ANNALES
POLONICI MATHEMATICI
104.2 (2012)

The rigidity theorem for Landsberg hypersurfaces of a Minkowski space

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Abstract. Let M^n be a compact Landsberg hypersurface of a Minkowski space $(V^{n+1}, \overline{F})$ with constant mean curvature H. Using the Gauss formula for the Chern connection of Finsler submanifolds, we prove that if M is convex, then M is Riemannian with constant curvature.

1. Introduction. Let M be an n-dimensional smooth manifold and $\pi : TM \to M$ be the natural projection from the tangent bundle. Let (x, Y) be a point of TM with $x \in M$, $Y \in T_xM$ and let (x^i, Y^i) be the local coordinates on TM with $Y = Y^i\partial/\partial x^i$. A *Finsler metric* on M is a function $F : TM \to [0, \infty)$ with the following properties:

(i) Regularity: $F(x, Y)$ is smooth in $TM \setminus 0$.
(ii) Positive homogeneity: $F(x, \lambda Y) = \lambda F(x, Y)$ for $\lambda > 0$.
(iii) Strong convexity: The fundamental quadratic form $g = g_{ij}(x, Y)dx^i \otimes dx^j$ is positive definite, where $g_{ij} = \frac{1}{2}\partial^2(F^2)/\partial Y^i\partial Y^j$.

Then (M, F) is called a *Finsler manifold*. The simplest class of Finsler manifolds is the Minkowski spaces. Let V be an n-dimensional real vector space. A *Minkowski norm* on V is a function $F : V \to [0, \infty)$ with the following properties:

(i) Regularity: F is smooth in $V \setminus \{0\}$.
(ii) Positive homogeneity: $F(\lambda Y) = \lambda F(Y)$ for $\lambda > 0$ and $Y \in V$.
(iii) Strong convexity: For any $Y \in V \setminus \{0\}$, the symmetric bilinear form g_Y is positive definite, where $g_Y(u, v) = \frac{1}{2}\frac{\partial^2}{\partial s\partial t}[F^2(Y + su + tv)]\big|_{s=t=0}$.

Then (V, F) is called a *Minkowski space*. Let $\{e_i\}$ be an arbitrary basis for V. From the above definition we find that

$$g_{ij}(Y) = g_Y(e_i, e_j) = \frac{1}{2}\frac{\partial^2}{\partial Y^i\partial Y^j}[F^2(Y)].$$

2010 *Mathematics Subject Classification*: Primary 53C60; Secondary 53C40.
Key words and phrases: Finsler manifold, Minkowski space, Landsberg hypersurfaces.

DOI: 10.4064/ap104-2-3 © Instytut Matematyczny PAN, 2012

We see that for any Finsler manifold (M, F), $F_x(Y) := F(x, Y)$ is a Minkowski norm on T_xM for every point $x \in M$. On the other hand, for a Minkowski space (V, F), the Finsler metric $F(x, Y) := F(Y) : TV \to [0, \infty)$ is a function of $Y \in V$ only.

Riemannian submanifolds are important in modern differential geometry and extensively studied. For a compact hypersurface M of the Euclidean space with constant mean curvature, Nomizu and Smyth [NS] proved that if M is convex, then M is a Riemannian sphere. There has been remarkable progress in recent studies on Finsler manifolds. For example, in [BRS], D. Bao, C. Robles and Z. Shen have completely classified strongly convex Randers metrics with constant flag curvature, and the geometry of Ingarden spaces has been described by R. Miron [M].

In this paper, by using the Gauss formula for the Chern connection, we study the Landsberg hypersurfaces of a Minkowski space $(V^{n+1}, \overline{F})$ and obtain the following

Main Theorem 1.1. *Let M^n be a compact Landsberg hypersurface of a Minkowski space $(V^{n+1}, \overline{F})$ with constant mean curvature H. If M is convex, then M is Riemannian with constant curvature.*

Remark. The Main Theorem generalizes the result of Nomizu and Smyth [NS] from the Riemannian to the Finsler case.

