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Analysis of a contact adhesive problem with normal compliance and nonlocal friction

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Abstract. The paper deals with the problem of a quasistatic frictional contact between a nonlinear elastic body and a deformable foundation. The contact is modelled by a normal compliance condition in such a way that the penetration is restricted with a unilateral constraint and associated to the nonlocal friction law with adhesion. The evolution of the bonding field is described by a first-order differential equation. We establish a variational formulation of the mechanical problem and prove an existence and uniqueness result under a smallness assumption on the friction coefficient by using arguments of time-dependent variational inequalities, differential equations and the Banach fixed-point theorem.

1. Introduction. Contact problems involving deformable bodies are quite frequent in industry as well as in daily life and play an important role in structural and mechanical systems. Contact processes involve complicated surface phenomena, and are modelled by highly nonlinear initial boundary value problems. Taking into account various contact conditions associated with behaviour laws becoming more and more complex leads to the introduction of new and nonstandard models, expressed with the aid of evolution variational inequalities. An early attempt to study contact problems within the framework of variational inequalities was made in [10]. The mathematical, mechanical and numerical state of the art can be found in [23]. Unilateral frictional contact problems involving Signorini's condition with or without adhesion were studied by several authors (see for instance the references in [1, 2, 5, 6, 8, 11, 16, 17, 20, 26, 29, 30]).

In this paper, we study a mathematical model which describes a frictional unilateral contact problem with adhesion between a nonlinear elastic body and a deformable foundation. Following [16, 27] the unilateral contact is modelled by a normal compliance condition in such a way that finite pen-

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etration is allowed. Recall that models for dynamic or quasistatic processes of frictionless adhesive contact between a deformable body and a foundation have been studied in [3, 4, 12, 17, 22–24, 27]. Also recently dynamic or quasistatic frictional contact problems with adhesion were studied in [7, 28].

Here as in [13, 14] we use the bonding field as an additional state variable β, defined on the contact surface of the boundary. The variable is restricted to values $0 \leq \beta \leq 1$; when $\beta = 0$ all the bonds are severed and there are no active bonds; when $\beta = 1$ all the bonds are active; if $0 < \beta < 1$ then β measures the fraction of active bonds and partial adhesion takes place. We refer the reader to the extensive bibliography on the subject in [2, 12–15, 21–23]. However, according to [16], the method presented in this work considers a compliance model in which the compliance term does not necessarily represent a compact perturbation of the original problem without contact. This will help us to study models where a strictly limited penetration is performed by carrying out a limit procedure to the Signorini contact problem. Here we extend the result established in [26] to the unilateral contact problem with a normal compliance condition in such a way that the penetration is limited and associated to the nonlocal friction law with adhesion. We establish a variational formulation of the mechanical problem for which we prove the existence of a unique weak solution if the friction coefficient is sufficiently small and obtain a partial regularity result for the solution.

The paper is structured as follows. In Section 2 we present some notation and give the variational formulation. In Section 3 we state and prove our main existence and uniqueness result, Theorem 3.1.

2. Problem statement and variational formulation. Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a Lipschitzian domain initially occupied by a nonlinear elastic body. The boundary Γ of Ω is partitioned into three measurable parts such that $\Gamma = \bar{\Gamma}_1 \cup \bar{\Gamma}_2 \cup \bar{\Gamma}_3$ where $\Gamma_1, \Gamma_2, \Gamma_3$ are disjoint open sets and $\mathrm{meas}(\Gamma_1) > 0$. The body is acted upon by a volume force of density φ_1 on Ω and a surface traction of density φ_2 on Γ_2. On Γ_3 the body is in adhesive frictional unilateral contact with a deformable foundation.

The classical formulation of this mechanical problem is as follows.

Problem P_1. Find a displacement $u : \Omega \times [0, T] \to \mathbb{R}^d$ and a bonding field $\beta : \Gamma_3 \times [0, T] \to [0, 1]$ such that

$$\operatorname{div} \sigma + \varphi_1 = 0 \quad \text{in } \Omega \times (0, T), \tag{2.1}$$

$$\sigma = F\varepsilon(u) \quad \text{in } \Omega \times (0, T), \tag{2.2}$$

$$u = 0 \quad \text{on } \Gamma_1 \times (0, T), \tag{2.3}$$

$$\sigma\nu = \varphi_2 \quad \text{on } \Gamma_2 \times (0, T), \tag{2.4}$$

$$\left.\begin{aligned}&u_\nu \le g, \quad \sigma_\nu + p(u_\nu) - c_\nu\beta^2 R_\nu(u_\nu) \le 0,\\ &(\sigma_\nu + p(u_\nu) - c_\nu\beta^2 R_\nu(u_\nu))(u_\nu - g) = 0\end{aligned}\right\}\ \text{on } \Gamma_3 \times (0,T), \tag{2.5}$$

$$\left.\begin{aligned}&|\sigma_\tau + c_\tau R_\tau(u_\tau)\beta^2| \le \mu|R\sigma_\nu(u)|,\\ &|\sigma_\tau + c_\tau R_\tau(u_\tau)\beta^2| < \mu|R\sigma_\nu(u)| \Rightarrow u_\tau = 0,\\ &|\sigma_\tau + c_\tau R_\tau(u_\tau)\beta^2| = \mu|R\sigma_\nu(u)| \Rightarrow\\ &\qquad\qquad \exists\lambda \ge 0,\ \sigma_\tau + c_\tau R_\tau(u_\tau)\beta^2 = -\lambda u_\tau\end{aligned}\right\}\ \text{on } \Gamma_3 \times (0,T), \tag{2.6}$$

