last point α^j on the curve with $\alpha^j \in \partial U \cap D_1^j$. Let $\gamma^j(t_j) = \alpha^j$ and let σ^j be the subcurve of γ^j with end-points z^j and α^j. Then σ^j is contained in an ϵ-neighbourhood of ∂D_1^j for some fixed uniform $\epsilon > 0$ and for all j large. Arguing as in the proof of Proposition 4.1 using (5.11) we get

$$\int_0^1 F^K_{D_1^j}(\gamma^j(t), \dot\gamma^j(t))\, dt \ge \int_{t_j}^1 F^K_{D_1^j}(\sigma^j(t), \dot\sigma^j(t))\, dt \gtrsim -\tfrac{1}{2}\log d(z^j, \partial D_1^j) + C$$

for some uniform $C > 0$. Taking the infimum over all such γ^j shows that

$$d_{D_1^j}(z^j, (0,-1)) \gtrsim -\tfrac{1}{2}\log d(z^j, \partial D_1^j) + C.$$

This is however a contradiction since the left side is at most R while the right side becomes unbounded. This completes the proof of Lemma 5.5. ∎

Proof of Proposition 5.3. Let K be a compact subdomain of $D_{1,\infty}$ and suppose that the desired convergence does not occur. Then there exists an $\epsilon_0 > 0$ and a sequence of points $z^j \in K$ which is relatively compact in D_1^j for all j large such that

$$|d_{D_1^j}((0,-1), z^j) - d_{D_{1,\infty}}((0,-1), z^j)| > \epsilon_0.$$

By passing to a subsequence, we may assume that $z^j \to z^0 \in K$. Then using the continuity of $d_{D_{1,\infty}}(z^0, \cdot)$ we have

$$|d_{D_1^j}((0,-1), z^j) - d_{D_{1,\infty}}((0,-1), z^0)| > \epsilon_0/2$$

for all j large. Fix $\epsilon > 0$ and let $\gamma : [0,1] \to D_{1,\infty}$ be a piecewise C^1-path such that $\gamma(0) = (0,-1)$, $\gamma(1) = z^0$ and

$$\int_0^1 F^K_{D_{1,\infty}}(\gamma(t), \dot\gamma(t))\, dt < d_{D_{1,\infty}}((0,-1), z^0) + \epsilon/2.$$

Define $\gamma^j : [0,1] \to \mathbb{C}^2$ by

$$\gamma^j(t) = \gamma(t) + (z^j - z^0)t.$$

Since the image $\gamma([0,1])$ is compactly contained in $D_{1,\infty}$ and $z^j \to z^0$, it follows that $\gamma^j : [0,1] \to D_1^j$ for j large. Note that $\gamma^j \to \gamma$, $\dot\gamma^j \to \dot\gamma$ uniformly on $[0,1]$, $\gamma^j(0) = (0,-1)$ and $\gamma^j(1) = z^j$. Applying Lemma 5.4, we obtain

$$\int_0^1 F^K_{D_1^j}(\gamma^j(t), \dot\gamma^j(t))\, dt \le \int_0^1 F^K_{D_{1,\infty}}(\gamma(t), \dot\gamma(t))\, dt + \epsilon/2 < d_{D_{1,\infty}}((0,-1), z^0) + \epsilon$$

for all j large. By the definition of $d_{D_1^j}\big((0,-1),z^j\big)$ it follows that

$$d_{D_1^j}((0,-1),z^j) \le \int_0^1 F^K_{D_1^j}(\gamma^j(t),\dot{\gamma}^j(t))\,dt \le d_{D_{1,\infty}}((0,-1),z^0)+\epsilon.$$

Thus

$$\limsup_{j\to\infty} d_{D_1^j}((0,-1),z^j) \le d_{D_{1,\infty}}((0,-1),z^0). \tag{5.12}$$