2. Preliminaries. Let (M^n, F) be an n-dimensional Finsler manifold. F inherits the *Hilbert form*, the *fundamental tensor* and the *Cartan tensor* as follows:

$$\omega = \frac{\partial F}{\partial Y^i} dx^i, \qquad g_Y = g_{ij}(x, Y) dx^i \otimes dx^j,$$
$$A_Y = A_{ijk} dx^i \otimes dx^j \otimes dx^k, \qquad A_{ijk} := \frac{F \partial g_{ij}}{2 \partial Y^k}.$$

It is well known that there exists a unique *Chern connection* ∇ on $\pi^* TM$ with $\nabla \frac{\partial}{\partial x^i} = \omega_i^j \frac{\partial}{\partial x^j}$ and $\omega_i^j = \Gamma_{ik}^j dx^k$ satisfying

$$d(dx^i) - dx^j \wedge \omega_j^i = -dx^j \wedge \omega_j^i = 0,$$
$$dg_{ij} - g_{ik}\omega_j^k - g_{jk}\omega_i^k = 2A_{ijk}\frac{\delta Y^k}{F},$$

where $\delta Y^i = dY^i + N_j^i dx^j, N_j^i = \gamma_{jk}^i Y^k - \frac{1}{F} A_{jk}^i \gamma_{st}^k Y^s Y^t$ and γ_{jk}^i are the formal Christoffel symbols of the second kind for g_{ij}.

The curvature 2-forms of the Chern connection ∇ are

$$\omega_j^i - \omega_j^k \wedge \omega_k^i = \Omega_j^i = \frac{1}{2} R_{jkl}^i dx^k \wedge dx^l + \frac{1}{F} P_{jkl}^i dx^k \wedge \delta Y^l,$$

where R_{jkl}^i and P_{jkl}^i are the components of the hh-curvature tensor and the hv-curvature tensor of the Chern connection, respectively.

Let $\varphi : (M^n, F) \to (\overline{M}^{n+p}, \overline{F})$ be an isometric immersion from a Finsler manifold to a Finsler manifold. We have [S]

$$
\begin{aligned}
&F(Y) = \overline{F}(\varphi_*(Y)), \quad g_Y(U,V) = \overline{g}_{\varphi_*(Y)}(\varphi_*(U), \varphi_*(V)),\\
&A_Y(U,V,W) = \overline{A}_{\varphi_*(Y)}(\varphi_*(U), \varphi_*(V), \varphi_*(W)),
\end{aligned} \tag{2.1}
$$

where $Y, U, V, W \in TM$, and $\overline{g}$ and $\overline{A}$ are the fundamental tensor and the Cartan tensor of $\overline{M}$, respectively.

It can be seen from (2.1) that $\varphi^*(\overline{\omega}) = \omega$, where $\overline{\omega}$ is the Hilbert form of $\overline{M}$.

In the following we simplify A_Y and g_Y to A and g, respectively.

When $\overline{M}$ is a Minkowski space, the formal Christoffel symbols $\overline{\gamma}^a_{bc}$ of the second kind for $\overline{g}_{ab}$ must vanish and so $\overline{N}^a_b = \overline{\gamma}^a_{bc}\overline{Y}^c - \frac{1}{\overline{F}}\overline{A}^a_{bc}\overline{\gamma}^c_{df}\overline{Y}^d\overline{Y}^f = 0$; then the horizontal part $(\varphi_* e_i)^H$ of $\varphi_* e_i = u^j_i \varphi^A_j \frac{\partial}{\partial \overline{x}^A}$ can be written as

$$
(\varphi_* e_i)^H = u^j_i \varphi^A_j \frac{\delta}{\delta \overline{x}^A} = u^j_i \varphi^A_j \left(\frac{\partial}{\partial \overline{x}^A} - \overline{N}^B_A \frac{\partial}{\partial \overline{Y}^B} \right) = u^j_i \varphi^A_j \frac{\partial}{\partial \overline{x}^A} = \varphi_* e_i,
$$

which, together with $\overline{A}(\cdot,\cdot,\overline{Y}) = 0$ and $\varphi_* Y = \overline{Y}$, implies that

$$
\overline{A}(\cdot,\cdot,\overline{\nabla}_{e_i}\varphi_*\ell) = \overline{A}(\cdot,\cdot,\overline{\nabla}_{\varphi_* e_i}\overline{\ell}) = 0, \tag{2.2}
$$

where $\ell = Y/F$ and $\overline{\ell} = \overline{Y}/\overline{F}$.