$$\dot\beta = -[\beta(c_\nu(R_\nu(u_\nu))^2 + c_\tau|R_\tau(u_\tau)|^2) - \varepsilon_a]_+ \quad \text{on } \Gamma_3 \times (0,T), \tag{2.7}$$

$$\beta(0) = \beta_0 \ \text{ on } \Gamma_3. \tag{2.8}$$

Equation (2.1) is the equilibrium equation. Equation (2.2) represents the elastic constitutive law of the material in which F is a given function and $\varepsilon(u)$ denotes the small strain tensor; (2.3) and (2.4) are the displacement and traction boundary conditions, respectively, in which ν denotes the unit outward normal vector on Γ and $\sigma\nu$ is the Cauchy stress vector. Conditions (2.5) represent the unilateral contact with adhesion in which σ_ν and u_ν denote respectively the normal stress and the normal displacement; c_ν is a given adhesion coefficient and as in [20], R_ν is a truncation operator given by

$$R_\nu(s) = \begin{cases} L & \text{if } s < -L,\\ -s & \text{if } -L \le s \le 0,\\ 0 & \text{if } s > 0.\end{cases}$$

Here $L > 0$ is the characteristic length of the bond, beyond which the latter has no additional traction (see [23]), and p is a normal compliance function which satisfies assumptions (2.18) below. The positive constant g is the maximum value of the penetration. When $u_\nu < 0$, i.e. when there is separation between the body and the foundation, condition (2.5) combined with hypotheses (2.18) on the function p shows that $\sigma_\nu = c_\nu R_\nu(u_\nu)$ and does not exceed the value $L\|c_\nu\|_{L^\infty(\Gamma_3)}$. When $g > 0$, the body may interpenetrate into the foundation, but the penetration is limited, that is, $u_\nu \le g$. In the case of penetration (i.e. $u_\nu \ge 0$), when $0 \le u_\nu < g$ then $-\sigma_\nu = p(u_\nu)$, which means that the reaction of the foundation is uniquely determined by the normal displacement and $\sigma_\nu \le 0$. Since p is an increasing function, the reaction increases with the penetration. When $u_\nu = g$ then $-\sigma_\nu \ge p(g)$ and σ_ν is not uniquely determined. When $g > 0$, condition (2.5) becomes the Signorini contact condition with adhesion with a gap function,

$$u_\nu \le g, \quad \sigma_\nu - c_\nu\beta^2 R_\nu(u_\nu) \le 0, \quad (\sigma_\nu - c_\nu\beta^2 R_\nu(u_\nu))(u_\nu - g) = 0.$$

When $g = 0$, condition (2.5) combined with hypotheses (2.18) becomes the Signorini contact condition with adhesion with a zero gap function, given by

$$u_\nu \le 0, \quad \sigma_\nu - c_\nu \beta^2 R_\nu(u_\nu) \le 0, \quad (\sigma_\nu - c_\nu \beta^2 R_\nu(u_\nu))u_\nu = 0.$$

This contact condition was used in [6, 23, 24, 26, 27, 29]. Conditions (2.6) represent the nonlocal friction law with adhesion in which σ_τ denotes the tangential stress, u_τ denotes the tangential displacement on the boundary and μ is the friction coefficient; R is a continuous regularizing operator representing the averaging of the normal stress over a small neighbourhood of the contact point. Here, c_τ is the adhesion coefficient and R_τ (see [23]) is a truncation operator defined by

$$R_\tau(v) = \begin{cases} v & \text{if } |v| \le L, \\ Lv/|v| & \text{if } |v| > L. \end{cases}$$

Equation (2.7) is an ordinary differential equation which describes the evolution of the bonding field, in which ε_a is the adhesion coefficient and $r_+ = \max\{r, 0\}$; it was already used in [26]. Since $\dot\beta \le 0$ on $\Gamma_3 \times (0,T)$, once debonding occurs bonding cannot be reestablished and, indeed, the adhesive process is irreversible. Also from [18] it must be pointed out clearly that condition (2.7) does not allow for complete debonding in finite time. Finally, (2.8) is the initial condition, in which β_0 denotes the initial bonding field. In (2.7) the dot above a variable represents its derivative with respect to time.