A similar argument yields

$$\limsup_{j\to\infty} d_{D_1^j}((0,-1),\cdot) \le d_{D_{1,\infty}}((0,-1),\cdot),$$

which in turn implies

$$B_{D_{1,\infty}}((0,-1),R-\epsilon) \subset B_{D_1^j}((0,-1),R) \tag{5.13}$$

for any $R>0$ and for all j large. Exploiting the continuity of the distance function $d_{D_{1,\infty}}((0,-1),\cdot)$ and (5.13), we see that

$$z^j \in K \subset B_{D_{1,\infty}}((0,-1),\tilde{R}-\epsilon) \subset B_{D_1^j}((0,-1),\tilde{R})$$

for some uniform positive constant $\tilde{R}=\tilde{R}(K)$ and for all j large. Pick $R' \gg \tilde{R}$. Using Lemma 5.5, we have

$$B_{D_1^j}((0,-1),R') \subset D_{1,\infty}$$

and consequently

$$d_{D_{1,\infty}}((0,-1),\cdot) \le d_{B_{D_1^j}((0,-1),R')}((0,-1),\cdot),$$

which in particular implies that

$$d_{D_{1,\infty}}((0,-1),z^j) \le d_{B_{D_1^j}((0,-1),R')}((0,-1),z^j) \tag{5.14}$$

for all j large. Now, applying Lemma 5.1 to the domain D_1^j with the Kobayashi metric ball $B_{D_1^j}((0,-1),R')$ as the subdomain D' gives

$$d_{B_{D_1^j}((0,-1),R')}((0,-1),z^j) \le \frac{d_{D_1^j}((0,-1),z^j)}{\tanh(R'-d_{D_1^j}((0,-1),z^j))}. \tag{5.15}$$

The right side above cannot be greater than

$$\frac{d_{D_1^j}((0,-1),z^j)}{\tanh(R'-\tilde{R})}.$$

The above observation together with (5.14) and (5.15) yields

$$d_{D_{1,\infty}}((0,-1),z^j) \le \frac{d_{D_1^j}((0,-1),z^j)}{\tanh(R'-\tilde{R})}.$$

Let first $j \to \infty$ and then $R' \to \infty$ to get

$$d_{D_{1,\infty}}((0,-1), z^0) \le \limsup_{j\to\infty} d_{D_1^j}((0,-1), z^j). \tag{5.16}$$

From (5.12) and (5.16), we get

$$d_{D_1^j}((0,-1), z^j) \to d_{D_{1,\infty}}((0,-1), z^0).$$

This is a contradiction, and hence the result follows. ∎

The following result from [SV1] shows that the integrated Kobayashi distance is stable under scaling in the strongly pseudoconvex case. The proof uses Lempert's theorem ([L1]) that guarantees the existence of complex geodesics in strongly convex domains.

Proposition 5.6. *For $w \in D_{2,\infty}$, $d_{D_2^j}(w, \cdot) \to d_{D_{2,\infty}}(w, \cdot)$ uniformly on compact sets of $D_{2,\infty}$.*

Proposition 5.7. *For $w \in D_{2,\infty}$ and $R > 0$,*

$$B_{D_2^j}(w, R) \to B_{D_{2,\infty}}(w, R)$$

in the Hausdorff sense. Moreover, for any $\epsilon > 0$ and for all j large,

(i) $B_{D_{2,\infty}}(w, R) \subset B_{D_2^j}(w, R + \epsilon)$,
(ii) $B_{D_2^j}(w, R - \epsilon) \subset B_{D_{2,\infty}}(w, R)$.

Proof. It is not difficult to see that $B_{D_2^j}(w, R) \to B_{D_{2,\infty}}(w, R)$ from the stability of the Kobayashi metric on scaled domains, i.e., Proposition 5.6.