In the following any vector $U \in TM$ will be identified with the corresponding vector $\varphi_*(U) \in T\overline{M}$ and we will use the following convention:

$$
\begin{aligned}
&1 \le i, j, \ldots \le n; \qquad n+1 \le \alpha, \beta, \ldots \le n+p;\\
&1 \le \lambda, \mu, \ldots \le n-1; \qquad 1 \le a, b, \ldots \le n+p.
\end{aligned}
$$

Let $\varphi : (M^n, F) \to (\overline{M}^{n+p}, \overline{F})$ be an isometric immersion. Take a $\overline{g}$-orthonormal frame $\{e_a\}$ for each fibre of $\pi^* T\overline{M}$ and let $\{\omega^a\}$ be its local dual coframe such that $\{e_i\}$ is a frame field for each fibre of $\pi^* TM$ and ω^n is the Hilbert form, where $\pi : TM \to M$ denotes the natural projection. Let θ^a_b and ω^i_j denote the Chern connection 1-forms of $\overline{F}$ and F, respectively, i.e., $\overline{\nabla} e_a = \theta^b_a e_b$ and $\nabla e_i = \omega^j_i e_j$, where $\overline{\nabla}$ and ∇ are the Chern connections of $\overline{M}$ and M, respectively. We find that $A(e_i, e_j, e_n) = \overline{A}(e_a, e_b, e_n) = 0$, where $e_n = \frac{Y^i}{F}\frac{\partial}{\partial x^i}$ is the natural dual of the Hilbert form ω^n. Formula (2.2) implies

LEMMA 2.1. *Let $\varphi : (M^n, F) \to (\overline{M}^{n+p}, \overline{F})$ be an isometric immersion from a Finsler manifold to a Minkowski space. Then $\overline{A}(\cdot,\cdot,\overline{\nabla}_{e_i} e_n) = 0$.*

From $\omega^\alpha = 0$ and the structure equations of $\overline{M}$, we have $\theta^\alpha_j \wedge \omega^j = 0$, which implies that $\theta^\alpha_j = h^\alpha_{ij}\omega^i$, $h^\alpha_{ij} = h^\alpha_{ji}$. We obtain [L1]

$$
\omega^j_i = \theta^j_i - \Psi_{jik}\omega^k, \tag{2.3}
$$

where

$$\Psi_{jik} = h^{\alpha}_{jn}\overline{A}_{ki\alpha} - h^{\alpha}_{kn}\overline{A}_{ji\alpha} - h^{\alpha}_{in}\overline{A}_{kj\alpha} - h^{\alpha}_{nn}\overline{A}_{iks}\overline{A}_{sj\alpha} + h^{\alpha}_{nn}\overline{A}_{ijs}\overline{A}_{sk\alpha} + h^{\alpha}_{nn}\overline{A}_{jks}\overline{A}_{si\alpha}. \tag{2.4}$$

In particular,

$$\omega^n_i = \theta^n_i - h^{\alpha}_{nn}\overline{A}_{ki\alpha}\omega^k. \tag{2.5}$$

Using the almost $\overline{g}$-compatibility, we have

$$\theta^j_{\alpha} = (-h^{\alpha}_{ij} - 2h^{\beta}_{ni}\overline{A}_{j\alpha\beta} + 2h^{\beta}_{nn}\overline{A}_{j\lambda\alpha}\overline{A}_{i\lambda\beta})\omega^i - 2\overline{A}_{j\alpha\lambda}\omega^{\lambda}_n. \tag{2.6}$$

In particular, $\theta^n_{\alpha} = -h^{\alpha}_{ni}\omega^i$.