We denote by S_d the space of second order symmetric tensors on $\mathbb{R}^d$ ($d = 2,3$) and $|\cdot|$ represents the Euclidean norm on $\mathbb{R}^d$ and S_d. Thus, for every $u, v \in \mathbb{R}^d$, $u.v = u_i v_i$, $|v| = (v.v)^{1/2}$, and for every $\sigma, \tau \in S_d$, $\sigma.\tau = \sigma_{ij}\tau_{ij}$, $|\tau| = (\tau.\tau)^{1/2}$. Here and below, the indices i and j run between 1 and d and the summation convention over repeated indices is adopted. Now, to proceed with the variational formulation, we need the following function spaces:

$$H = (L^2(\Omega))^d, \quad H_1 = (H^1(\Omega))^d, \quad Q = \{\tau = (\tau_{ij});\ \tau_{ij} = \tau_{ji} \in L^2(\Omega)\},$$
$$Q_1 = \{\sigma \in Q;\ \operatorname{div}\sigma \in H\}.$$

Note that H and Q are real Hilbert spaces endowed with the respective canonical inner products

$$(u,v)_H = \int_\Omega u_i v_i \, dx, \qquad (\sigma,\tau)_Q = \int_\Omega \sigma_{ij}\tau_{ij}\, dx.$$

The strain tensor is

$$\varepsilon(u) = (\varepsilon_{ij}(u)) = \tfrac{1}{2}(u_{i,j} + u_{j,i});$$

$\operatorname{div}\sigma = (\sigma_{ij,j})$ is the divergence of σ. For every $v \in H_1$ we denote by v_ν and v_τ the normal and the tangential components of v on the boundary Γ given by

$$v_\nu = v.\nu, \qquad v_\tau = v - v_\nu \nu.$$

We also denote by σ_ν and σ_τ the normal and the tangential traces of a function $\sigma \in Q_1$; when σ is a regular function then

$$\sigma_\nu = (\sigma\nu).\nu, \qquad \sigma_\tau = \sigma\nu - \sigma_\nu \nu.$$

The following Green's formula holds:

$$(\sigma, \varepsilon(v))_Q + (\operatorname{div}\sigma, v)_H = \int_\Gamma \sigma\nu.v\, da \qquad \forall v \in H_1,$$

where da is the surface measure element.

Now, let V be the closed subspace of H_1 defined by

$$V = \{v \in H_1;\ v = 0 \text{ on } \Gamma_1\},$$

and let

$$K = \{v \in V;\ v_\nu \le g \text{ a.e. on } \Gamma_3\}$$

be the convex subset of admissible displacements. Since $\text{meas}(\Gamma_1) > 0$, the following Korn inequality holds [10]:

$$\|\varepsilon(v)\|_Q \ge c_\Omega \|v\|_{H_1} \qquad \forall v \in V, \tag{2.9}$$

where $c_\Omega > 0$ is a constant which depends only on Ω and Γ_1. We equip V with the inner product

$$(u, v)_V = (\varepsilon(u), \varepsilon(v))_Q$$

and $\|\cdot\|_V$ is the associated norm. It follows from Korn's inequality (2.9) that the norms $\|\cdot\|_{H_1}$ and $\|\cdot\|_V$ are equivalent on V. Thus $(V, \|\cdot\|_V)$ is a real Hilbert space. Moreover by Sobolev's trace theorem, there exists $d_\Omega > 0$ which only depends on the domain Ω, Γ_1 and Γ_3 such that

$$\|v\|_{(L^2(\Gamma_3))^d} \le d_\Omega \|v\|_V \qquad \forall v \in V. \tag{2.10}$$

We use the standard norm on $L^\infty(0,T;V)$ and equip the Sobolev space $W^{1,\infty}(0,T;V)$ with the norm

$$\|v\|_{W^{1,\infty}(0,T;V)} = \|v\|_{L^\infty(0,T;V)} + \|\dot{v}\|_{L^\infty(0,T;V)}.$$

For every real Banach space $(X, \|\cdot\|_X)$ and $T > 0$ we use the notation $C([0,T];X)$ for the space of continuous functions from $[0,T]$ to X; recall that $C([0,T];X)$ is a real Banach space with the norm

$$\|x\|_{C([0,T];X)} = \max_{t\in[0,T]} \|x(t)\|_X.$$

We suppose that the body forces and surface tractions have the regularity

$$\varphi_1 \in W^{1,\infty}(0,T;H), \qquad \varphi_2 \in W^{1,\infty}(0,T;(L^2(\Gamma_2))^d), \tag{2.11}$$

and, using Riesz's representation theorem, we define the function $f : [0,T] \to V$ by

$$(f(t), v)_V = \int_\Omega \varphi_1(t).v\, dx + \int_{\Gamma_2} \varphi_2(t).v\, da \qquad \forall v \in V,\ t \in [0,T]. \tag{2.12}$$

We note that (2.11) and (2.12) imply $f \in W^{1,\infty}(0,T;V)$.

In the study of the mechanical problem P_1 we assume that the nonlinear elasticity operator satisfies

$$(2.13)\quad \begin{cases} \text{(a)}\ F:\Omega\times S_d \to S_d;\\ \text{(b) there exists } M>0 \text{ such that}\\ \quad |F(x,\varepsilon_1)-F(x,\varepsilon_2)|\le M|\varepsilon_1-\varepsilon_2| \ \text{ for all } \varepsilon_1,\varepsilon_2 \text{ in } S_d,\\ \quad \text{a.e. } x\in\Omega;\\ \text{(c) there exists } m>0 \text{ such that}\\ \quad (F(x,\varepsilon_1)-F(x,\varepsilon_2)).(\varepsilon_1-\varepsilon_2)\ge m|\varepsilon_1-\varepsilon_2|^2\\ \quad \text{for all } \varepsilon_1,\varepsilon_2 \text{ in } S_d,\ \text{a.e. } x\in\Omega;\\ \text{(d) the mapping } x\mapsto F(x,\varepsilon) \text{ is Lebesgue measurable on } \Omega\\ \quad \text{for any } \varepsilon \text{ in } S_d;\\ \quad F(x,0)=0 \text{ for a.e. } x\in\Omega. \end{cases}$$