To verify (i), first observe that the closure of $B_{D_{2,\infty}}(w, R)$ is compact since $D_{2,\infty}$ is Kobayashi complete. Then using Proposition 5.6, we get

$$d_{D_2^j}(w, \tilde{w}) \le d_{D_{2,\infty}}(w, \tilde{w}) + \epsilon$$

for all $\tilde{w}$ in the closure of $B_{D_{2,\infty}}(w, R)$ and for all j large. Said differently,

$$B_{D_{2,\infty}}(w, R) \subset B_{D_2^j}(w, R + \epsilon)$$

for all j large. For (ii) suppose that the desired result is not true. Then there exists an $\epsilon_0 > 0$ and a sequence of points w'^j in $\partial B_{D_{2,\infty}}(w, R)$ such that $w'^j \in B_{D_2^j}(w, R - \epsilon_0)$. In view of compactness of $\partial B_{D_{2,\infty}}(w, R)$, we may assume that $w'^j \to w' \in \partial B_{D_{2,\infty}}(w, R)$. It follows from Proposition 5.6 that

$$d_{D_2^j}(w'^j, w) \to d_{D_{2,\infty}}(w', w).$$

Consequently, $d_{D_{2,\infty}}(w', w) \le R - \epsilon_0$. This contradicts the fact that

$$d_{D_{2,\infty}}(w', w) = R,$$

thereby proving (ii). ∎

Step III (of the proof of Theorem 1.3). Let $\{K_\nu\}$ be an increasing sequence of relatively compact subsets of $D_{1,\infty}$ that exhausts $D_{1,\infty}$ such that each contains $(0,-1)$. Fix a pair of K_{ν_0} compactly contained in K_{ν_0+1} and write $K_1 = K_{\nu_0}$ and $K_2 = K_{\nu_0+1}$ for brevity. Let $\omega(K_1)$ be a neighbourhood of K_1 such that $\omega(K_1) \subset K_2$. Since $D_1^j \to D_{1,\infty}$, it follows that $K_1 \subset \omega(K_1) \subset K_2$, which in turn is relatively compact in D_1^j for all j large. Then the sequence f^j is equicontinuous at each point of $\omega(K_1)$.

Since each f^j is a Kobayashi isometry,

$$d_{D_2^j}(f^j(z), f^j(\tilde{z})) = d_{D_1^j}(z, \tilde{z})$$

for $z, \tilde{z}$ in K_2. In particular,

$$d_{D_2^j}(f^j(z), (0,-1)) = d_{D_1^j}(z, (0,-1))$$

for all z in K_2. It follows from Proposition 5.3 that

$$d_{D_1^j}(\cdot, (0,-1)) \to d_{D_{1,\infty}}(\cdot, (0,-1))$$

uniformly on K_2 and hence for all z in K_2,

$$d_{D_2^j}(f^j(z), (0,-1)) = d_{D_1^j}(z, (0,-1)) \leq d_{D_{1,\infty}}(z, (0,-1)) + \epsilon$$

for $\epsilon > 0$ fixed. The right hand side is bounded above by a uniform constant $\tilde{R} > 0$. This observation together with Proposition 5.7 implies that

$$f^j(K_2) \in B_{D_2^j}((0,-1), \tilde{R}) \subset B_{D_{2,\infty}}((0,-1), \tilde{R} + \epsilon).$$

This is just the assertion that $\{f^j(K_2)\}$ is uniformly bounded.

Further, for each $z \in \omega(K_1)$ fixed, there exists an $r > 0$ such that $B(z,r)$ is compactly contained in $\omega(K_1)$. The distance decreasing property of the Kobayashi metric together with its explicit form on $B(z,r)$ gives

$$d_{D_2^j}(f^j(z), f^j(\tilde{z})) = d_{D_1^j}(z, \tilde{z}) \leq d_{B(z,r)}(z, \tilde{z}) \leq |z - \tilde{z}|/c \tag{5.17}$$

for all j large, $\tilde{z} \in B(z,r)$ and a uniform constant $c > 0$. For $R' \gg 2\tilde{R}$, apply Lemma 5.1 to the domain D_2^j and argue as in the proof of Proposition 5.3 to get

$$\begin{aligned} d_{B_{D_2^j}((0,-1),R')}(f^j(z), f^j(\tilde{z})) &\leq \frac{d_{D_2^j}(f^j(z), f^j(\tilde{z}))}{\tanh(R'/2 - d_{D_2^j}(f^j(z), f^j(\tilde{z})))} \\ &\leq \frac{d_{D_2^j}(f^j(z), f^j(\tilde{z}))}{\tanh(R'/2 - 2\tilde{R})} \end{aligned} \tag{5.18}$$