We quote the following propositions:

PROPOSITION 2.2 (Gauss equations, [L1, Theorem 3.1]). *Let $\varphi : (M^n, F) \to (\overline{M}^{n+p}, \overline{F})$ be an isometric immersion from a Finsler manifold to a Finsler manifold. Then*

$$\begin{aligned}
P^j_{ik\lambda} &= \overline{P}^j_{ik\lambda} + \Psi_{jik;\lambda} - 2\Psi_{sik}A_{js\lambda} - 2h^{\alpha}_{ik}\overline{A}_{j\lambda\alpha},\\
R^j_{ikl} &= \overline{R}^j_{ikl} - h^{\alpha}_{ik}h^{\alpha}_{jl} + h^{\alpha}_{il}h^{\alpha}_{jk} + \Psi_{jik|l} - \Psi_{jil|k}\\
&\quad + \Psi_{sik}\Psi_{jsl} - \Psi_{sil}\Psi_{jsk} - 2h^{\alpha}_{ik}h^{\beta}_{nl}\overline{A}_{j\alpha\beta} + 2h^{\alpha}_{il}h^{\beta}_{nk}\overline{A}_{j\alpha\beta}\\
&\quad + 2h^{\alpha}_{ik}h^{\beta}_{nn}\overline{A}_{js\alpha}\overline{A}_{ls\beta} - 2h^{\alpha}_{il}h^{\beta}_{nn}\overline{A}_{js\alpha}\overline{A}_{ks\beta} - h^{\alpha}_{nn}\overline{A}_{sl\alpha}\overline{P}^j_{iks}\\
&\quad + h^{\alpha}_{nn}\overline{A}_{sk\alpha}\overline{P}^j_{ils} + h^{\alpha}_{nl}\overline{P}^j_{ik\alpha} - h^{\alpha}_{nk}\overline{P}^j_{il\alpha},
\end{aligned}$$

where ";" and "|" denote the vertical and the horizontal covariant differentials with respect to the Chern connection ∇ respectively.

PROPOSITION 2.3 (Codazzi equations, [L1, Theorem 3.2]). *Let $\varphi : (M^n, F) \to (\overline{M}^{n+p}, \overline{F})$ be an isometric immersion from a Finsler manifold to a Finsler manifold. Then*

$$\begin{aligned}
h^{\alpha}_{ij;\lambda} &= -\overline{P}^{\alpha}_{ij\lambda},\\
h^{\alpha}_{ij|k} - h^{\alpha}_{ik|j} &= -\overline{R}^{\alpha}_{ijk} + h^{\beta}_{nj}\overline{P}^{\alpha}_{ik\beta} - h^{\beta}_{nk}\overline{P}^{\alpha}_{ij\beta}\\
&\quad - h^{\alpha}_{lk}\Psi_{lij} + h^{\alpha}_{lj}\Psi_{lik} - h^{\beta}_{nn}\overline{A}_{lj\beta}\overline{P}^{\alpha}_{ikl} + h^{\beta}_{nn}\overline{A}_{lk\beta}\overline{P}^{\alpha}_{ijl}.
\end{aligned}$$

PROPOSITION 2.4 ([L2, Theorem 4.4]). *An isometric immersion $\varphi : (M, F) \to (\overline{M}, \overline{F})$ is minimal if and only if*

$$\int_{SM} \langle V, \mathcal{H}\rangle \, dV_{SM} = 0$$

for any vector $V \in \Gamma(TM)^{\perp}$, where

$$\mathcal{H} = \sum_i \Big\{ B(e_i, e_i) + \sum_\alpha [2\overline{C}(e_\alpha, e_i, B(e_i, Fe_n)) + (\overline{\nabla}_{Fe_n^H}\overline{C})(e_i, e_i, e_\alpha) + 2\overline{C}(\overline{\nabla}_{Fe_n^H} e_i, e_i, e_\alpha)]e_\alpha \Big\}, \tag{2.7}$$

$\overline{C} = \overline{A}/\overline{F}$, e_i^H *and denotes the horizontal part of* e_i.

Definition 2.5. M^n is called of *constant mean curvature* if $H = |\mathcal{H}| =$ constant.