The adhesion coefficients are assumed to satisfy

$$(2.14)\quad c_\nu, c_\tau\in L^\infty(\Gamma_3),\quad \varepsilon_a\in L^2(\Gamma_3)\quad \text{and}\quad c_\nu,c_\tau,\varepsilon_a\ge 0\quad \text{a.e. on } \Gamma_3,$$

and the friction coefficient μ satisfies

$$(2.15)\qquad \mu\in L^\infty(\Gamma_3)\quad\text{and}\quad \mu\ge 0 \text{ a.e. on } \Gamma_3.$$

We define the normal stress $\sigma_\nu(u)$ on Γ at time t as follows. Let $u\in H_1$ be such that $\operatorname{div}\sigma(u)=-\varphi_1(t)$. Then $\sigma_\nu(u)\in H^{-1/2}(\Gamma)$ is given by

$$(2.16)\qquad \langle\sigma_\nu(u),v_\nu\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)}=\langle F\varepsilon(u),\varepsilon(v)\rangle_Q-(\varphi_1(t),v)_H$$

for all $v\in H_1$ such that $v_\tau=0$ on Γ.

Next at time $t\in[0,T]$, we define the set V_t as

$$V_t=\{v\in H_1;\ \operatorname{div}\sigma(v)+\varphi_1(t)=0 \text{ in } \Omega\}$$

and the functional $j_t: V_t\times V\to\mathbb{R}$ by

$$j_t(u,v)=\int_{\Gamma_3}\mu|R\sigma_\nu(u)|\,|v_\tau|\,da\qquad \forall(u,v)\in V_t\times V,$$

where $R: H^{-1/2}(\Gamma)\to L^2(\Gamma_3)$ is a continuous linear operator (see [9]). That is, there exists a constant $c_R>0$ such that

$$(2.17)\qquad \|R\tau\|_{L^2(\Gamma_3)}\le c_R\|\tau\|_{H^{-1/2}(\Gamma)}\qquad \forall\tau\in H^{-1/2}(\Gamma).$$

We also define the functional $r: L^2(\Gamma_3)\times V\times V\to\mathbb{R}$ by

$$r(\beta,u,v)=\int_{\Gamma_3}[(-c_\nu\beta^2R_\nu(u_\nu)+p(u_\nu))v_\nu+c_\tau\beta^2R_\tau(u_\nu).v_\tau]\,da$$

for $(\beta,u,v)\in L^2(\Gamma_3)\times V\times V$, where as in [16] we assume that the normal compliance function p satisfies

$$(2.18)\qquad \begin{cases} \text{(a)}\ \ p :]-\infty, g] \to \mathbb{R}; \\ \text{(b)}\ \ \text{there exists } L_p > 0 \text{ such that} \\ \qquad |p(r_1) - p(r_2)| \le L_p|r_1 - r_2| \text{ for all } r_1, r_2 \le g; \\ \text{(c)}\ \ (p(r_1) - p(r_2))(r_1 - r_2) \ge 0 \text{ for all } r_1, r_2 \le g; \\ \text{(d)}\ \ p(r) = 0 \text{ for all } r < 0. \end{cases}$$

Finally we introduce the following set of bonding fields:

$$B = \{\theta : [0,T] \to L^2(\Gamma_3);\ 0 \le \theta(t) \le 1,\ \forall t \in [0,T],\ \text{a.e. on } \Gamma_3\}$$

and we suppose that the initial bonding field satisfies

$$(2.19)\qquad \beta_0 \in L^2(\Gamma_3), \qquad 0 \le \beta_0 \le 1 \text{ a.e. on } \Gamma_3.$$

Now by assuming the solution to be sufficiently regular, we find by using Green's formula that Problem P_1 has the following variational formulation in terms of displacements and bonding fields.

Problem P_2. Find a displacement field $u \in W^{1,\infty}(0,T;V)$ and a bonding field $\beta \in W^{1,\infty}(0,T;L^2(\Gamma_3)) \cap B$ such that

$$(2.20)\qquad \begin{aligned} &u(t) \in K \cap V_t, \\ &(F\varepsilon(u(t)), \varepsilon(v) - \varepsilon(u(t)))_Q + r(\beta(t), u(t), v - u(t)) \\ &\quad + j_t(u(t), v) - j_t(u(t), u(t)) \ge (f(t), v - u(t))_V \qquad \forall v \in K,\ t \in [0,T], \end{aligned}$$

$$(2.21)\qquad \dot{\beta}(t) = -[\beta(t)(c_\nu(R_\nu(u_\nu(t)))^2 + c_\tau|R_\tau(u_\tau(t))|^2) - \varepsilon_a]_+ \qquad \text{a.e. } t \in (0,T),$$