for all j large and $\tilde{z}$ in $B(z,r)$. Now, for a sufficiently small neighbourhood W of $q^0 \in \partial D_2$, we see that

$$B_{D_{2,\infty}}((0,-1), R' + \epsilon) \subset (T^j \circ h^j)(W \cap D_2)$$

for all j large. If W is small enough, there exists $R > 1$ such that for all j

large

$$h^j(W \cap D_2) \subset \{w \in \mathbb{C}^2 : |w_1|^2 + |w_2 + R|^2 < R^2\}$$
$$\subset \{w \in \mathbb{C}^2 : 2R\Re w_2 + |w_1|^2 < 0\} = \Omega_0.$$

Note that Ω_0 is invariant under T^j and Ω_0 is biholomorphically equivalent to $\mathbb{B}^2$. Hence

$$B_{D_{2,\infty}}((0,-1), R' + \epsilon) \subset (T^j \circ h^j)(W \cap D_2) \subset \Omega_0.$$

The above observation together with Proposition 5.7 gives

$$d_{\Omega_0}(f^j(z), f^j(\tilde{z})) \le d_{B_{D_{2,\infty}}((0,-1),R'+\epsilon)}(f^j(z), f^j(\tilde{z}))$$
$$\le d_{B_{D_2^j}((0,-1),R')}(f^j(z), f^j(\tilde{z}))$$

for all $\tilde{z}$ in $B(z,r)$ and j large. Hence, from (5.18) and (5.17) we get

$$d_{\Omega_0}(f^j(z), f^j(\tilde{z})) \le d_{B_{D_2^j}((0,-1),R')}(f^j(z), f^j(\tilde{z})) \le \frac{|z - \tilde{z}|}{c \tanh(R'/2 - 2\tilde{R})}$$

Now, using the fact that $\Omega_0 \simeq \mathbb{B}^2$ and the explicit form of the metric on $\mathbb{B}^2$ we obtain

$$|f^j(z) - f^j(\tilde{z})| \lesssim |z - \tilde{z}|$$

for $\tilde{z} \in B(z,r)$. This shows that $\{f^j\}$ is equicontinuous at each point of $\omega(K_1)$. The diagonal subsequence, still denoted by the same symbols, then converges uniformly on compact subsets of $D_{1,\infty}$ to a limit mapping $\tilde{f} : D_{1,\infty} \to \overline{D}_{2,\infty}$ which is continuous.

STEP IV. We claim that $d_{D_{1,\infty}}(z, \tilde{z}) = d_{D_{2,\infty}}(\tilde{f}(z), \tilde{f}(\tilde{z}))$ for all $z, \tilde{z} \in \Omega_1$ where $\Omega_1 = \{z \in D_{1,\infty} : \tilde{f}(z) \in D_{2,\infty}\}$. Note that $(0,-1) \in \Omega_1$ and hence Ω_1 is non-empty. Recall that for $z, \tilde{z}$ in Ω_1,

$$d_{D_1^j}(z, \tilde{z}) = d_{D_2^j}(f^j(z), f^j(\tilde{z}))$$

for all j. Now $d_{D_1^j}(z, \tilde{z}) \to d_{D_{1,\infty}}(z, \tilde{z})$ as can be seen from the arguments presented in Proposition 5.3. To show that the right side above converges to $d_{D_2^j}(\tilde{f}(z), \tilde{f}(\tilde{z}))$, notice that

$$|d_{D_2^j}(f^j(z), f^j(\tilde{z})) - d_{D_2^j}(\tilde{f}(z), \tilde{f}(\tilde{z}))| \le d_{D_2^j}(f^j(z), \tilde{f}(z)) + d_{D_2^j}(\tilde{f}(\tilde{z}), f^j(\tilde{z}))$$