Proposition 2.6 ([L2, Theorem 5.2]). *Let* M^n *be a hypersurface of a Minkowski space* $\overline{V}^{n+1} = (V^{n+1}, \overline{F})$. *If* M *is Landsberg, then* $h_{in}^{n+1}\overline{A}_{jl\,n+1} = 0$ *and* $\Psi_{ijk} = 0$.

It follows from Lemma 2.1 that

$$\overline{A}(\cdot,\cdot,e_j)\omega_n^j(e_i) + \overline{A}(\cdot,\cdot,e_\lambda)\Psi_{\lambda ni} + \overline{A}(\cdot,\cdot,e_{n+1})h_{ni}^{n+1} = 0. \tag{2.8}$$

Combining (2.8) and Proposition 2.5 immediately yields

Proposition 2.7. *Let* M^n *be a Landsberg hypersurface of a Minkowski space* $(V^{n+1}, \overline{F})$. *Then* $h_{ni}^{n+1}\overline{A}_{j\,n+1\,n+1} = h_{ni}^{n+1}\overline{A}_{n+1\,n+1\,n+1} = 0$.

Proposition 2.8. *If* M^n *is a Landsberg hypersurface of a Minkowski space* $\overline{V}^{n+1}$ *with constant mean curvature, then* $\sum_i h_{ii}^{n+1} =$ constant.

Proof. It follows from Propositions 2.5 and 2.6 and the first formula of Proposition 2.3 that

$$h_{nn|j}^{n+1}\overline{A}_{i\,n+1\,n+1}\omega^j + h_{nn}^{n+1}\overline{A}_{i\,n+1\,n+1;\lambda}\omega_n^\lambda + h_{nn}^{n+1}\overline{A}_{i\,n+1\,n+1;n+1}h_{nj}^{n+1}\omega^j = 0,$$

which gives

$$h_{nn}^{n+1}\overline{A}_{i\,n+1\,n+1;\lambda} = 0. \tag{2.9}$$

It follows from (2.9) and $\overline{A}_{abc;d} = \overline{A}_{abd;c}$ that

$$(\overline{\nabla}_{Fe_n^H}\overline{C})(e_i, e_i, e_{n+1}) = \overline{C}_{i\,i\,n+1;\lambda}\theta_n^\lambda(Fe_n^H) + \overline{C}_{i\,i\,n+1;n+1}\theta_n^{n+1}(Fe_n^H) = 0. \tag{2.10}$$

From Propositions 2.6 and 2.7 we can deduce that $\overline{C}(\overline{\nabla}_{Fe_n^H} e_i, e_i, e_{n+1}) = 0$. Therefore by (2.10) and (2.7), we have $H = B(e_i, e_i) = \sum_i h_{ii}^{n+1} e_{n+1}$. Since we are assuming that H is constant, it follows that $\sum_i h_{ii}^{n+1} =$ constant. ∎

Proposition 2.9. *If a hypersurface* M^n *of a Minkowski space* $\overline{V}^{n+1}$ *is a Landsberg manifold, then*

$$h_{ij;\lambda;\mu}^{n+1} = h_{ij;\mu;\lambda}^{n+1},$$
$$h_{ij|k;\lambda}^{n+1} = h_{ij;\lambda|k}^{n+1} - h_{sj}^{n+1}P_{ik\lambda}^s - h_{is}^{n+1}P_{jk\lambda}^s + h_{ij|k}^{n+1}\overline{A}_{n+1\,n+1\,\lambda},$$
$$h_{ij|k|l}^{n+1} = h_{ij|k|l}^{n+1} + h_{sj}^{n+1}R_{ikl}^s + h_{is}^{n+1}R_{jkl}^s.$$

Proof. For a hypersurface (M^n, F) of a Minkowski space $(V^{n+1}, \overline{F})$, we have

$$h^{n+1}_{ij|k}\omega^k + h^{n+1}_{ij;\lambda}\omega^\lambda_n = dh^{n+1}_{ij} - h^{n+1}_{kj}\omega^k_i - h^{n+1}_{ik}\omega^k_j + h^{n+1}_{ij}\theta^{n+1}_{n+1}. \tag{2.11}$$