$$(2.22)\qquad \beta(0) = \beta_0.$$

3. Existence and uniqueness result. Our main result to be established in this section is

Theorem 3.1. *Let* (2.11), (2.13), (2.14), (2.15), (2.18) *and* (2.19) *hold. Then there exists a constant* $\mu_0 > 0$ *such that Problem* P_2 *has a unique solution if*

$$\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0.$$

The proof of Theorem 3.1 is carried out in several steps. In the first step, let X denote the closed subset of the space $C([0,T];L^2(\Gamma_3))$ defined by

$$X = \{\theta \in C([0,T];L^2(\Gamma_3)) \cap B; \theta(0) = \beta_0\},$$

where $C([0,T];L^2(\Gamma_3))$ is endowed with the norm

$$\|\theta\|_k = \max_{t\in[0,T]} \exp(-kt)\|\theta(t)\|_{L^2(\Gamma_3)} \qquad \text{for all } \theta \in C([0,T];L^2(\Gamma_3)),\ k > 0.$$

Next for a given $\beta \in X$, we consider the following variational problem.

PROBLEM $P_{1\beta}$. Find $u_\beta \in C([0,T];V)$ such that

$$(3.1)\quad \begin{aligned} &u_\beta(t) \in K \cap V_t,\\ &(F\varepsilon(u_\beta(t)), \varepsilon(v - u_\beta(t)))_Q + r(\beta(t), u_\beta(t), v - u_\beta(t))\\ &+ j_t(u_\beta(t), v) - j_t(u_\beta(t), u_\beta(t)) \geq (f(t), v - u_\beta(t))_V \quad \forall v \in K,\ t \in [0,T]. \end{aligned}$$

PROPOSITION 3.2. *There exists a constant $\mu_0 > 0$ such that Problem $P_{1\beta}$ has a unique solution if $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0$.*

The proof of Proposition 3.2 will be carried out in several steps. In the first step for each $t \in [0,T]$ and a given $\eta \in K \cap V_t$, we consider the following intermediate problem.

PROBLEM P_η. Find $u_{\beta\eta}(t) \in K \cap V_t$ such that

$$(3.2)\quad \begin{aligned} &(F\varepsilon(u_{\beta\eta}(t)), \varepsilon(v - u_{\beta\eta}(t)))_Q + r(\beta(t), u_{\beta\eta}(t), v - u_{\beta\eta}(t))\\ &\qquad + j_t(\eta, v) - j_t(\eta, u_{\beta\eta}(t)) \geq (f(t), v - u_{\beta\eta}(t))_V \quad \forall v \in K. \end{aligned}$$

LEMMA 3.3. *Problem P_η has a unique solution.*

Proof. Let $t \in [0,T]$ and let $A_{\beta(t)} : V \to V$ be defined by

$$(A_{\beta(t)}u, v)_V = (F\varepsilon(u), \varepsilon(v))_Q + r(\beta(t), u, v) \quad \forall u, v \in V.$$

We use (2.10), (2.13)(b)&(c), (2.18)(b)&(c) and the properties of the operators R_ν and R_τ (see [23]) to show that the operator $A_{\beta(t)}$ is strongly monotone and Lipschitz continuous; the functional $j_t(\eta,\cdot) : V \to \mathbb{R}$ is a continuous seminorm, therefore since K is a nonempty closed convex subset of V, it follows from a standard existence and uniqueness result for elliptic quasivariational inequalities (see [25]) that there exists a unique element $u_{\beta\eta}(t) \in K$ which satisfies the inequality (3.2). On the other hand taking $v = u_{\beta\eta}(t) \pm \theta$ in (3.2) where $\theta \in (C_0^\infty(\Omega))^d$, we obtain

$$\operatorname{div} \sigma(u_{\beta\eta}(t)) + \varphi_1(t) = 0 \quad \text{for all } t \in [0,T].$$

This implies that $u_{\beta\eta}(t) \in V_t$ for all $t \in [0,T]$. ∎

In the second step, let $t \in [0,T]$ and consider the mapping $\Phi : K \cap V_t \to K \cap V_t$ defined by

$$\Phi(\eta) = u_{\beta\eta}(t).$$

LEMMA 3.4. *The mapping Φ has a unique fixed point η^* and $u_{\beta\eta^*}(t)$ is a unique solution to the inequality* (3.1), *for any $t \in [0,T]$.*

Proof. Let $t \in [0,T]$ and $\eta_1, \eta_2 \in K \cap V_t$. We write (3.2) for $\eta = \eta_1$ and $v = u_{\beta\eta_2}(t)$, and then for $\eta = \eta_2$ and $v = u_{\beta\eta_1}(t)$; using (2.18)(c) and the properties of R_ν and R_τ, we add the resulting inequalities to obtain

$$(3.3)\quad \begin{aligned} &\big(F\varepsilon(u_{\beta\eta_1}(t)) - F\varepsilon(u_{\beta\eta_2}(t)), \varepsilon(u_{\beta\eta_1}(t) - u_{\beta\eta_2}(t))\big)_Q\\ &\qquad \leq j_t(\eta_1, u_{\beta\eta_2}(t)) - j_t(\eta_2, u_{\beta\eta_1}(t)) + j_t(\eta_2, u_{\beta\eta_1}(t)) - j_t(\eta_2, u_{\beta\eta_2}(t)). \end{aligned}$$