by the triangle inequality. Since $f^j(z) \to \tilde{f}(z)$ and $D_2^j \to D_{2,\infty}$, there is a small ball $B\big(\tilde{f}(z), r\big)$ around $\tilde{f}(z)$ which contains $f^j(z)$ and which is contained in D_2^j for all large j, where $r > 0$ is independent of j. Thus

$$d_{D_2^j}(f^j(z), \tilde{f}(z)) \lesssim |f^j(z) - \tilde{f}(z)|.$$

The same argument shows that $d_{D_2^j}(\tilde{f}(\tilde{z}), f^j(\tilde{z}))$ is small. So to verify the claim, it is enough to prove that

$$d_{D_2^j}(\tilde{f}(z), \tilde{f}(\tilde{z})) \to d_{D_{2,\infty}}(\tilde{f}(z), \tilde{f}(\tilde{z})).$$

But this is immediate from Proposition 5.6.

STEP V. The limit map $\tilde{f}$ is a surjection onto $D_{2,\infty}$. To establish this, we first show that $\tilde{f}(D_{1,\infty}) \subset D_{2,\infty}$. Indeed, $\Omega_1 = D_{1,\infty}$. If $z^0 \in \partial\Omega_1 \cap D_{1,\infty}$, choose a sequence $z^j \in \Omega_1$ that converges to z^0. It follows from Step IV that

$$d_{D_{1,\infty}}(z^j, (0,-1)) = d_{D_{2,\infty}}(\tilde{f}(z^j), (0,-1))$$

for all j. Since $z^0 \in \partial\Omega_1$, the sequence $\tilde{f}(z^j)$ converges to a point on $\partial D_{2,\infty}$ and, as $D_{2,\infty}$ is complete in the Kobayashi distance, the right hand side above becomes unbounded. However, the left hand side remains bounded again because of completeness of $D_{1,\infty}$. This contradiction shows that $\Omega_1 = D_{1,\infty}$, which exactly means that $\tilde{f}(D_{1,\infty}) \subset D_{2,\infty}$. The above observation coupled with Step IV forces that

$$d_{D_{1,\infty}}(z, \tilde{z}) = d_{D_{2,\infty}}(\tilde{f}(z), \tilde{f}(\tilde{z}))$$

for all $z, \tilde{z} \in D_{1,\infty}$. To establish the surjectivity of $\tilde{f}$, consider any point $w^0 \in \partial(\tilde{f}(D_{1,\infty})) \cap D_{2,\infty}$ and choose a sequence $w^j \in \tilde{f}(D_{1,\infty})$ that converges to w^0. Let $\{\tilde{z}^j\}$ be a sequence of points in $D_{1,\infty}$ such that $\tilde{f}(\tilde{z}^j) = w^j$. Then for all j and $z \in D_{1,\infty}$,

$$d_{D_{1,\infty}}(z, \tilde{z}^j) = d_{D_{2,\infty}}(\tilde{f}(z), \tilde{f}(\tilde{z}^j)). \tag{5.19}$$

There are two cases to consider. Possibly after passing to a subsequence, we have either

(i) $\tilde{z}^j \to \tilde{z}^0 \in \partial D_{1,\infty}$, or
(ii) $\tilde{z}^j \to \tilde{z}^1 \in D_{1,\infty}$ as $j \to \infty$.

In case (i), observe that the right hand side of (5.19) remains bounded because of the completeness of $D_{2,\infty}$. Moreover, since $D_{1,\infty}$ is complete in the Kobayashi metric, the left hand side of (5.19) becomes unbounded. This contradiction shows that $\tilde{f}(D_{1,\infty}) = D_{2,\infty}$.