Differentiating (2.11), we obtain

$$\begin{aligned}\{&h^{n+1}_{ij|k|l} - \tfrac{1}{2}h^{n+1}_{sj}R^s_{ikl} - \tfrac{1}{2}h^{n+1}_{is}R^s_{jkl} - \tfrac{1}{2}h^{n+1}_{ij;\lambda}R^\lambda_{nkl} \\ &+ 2h^{n+1}_{ij}h^{n+1}_{sk}h^{n+1}_{nl}\overline{A}_{s\,n+1\,n+1} - 2h^{n+1}_{ij}h^{n+1}_{sk}h^{n+1}_{nn}\overline{A}_{s\,t\,n+1}\overline{A}_{t\,l\,n+1}\}\omega^k\wedge\omega^l \\ &+ \{h^{n+1}_{ij|k;\lambda} - h^{n+1}_{ij;\lambda|k} + h^{n+1}_{ij;\mu}P^\mu_{nk\lambda} + h^{n+1}_{is}P^s_{jk\lambda} + h^{n+1}_{sj}P^s_{ik\lambda} - h^{n+1}_{ij|k}\overline{A}_{n+1\,n+1\,\lambda} \\ &+ 2h^{n+1}_{ij}h^{n+1}_{sk}\overline{A}_{s\,n+1\,\lambda}\}\omega^k\wedge\omega^\lambda_n + h^{n+1}_{ij;\lambda;\mu}\omega^\lambda_n\wedge\omega^\mu_n = 0.\end{aligned} \tag{2.12}$$

We obtain the conclusion immediately from (2.12), Propositions 2.5 and 2.6, and the first formula of Proposition 2.3. ■

3. Landsberg hypersurfaces of a Minkowski space. Let M^n be a Landsberg hypersurface with constant mean curvature of a Minkowski space $\overline{V}^{n+1}$. By Proposition 2.7, we have

$$\sum_i h^{n+1}_{ii|j}\omega^j + \sum_i h^{n+1}_{ii;\lambda}\omega^\lambda_n = 2\sum_{ij} h^{n+1}_{ij}A_{ij\lambda}\omega^\lambda_n. \tag{3.1}$$

It follows from (3.1) that

$$\sum_i h^{n+1}_{ii|j} = 0 \quad\text{and}\quad \sum_i h^{n+1}_{ii;\lambda} = 2\sum_{ik} h^{n+1}_{ik}A_{ik\lambda}. \tag{3.2}$$

Differentiating the first formula of (3.2), we obtain

$$\sum_i h^{n+1}_{ii|j|k}\omega^k + \sum_i h^{n+1}_{ii|j;\lambda}\omega^\lambda_n = 2\sum_{ik} h^{n+1}_{ik|l}A_{ik\lambda}\omega^\lambda_n,$$

which implies that

$$\sum_i h^{n+1}_{ii|j|k} = 0. \tag{3.3}$$

DEFINITION 3.1. M is called *convex* if the second fundamental form h^{n+1}_{ij} of M is positive semi-definite.

Define $\delta Y^i = dY^i + N^i_j dx^j$. The pull-back of the Sasaki metric $g_{ij}dx^i \otimes dx^j + g_{ij}\delta Y^i \otimes \delta Y^j$ from $TM\setminus\{0\}$ to the sphere bundle SM is a Riemannian metric

$$\widehat{g} = g_{ij}dx^i \otimes dx^j + \delta_{ab}\omega^a_n \otimes \omega^b_n.$$

We now quote two lemmas:

LEMMA 3.2 ([Mo, Lemma 2.2]). *For* $X = \sum_i x_i\omega^i \in \Gamma(\pi^*T^*M)$, $\operatorname{div}_{\widehat{g}} X = \sum_i x_{i|i} + \sum_{\mu,\lambda} x_\mu P^n_{\lambda\lambda\mu}$.