Using (2.17) and (2.10) we get

$$(3.4)\qquad j_t(\eta_1, u_{\beta\eta_2}(t)) - j_t(\eta_2, u_{\beta\eta_1}(t)) + j_t(\eta_2, u_{\beta\eta_1}(t)) - j_t(\eta_2, u_{\beta\eta_2}(t)) \\ \le \|\mu\|_{L^\infty(\Gamma_3)} d_\Omega \|\sigma_\nu(\eta_1) - \sigma_\nu(\eta_2)\|_{H^{-1/2}(\Gamma)} \|u_{\beta\eta_1}(t) - u_{\beta\eta_2}(t)\|_V.$$

Moreover it follows from (2.16) that there exists a constant $c_d > 0$ such that

$$\|\sigma_\nu(\eta_1) - \sigma_\nu(\eta_2)\|_{H^{-1/2}(\Gamma)} \le M c_d \|\eta_1 - \eta_2\|_V.$$

We now use (2.13)(c) to deduce from (3.3) and (3.4) that

$$\|\Phi(\eta_1) - \Phi(\eta_2)\|_V \le \frac{c_d c_R M d_\Omega}{m} \|\mu\|_{L^\infty(\Gamma_3)} \|\eta_1 - \eta_2\|_V.$$

Thus if we take $\mu_0 = m/c_R c_d M d_\Omega$, then the mapping Φ is a contraction for $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0$; it has a unique fixed point η^* and $u_{\beta\eta^*}(t)$ is a unique element which solves (3.1), for any $t \in [0, T]$. ∎

Next, denote $u_{\beta\eta^*}(t) = u_\beta(t)$ for all $t \in [0, T]$. To end the proof of Proposition 3.2 we prove the following result.

Lemma 3.5. *We have $u_\beta \in C([0, T]; V)$.*

Proof. Let $t_1, t_2 \in [0, T]$. Set $t = t_1$ and $v = u_\beta(t_2)$ in inequality (3.1), and then $t = t_2$ and $v = u_\beta(t_1)$; by adding the resulting inequalities we find

$$(3.5)\qquad \big(F\varepsilon(u_\beta(t_1)) - F\varepsilon(u_\beta(t_2)), \varepsilon(u_\beta(t_1)) - \varepsilon(u_\beta(t_2))\big)_Q \\ \le r(\beta(t_1), u_\beta(t_1), u_\beta(t_2) - u_\beta(t_1)) + r(\beta(t_2), u_\beta(t_2), u_\beta(t_1) - u_\beta(t_2)) \\ + j_{t_1}(u_\beta(t_1), u_\beta(t_2)) - j_{t_1}(u_\beta(t_1), u_\beta(t_1)) + j_{t_2}(u_\beta(t_2), u_\beta(t_1)) \\ - j_{t_2}(u_\beta(t_2), u_\beta(t_2)) + (f(t_1) - f(t_2), u_\beta(t_1) - u_\beta(t_2))_V.$$

We have

$$r(\beta(t_1), u_\beta(t_1), u_\beta(t_2) - u_\beta(t_1)) + r(\beta(t_2), u_\beta(t_2), u_\beta(t_1) - u_\beta(t_2)) \\ = \int_{\Gamma_3} (p(u_{\beta\nu}(t_1)) - p(u_{\beta\nu}(t_2)))(u_{\beta\nu}(t_2) - u_{\beta\nu}(t_1)) \\ - \int_{\Gamma_3} c_\nu(\beta^2(t_1) R_\nu(u_{\beta\nu}(t_1)) - \beta^2(t_2) R_\nu(u_{\beta\nu}(t_2)))(u_{\beta\nu}(t_2) - u_{\beta\nu}(t_1))\, da \\ + \int_{\Gamma_3} c_\tau(\beta^2(t_1) R_\tau(u_{\beta\tau}(t_1)) - \beta^2(t_2) R_\tau(u_{\beta\tau}(t_2))).(u_{\beta\tau}(t_2) - u_{\beta\tau}(t_1))\, da.$$

Using (2.18)(c), we get

$$(3.6)\qquad \int_{\Gamma_3} (p(u_{\beta\nu}(t_1)) - p(u_{\beta\nu}(t_2)))(u_{\beta\nu}(t_2) - u_{\beta\nu}(t_1)) \le 0.$$

We now write the second term on the right hand side of the last equality as

$$
\begin{aligned}
-\int_{\Gamma_3} c_\nu(\beta^2(t_1)R_\nu(u_{\beta\nu}(t_1)) - \beta^2(t_2)R_\nu(u_{\beta\nu}(t_2)))(u_{\beta\nu}(t_2) - u_{\beta\nu}(t_1))\,da \\
= -\int_{\Gamma_3} c_\nu(\beta^2(t_1) - \beta^2(t_2))R_\nu(u_{\beta\nu}(t_1))(u_{\beta\nu}(t_2) - u_{\beta\nu}(t_1))\,da \\
-\int_{\Gamma_3} c_\nu\beta^2(t_2)(R_\nu(u_{\beta\nu}(t_1)) - R_\nu(u_{\beta\nu}(t_2)))(u_{\beta\nu}(t_2) - u_{\beta\nu}(t_1))\,da.
\end{aligned}
$$