For (ii), firstly, the continuity of $\tilde{f}$ implies that $\tilde{f}(\tilde{z}^j) \to \tilde{f}(\tilde{z}^1)$ and consequently $\tilde{f}(\tilde{z}^1) = w^0$. Consider the mappings $(f^j)^{-1} : D_2^j \to D_1^j$. Now, an argument similar to the one employed in Step II shows that $(f^j)^{-1}$ admits a subsequence that converges uniformly on compact sets of $D_{2,\infty}$ to a continuous mapping $\tilde{g} : D_{2,\infty} \to \overline{D}_{1,\infty}$. Then $\tilde{g} \circ \tilde{f} \equiv \mathrm{id}_{D_{1,\infty}}$. Therefore,

$$\tilde{z}^1 = \tilde{g} \circ \tilde{f}(\tilde{z}^1) = \tilde{g}(w^0) = \lim_{j\to\infty} (f^j)^{-1}(w^0)$$

and consequently the sequence $(f^j)^{-1}(w^0)$ is compactly contained in $D_{1,\infty}$. Now, repeating the earlier argument for $(f^j)^{-1}$, it follows that $\tilde{g}(D_{2,\infty}) \subset D_{1,\infty}$ and $\tilde{f} \circ \tilde{g} \equiv \mathrm{id}_{D_{2,\infty}}$. In particular, $\tilde{f}$ is surjective.

Hence $\tilde{f}$ is an isometry between $D_{1,\infty}$ and $D_{2,\infty}$ in the Kobayashi metric. The goal now is to show that this isometry is indeed a biholomorphic mapping. To do this, observe that $\tilde{f}$ is differentiable almost everywhere (see [SV2]).

STEP VI. By [M2], the infinitesimal Kobayashi metric $F^K_{D_{1,\infty}}$ is C^1-smooth on $D_{1,\infty} \times (\mathbb{C}^2 \setminus \{0\})$. Further, $F^K_{D_{1,\infty}}$ is the quadratic form associated to a Riemannian metric, $\tilde{f}$ is C^1 on $D_{1,\infty}$ and finally $\tilde{f}$ is holomorphic or antiholomorphic. These statements can be deduced from the arguments in [SV2] without any additional difficulties. It follows that

$$\begin{aligned}\mathbb{B}^2 \simeq D_{1,\infty} &= \{(z_1, z_2) \in \mathbb{C}^2 : 2\Re z_2 + |z_1|^{2m} < 0\} \\ \simeq \tilde{D} &= \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^{2m} + |z_2|^2 < 1\}.\end{aligned}$$

Let $F : \mathbb{B}^2 \to \tilde{D}$ be a biholomorphism which in addition may be assumed to preserve the origin. Since $\mathbb{B}^2$ and $\tilde{D}$ are both circular domains, it follows that G is linear. This forces that $2m = 2$. But this exactly means that there exists a local coordinate system in a neighbourhood of the origin which can be written as

$$\{(z_1, z_2) \in \mathbb{C}^2 : 2\Re z_2 + |z_1|^2 + o(|z_1|^2 + \Im z_2) < 0\}.$$

This contradicts the assumption that $p^0 = (0, 0)$ is a weakly pseudoconvex point and proves Theorem 1.3. ∎

REMARK 5.8. Theorem 1.3 is to be interpreted as a version of Bell's result (cf. [BEL]) on biholomorphic inequivalence of a strongly pseudoconvex domain and a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$. Here, the end conclusion of non-existence of a global biholomorphism is replaced by a global isometry. The question of recovering the theorem for arbitrary weakly pseudoconvex finite type domains for isometries seems interesting.

THEOREM 5.9. *Let $f : D_1 \to D_2$ be a Kobayashi isometry between two bounded domains in $\mathbb{C}^2$. Let p^0 and q^0 be points on ∂D_1 and ∂D_2 respectively. Assume that the boundaries ∂D_1 and ∂D_2 are both C^∞-smooth weakly pseudoconvex and of finite type near p^0 and q^0 respectively. Suppose that q^0 belongs to the cluster set of p^0 under f. Then f extends continuously to a neighbourhood of p^0 in $\overline{D}_1$.*