LEMMA 3.3 ([N, Theorem 1]). *All Landsberg spaces of nonzero constant flag curvature are Riemannian.*

Proof of Main Theorem. According to Propositions 2.2, 2.3, and 2.6–2.8,

$$R^i_{jkl} = h^{n+1}_{ik} h^{n+1}_{jl} - h^{n+1}_{il} h^{n+1}_{jk}, \tag{3.4}$$

$$h^{n+1}_{ij;\lambda} = 0, \tag{3.5}$$

$$h^{n+1}_{ij|k} = h^{n+1}_{ik|j}, \tag{3.6}$$

$$h^{n+1}_{ij|k|l} = h^{n+1}_{ij|l|k} + h^{n+1}_{sj} R^s_{ikl} + h^{n+1}_{is} R^s_{jkl}. \tag{3.7}$$

Let $\omega = dS = S_{|i}\omega^i + S_{;i}\omega^i_n$. Then ω is a global section on π^*T^*M. By (3.5), i.e., $S_{;i} = 0$, and Lemma 3.1, we have

$$\operatorname{div}_{\widehat{g}} \omega = 2\sum_{i,j,k} (h^{n+1}_{ij|k})^2 + 2\sum_{i,j,k} h^{n+1}_{ij} h^{n+1}_{ij|k|k}. \tag{3.8}$$

It can be seen from (3.3)–(3.8) that

$$\begin{aligned} \operatorname{div}_{\widehat{g}} \omega &= 2\sum_{i,j,k} (h^{n+1}_{ij|k})^2 + 2\sum_{i,j,k,s} h^{n+1}_{ij} \{h^{n+1}_{kk|i|j} + h^{n+1}_{si} R^s_{kjk} + h^{n+1}_{ks} R^s_{ijk}\} \\ &= 2\sum_{i,j,k} (h^{n+1}_{ij|k})^2 + 2\sum_{i,j,k,l} \{h^{n+1}_{ij} h^{n+1}_{ki} h^{n+1}_{jk} h^{n+1}_{ll} - 2(h^{n+1}_{ij} h^{n+1}_{kl})^2\}. \end{aligned} \tag{3.9}$$

Let λ_i be the eigenvalues of the second fundamental tensor h^{n+1}_{ij} of M. It is easy to see from (3.9) that

$$\frac{1}{2} \operatorname{div}_{\widehat{g}} \omega = \sum_{i,j,k} (h^{n+1}_{ij|k})^2 + \frac{1}{2} \sum_{i,j} (\lambda_i - \lambda_j)^2 \lambda_i \lambda_j. \tag{3.10}$$

Since M is convex, i.e., $\lambda_i\lambda_j \geq 0$ for all i, j, the right hand side of (3.10) is nonnegative. Because of the compactness of M, we infer that h^{n+1}_{ij} is constant and $h^{n+1}_{ij} = 0$ for all $i \neq j$ on M. Differentiating $h^{n+1}_{na} = 0$ yields $h^{n+1}_{aa} = h^{n+1}_{nn}$ for all $a = 1, \dots, n-1$, i.e., $h^{n+1}_{ii} = H$ for all i. It is easy to see from (3.4) that

$$R^j_{ikl} = H(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}). \tag{3.11}$$

On the other hand, let $x = \overline{x}^a \partial/\partial \overline{x}^a$ be the position vector field of the Minkowski space V^{n+1} with respect to the origin. By a simple direct computation, we get $\overline{\nabla}_Z x = Z$ for all $Z = z^a \partial/\partial \overline{x}^a$ on V^{n+1}, which, together with Lemma 2.1, implies that $\nabla_{e_i} x^2 = 2\langle e_i, x\rangle$ and $\nabla_{e_i}\langle e_i, x\rangle = \theta^j_i(e_i)\langle e_j, x\rangle + h^{n+1}_{ii}\langle e_{n+1}, x\rangle + 1$. As M is compact, there exists a point $P \in M$ such that $h^{n+1}_{ii}(P) > 0$ for all i, so $H > 0$. Thus by (3.11), M is a Landsberg space with nonzero constant flag curvature H, which together with Lemma 3.2 finishes the proof of Main Theorem.

Acknowledgements. The author wishes to thank the referee for a careful reading of the manuscript and for very helpful suggestions.

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Received 28.2.2011
and in final form 10.9.2011 (2401)