From the property

$$
(R_\nu(u_{\beta\nu}(t_1)) - R_\nu(u_{\beta\nu}(t_2)))(u_{\beta\nu}(t_2) - u_{\beta\nu}(t_1)) \geq 0 \quad \text{a.e. on } \Gamma_3,
$$

it follows that

$$
\begin{aligned}
(3.7)\quad -\int_{\Gamma_3} c_\nu(\beta^2(t_1)R_\nu(u_{\beta\nu}(t_1)) - \beta^2(t_2)R_\nu(u_{\beta\nu}(t_2)))(u_{\beta\nu}(t_2) - u_{\beta\nu}(t_1))\,da \\
\leq -\int_{\Gamma_3} c_\nu(\beta^2(t_1) - \beta^2(t_2))R_\nu(u_{\beta\nu}(t_1))(u_{\beta\nu}(t_2) - u_{\beta\nu}(t_1))\,da.
\end{aligned}
$$

Analogously, as

$$
(R_\tau(u_{\beta\tau}(t_1)) - R_\tau(u_{\beta\tau}(t_2))).(u_{\beta\tau}(t_2) - u_{\beta\tau}(t_1)) \leq 0 \quad \text{a.e. on } \Gamma_3,
$$

we find

$$
\begin{aligned}
(3.8)\quad \int_{\Gamma_3} c_\tau(\beta^2(t_1)R_\tau(u_{\beta\tau}(t_1)) - \beta^2(t_2)R_\tau(u_{\beta\tau}(t_2))).(u_{\beta\tau}(t_2) - u_{\beta\tau}(t_1))\,da \\
\leq \int_{\Gamma_3} c_\tau(\beta^2(t_1) - \beta^2(t_2))R_\tau(u_{\beta\tau}(t_1)).(u_{\beta\tau}(t_2) - u_{\beta\tau}(t_1))\,da.
\end{aligned}
$$

As a consequence of (3.5)–(3.8) we obtain

$$
\begin{aligned}
(3.9)\quad &\big(F\varepsilon(u_\beta(t_1)) - F\varepsilon(u_\beta(t_2)), \varepsilon(u_\beta(t_1)) - \varepsilon(u_\beta(t_2))\big)_Q \\
&\leq -\int_{\Gamma_3} c_\nu(\beta^2(t_1) - \beta^2(t_2))R_\nu(u_{\beta\nu}(t_1))(u_{\beta\nu}(t_2) - u_{\beta\nu}(t_1))\,da \\
&\quad + \int_{\Gamma_3} c_\tau(\beta^2(t_1) - \beta^2(t_2))R_\tau(u_{\beta\tau}(t_1)).(u_{\beta\tau}(t_2) - u_{\beta\tau}(t_1))\,da \\
&\quad + j_{t_1}(u_\beta(t_1), u_\beta(t_2)) - j_{t_1}(u_\beta(t_1), u_\beta(t_1)) \\
&\quad + j_{t_2}(u_\beta(t_2), u_\beta(t_1)) - j_{t_2}(u_\beta(t_2), u_\beta(t_2)).
\end{aligned}
$$

On the other hand, as in the proof of Lemma 3.4 we derive

$$
\begin{aligned}
(3.10)\quad j_{t_1}(u_\beta(t_1), u_\beta(t_2)) - j_{t_1}(u_\beta(t_1), u_\beta(t_1)) + j_{t_2}(u_\beta(t_2), u_\beta(t_1)) \\
- j_{t_2}(u_\beta(t_2), u_\beta(t_2)) \leq \|\mu\|_{L^\infty(\Gamma_3)} c_R c_d d_\Omega M \|u_\beta(t_1) - u_\beta(t_2)\|_V^2.
\end{aligned}
$$

Then, using (2.10), (2.13)(c), (2.14), $|\beta(t_i)| \le 1$, $i = 1,2$, and $|R_s(u_s)| \le L$ for $s = \nu, \tau$, from (3.9) and (3.10) we obtain

$$(m - \|\mu\|_{L^\infty(\Gamma_3)} c_R c_d d_\Omega M)\|u_\beta(t_1) - u_\beta(t_2)\|_V$$
$$\le 2(\|c_\nu\|_{L^\infty(\Gamma_3)} + \|c_\tau\|_{L^\infty(\Gamma_3)}) L d_\Omega \|\beta(t_1) - \beta(t_2)\|_{L^2(\Gamma_3)} + \|f(t_1) - f(t_2)\|_V.$$

Since $m - \|\mu\|_{L^\infty(\Gamma_3)} c_R c_d d_\Omega M > 0$, we deduce the estimate

$$(3.11)\qquad \|u_\beta(t_1) - u_\beta(t_2)\|_V$$
$$\le \frac{2Ld_\Omega}{m - \|\mu\|_{L^\infty(\Gamma_3)} c_R c_d d_\Omega M}(\|c_\nu\|_{L^\infty(\Gamma_3)} + \|c_\tau\|_{L^\infty(\Gamma_3)})\|\beta(t_1) - \beta(t_2)\|_{L^2(\Gamma_3)}$$
$$+ \frac{1}{m - \|\mu\|_{L^\infty(\Gamma_3)} c_R c_d d_\Omega M}\|f(t_1) - f(t_2)\|_V.$$

Therefore from (3.11), as $f \in C([0,T]; V)$ and $\beta \in C([0,T]; L^2(\Gamma_3))$ we immediately conclude the proof. ∎

We now consider the following problem.