Proof. We first show that f is continuous up to p^0. If f does not extend continuously to p^0, there exists a sequence s^j in D_1 with $s^j \to p^0 \in \partial D_1$ such that $f(s^j)$ does not converge to $q^0 \in \partial D_2$. Note that there is a sequence $p^j \in D_1$ converging to $p^0 \in \partial D_1$ such that $f(p^j) \to q^0 \in \partial D_2$. Then for polygonal paths γ^j in D_1 joining p^j and s^j defined as in the proof of Theorem 1.2 and points $p^{j0}, s^{j0}, t^j, u^j, u^0$ chosen analogously, it follows from [FR] that

$$\begin{aligned} d_{D_1}(p^j, t^j) \leq & -\tfrac{1}{2}\log d(p^j, \partial D_1) \\ & + \tfrac{1}{2}\log(d(p^j, \partial D_1) + |p^j - t^j|) \\ & + \tfrac{1}{2}\log(d(t^j, \partial D_1) + |p^j - t^j|) \\ & - \tfrac{1}{2}\log d(t^j, \partial D_1) + C_1. \end{aligned} \tag{5.20}$$

Applying Proposition 4.2 yields

$$\begin{aligned} d_{D_2}(f(p^j), f(t^j)) \geq & -\tfrac{1}{2}\log d(f(p^j), \partial D_2 e\big) \\ & - \tfrac{1}{2}\log d(f(t^j), \partial D_2) - C_2 \end{aligned} \tag{5.21}$$

for all j large and a uniform positive constant C_2. Next, we claim that

$$d\big(f(p^j), \partial D_2\big) \leq C_4 d(p^j, \partial D_2) \quad \text{and} \quad d\big(f(t^j), \partial D_2\big) \leq C_4 d(t^j, \partial D_2)$$

for some uniform positive constant C_4. Assume this for now. Using the fact that $d_{D_1}(p^j, t^j) = d_{D_2}\big(f(p^j), f(t^j)\big)$ and comparing (5.20) and (5.21), it follows from the above claim that for all j large,

$$\begin{aligned} -(C_1 + C_2 + \log C_4) \leq & \tfrac{1}{2}\log(d(p^j, \partial D_1) + |p^j - t^j|) \\ & + \tfrac{1}{2}\log(d(t^j, \partial D_1) + |p^j - t^j|), \end{aligned}$$

which is impossible.

To prove the claim, fix $a \in D_1$. By Proposition 4.1, we have

$$d_{D_1}(a, p^j) \geq -\tfrac{1}{2}\log d(p^j, \partial D_1) - C_5, \tag{5.22}$$

$$d_{D_2}(f(p^j), f(a)) \leq -\tfrac{1}{2}\log d(f(p^j), \partial D_2) + C_6, \tag{5.23}$$

for all j large and uniform positive constants C_5 and C_6. Fix $a \in D_1$. Using $d_{D_1}(a, p^j) = d_{D_2}\big(f(p^j), f(a)\big)$, and comparing (5.22) and (5.23), we get the required estimates. Hence the claim. From this point, using an argument similar to the one in the proof of Theorem 1.2, we infer that f is continuous on a neighbourhood of p^0 in $\overline{D}_1$. ∎

THEOREM 5.10. *Let $f : D_1 \to D_2$ be a Kobayashi isometry between two bounded domains in $\mathbb{C}^n$. Let p^0 and q^0 be points on ∂D_1 and ∂D_2 respectively. Assume that the boundaries ∂D_1 and ∂D_2 are both C^2-smooth strongly pseudoconvex near p^0 and q^0 respectively. Suppose that q^0 belongs to the cluster set of p^0 under f. Then f extends as a continuous mapping to a neighbourhood of p^0 in $\overline{D}_1$.*

The proof of the above theorem is along the same lines as that of Theorems 1.2 and 5.9 and is hence omitted.

It turns out that versions of the above mentioned results hold for the inner Carathéodory distance. More concretely, the following global statements can be proved:

Theorem 5.11. *Let $f : D_1 \to D_2$ be an isometry between two bounded domains in $\mathbb{C}^n$ with respect to the inner Carathéodory distances on these domains.*

(i) *Assume that D_1 and D_2 are both C^3-smooth strongly pseudoconvex domains in $\mathbb{C}^n$. Then f extends continuously up to the boundary.*
(ii) *Assume that $D_1 \subset \mathbb{C}^2$ is a C^3-smooth strongly pseudoconvex domain and $D_2 \subset \mathbb{C}^2$ is a C^∞-smooth weakly pseudoconvex finite type domain. Then f extends continuously up to the boundary.*
(iii) *Assume that D_1 and D_2 are both C^∞-smooth weakly pseudoconvex finite type domains in $\mathbb{C}^2$. Then f extends continuously up to the boundary.*

To establish the above theorem, the following result due to Balogh–Bonk ([BB]) will be needed. This in turn relies on estimates for the infinitesimal Carathéodory metric given by D. Ma ([M1]).

Theorem 5.12. *Let $D \subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain with C^3-smooth boundary. Then there exists a constant $C > 0$ such that for all $a, b \in D$,*

$$g(a,b) - C \le c^i_D(a,b) \le g(a,b) + C$$

where

$$g(a,b) = 2\log\left[\frac{d_H(\pi(a),\pi(b)) + \max\{(d(a,\partial D))^{1/2},(d(b,\partial D))^{1/2}\}}{(d(a,\partial D))^{1/4}(d(b,\partial D))^{1/4}}\right],$$

d_H is the Carnot–Carathéodory metric on ∂D and $\pi(z) \in \partial D$ is such that $|\pi(z) - z| = d(z,\partial D)$.

Proof of Theorem 5.11. Part (i) is in Lemma 2.2 of [SV1]. Parts (ii) and (iii) can be verified by making appropriate changes in the proof of Theorem 5.9—using the inequality $c^i_D \le d_D$ to get the upper bounds on the inner Carathéodory distance and the following consequences of the results of Balogh–Bonk ([BB]) and Herbort ([HERB]):

- If a belongs to some fixed compact set L in D, then
$$g(a,b) \approx -\tfrac{1}{2}\log d(b,\partial D) \pm C(L).$$
Thus
$$c^i_D(a,b) \approx -\tfrac{1}{2}\log d(b,\partial D) \pm C(L).$$
- If $a, b \in D$ are close to two distinct points on ∂D, then
$$g(a,b) \approx -\tfrac{1}{2}\log d(a,\partial D) - \tfrac{1}{2}\log d(b,\partial D) \pm C.$$
Consequently,
$$c^i_D(a,b) \approx -\tfrac{1}{2}\log d(a,\partial D) - \tfrac{1}{2}\log d(b,\partial D) \pm C.$$

- If a, b are close to the same boundary point, then
$$\begin{aligned} g(a,b) \le & -\tfrac{1}{2}\log d(a,\partial D) + \tfrac{1}{2}\log(d(a,\partial D) + |a-b|) \\ & + \tfrac{1}{2}\log(d(b,\partial D) + |a-b|) - \tfrac{1}{2}\log d(b,\partial D) + C. \end{aligned}$$
Therefore,
$$\begin{aligned} c_D^i(a,b) \le & -\tfrac{1}{2}\log d(a,\partial D) + \tfrac{1}{2}\log(d(a,\partial D) + |a-b|) \\ & + \tfrac{1}{2}\log(d(b,\partial D) + |a-b|) - \tfrac{1}{2}\log d(b,\partial D) + C. \end{aligned}$$
- If a, b are sufficiently close to two distinct points on ∂D for a weakly pseudoconvex finite domain in $\mathbb{C}^2$, then $\tilde{d}(a,b) \gtrsim 1$ and $\tilde{d}(b,a) \gtrsim 1$ so that
$$c_D^i(a,b) \gtrsim -\tfrac{1}{2}\log d(a,\partial D) - \tfrac{1}{2}\log d(b,\partial D) \pm C.$$

These bounds on $c_D^i(a,b)$ are exactly the ones that are needed to reprove Theorem 5.9 for the inner Carathéodory distance. ■

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Prachi Mahajan
Department of Mathematics
Indian Institute of Science
Bangalore 560 012, India
E-mail: mittal@math.iisc.ernet.in

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