Problem $P_{2\beta}$. Find $\beta^* : [0,T] \to L^2(\Gamma_3)$ such that

(3.12)
$$\dot\beta^*(t) = -[\beta^*(t)(c_\nu(R_\nu(u_{\beta^*\nu}(t)))^2 + c_\tau|R_\tau(u_{\beta^*\tau}(t))|^2) - \varepsilon_a]_+ \text{ a.e. } t \in (0,T),$$

$$(3.13)\qquad \beta^*(0) = \beta_0.$$

Lemma 3.6. *Problem $P_{2\beta}$ has a unique solution β^* which satisfies*
$$\beta^* \in W^{1,\infty}(0,T; L^2(\Gamma_3)) \cap B.$$

Proof. Let $T : X \to X$ be defined by

$$T\beta(t) = \beta_0 - \int_0^t [\beta(s)(c_\nu(R_\nu(u_{\beta\nu}(s)))^2 + c_\tau|R_\tau(u_{\beta\tau}(s))|^2) - \varepsilon_a]_+ \, ds$$

for $\beta \in X$ and $t \in [0,T]$, where u_β is the solution of Problem $P_{1\beta}$. Then for $\beta_1, \beta_2 \in X$, we have

$$\|T\beta_1(t) - T\beta_2(t)\|_{L^2(\Gamma_3)}$$
$$\le c_1 \int_0^t \|\beta_1(s)(R_\nu(u_{\beta_1\nu}(s)))^2 - \beta_2(s)(R_\nu(u_{\beta_2\nu}(s)))^2\|_{L^2(\Gamma_3)} \, ds$$
$$+ c_1 \int_0^t \|\beta_1(s)|R_\tau(u_{\beta_1\tau}(s))|^2 - \beta_2(s)|R_\tau(u_{\beta_2\tau}(s))|^2\|_{L^2(\Gamma_3)} \, ds,$$

where $c_1 > 0$. Using the definition of the truncation operators R_ν, R_τ and writing

$$\beta_1 = \beta_1 - \beta_2 + \beta_2,$$

we get

$$\|T\beta_1(t) - T\beta_2(t)\|_{L^2(\Gamma_3)} \le c_2 \int_0^t \|\beta_1(s) - \beta_2(s)\|_{L^2(\Gamma_3)}\, ds + c_2 \int_0^t \|u_{\beta_1}(s) - u_{\beta_2}(s)\|_{L^2(\Gamma_3)}\, ds$$

for some positive constant c_2. Moreover using (2.10), we obtain

$$\|T\beta_1(t) - T\beta_2(t)\|_{L^2(\Gamma_3)} \le c_2 \int_0^t \|\beta_1(s) - \beta_2(s)\|_{L^2(\Gamma_3)} ds + c_2 d_\Omega \int_0^t \|u_{\beta_1}(s) - u_{\beta_2}(s)\|_V\, ds.$$

Now for each $t \in [0, T]$, using (3.1), (2.13)(c), (2.17) and (2.18)(c), we find for $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0$ (see [26]) the inequality

$$\|u_{\beta_1}(t) - u_{\beta_2}(t)\|_V \le c_3 \|\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma_3)},$$

where $c_3 > 0$. Hence, we deduce that there exists a constant $c_4 > 0$ such that

$$\|T\beta_1(t) - T\beta_2(t)\|_{L^2(\Gamma_3)} \le c_4 \int_0^t \|\beta_1(s) - \beta_2(s)\|_{L^2(\Gamma_3)}\, ds.$$

Therefore

$$\|T\beta_1(t) - T\beta_2(t)\|_{L^2(\Gamma_3)} \le c_4 \|\beta_1 - \beta_2\|_k \frac{\exp(kt)}{k} \qquad \forall t \in [0, T],$$

which implies

$$\|T\beta_1 - T\beta_2\|_k \le \frac{c_4}{k} \|\beta_1 - \beta_2\|_k. \tag{3.14}$$

Hence, for $k > c_4$, T is a contraction. Then it has a unique fixed point β^* which satisfies (3.12) and (3.13). Taking into account (3.11), we obtain $u_{\beta^*} \in W^{1,\infty}(0, T; V)$.

Proof of Theorem 3.1. Let $\beta = \beta^*$ and let u_{β^*} be the solution to Problem $P_{1\beta}$. We conclude from (3.1), (3.12) and (3.13) that (u_{β^*}, β^*) is a solution of Problem P_2. Now to prove the uniqueness of the solution, suppose that (u, β) is a solution of Problem P_2. It follows from (2.20) that u is a solution of Problem $P_{1\beta}$ and by Proposition 3.2 we get $u = u_\beta$. Taking $u = u_\beta$ in (2.21) and using the initial condition (2.22), we deduce that β is a solution of Problem $P_{2\beta}$. Finally, using Lemma 3.6, we obtain $\beta = \beta^*$ and so (u_{β^*}, β^*) is a unique solution to Problem P_2.

Remark. For a large friction coefficient the problem of existence and uniqueness of the solution is not studied here and remains open.

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