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Boundary subordination

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Abstract. We study the idea of the boundary subordination of two analytic functions. Some basic properties of the boundary subordination are discussed. Applications to classes of univalent functions referring to a boundary point are demonstrated.

1. Introduction. The aim of this paper is to study the concept of *boundary subordination* of two analytic functions, which appeared first in [Le3, p. 182]. The source of this idea is in the geometrical concepts referring to a boundary point, e.g. starlikeness with respect to a boundary point or convexity in the positive direction of the real axis. In the classical subordination theory two subordinated analytic functions are connected by a Schwarz function. Two boundary subordinated analytic functions are related by a function of the class $\mathcal{B}_1$ defined here, i.e. by a function satisfying the assumptions of the Julia lemma. By analogy to Schwarz functions, let us call functions in $\mathcal{B}_1$ *Julia functions*. Using the assertion of the Julia lemma the class $\mathcal{B}_1$ can be considered as the union of its subclasses $\mathcal{B}_1(\lambda),\ 0 < \lambda \le \infty$. When two analytic functions are connected by a function of the class $\mathcal{B}_1(\lambda)$, we say that they are *λ-boundary subordinated.* The essential difference between Schwarz and Julia functions is in the behavior of their fixed points: zero lying in $\mathbb{D}$ is a nonexpansive fixed point for Schwarz functions, while 1 which is a boundary fixed point in $\mathbb{D}$ admits expansion for Julia functions. Therefore this concept may have some deeper meaning and be useful for the study of the boundary behavior of analytic functions.

In the last section we reformulate some known results describing some subclasses of univalent functions defined by a boundary point in terms of boundary subordination.

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2. Preliminaries. Let $\overline{\mathbb{C}} = \mathbb{C}\cup\{\infty\}$. For $z_0 \in \mathbb{C}$ and $r > 0$ let $\mathbb{D}(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$, $\mathbb{D}_r = \mathbb{D}(0, r)$, $\mathbb{D} = \mathbb{D}_1$ and $\mathbb{T} = \partial\mathbb{D}$. Let $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. For each $k > 0$, let

$$\mathbb{O}_k = \left\{ z \in \mathbb{D} : \frac{|1-z|^2}{1-|z|^2} < k \right\}$$

denote the disk in $\mathbb{D}$ called the *orocycle*. Note that $\mathbb{O}_k = \mathbb{D}(1/(1+k), k/(1+k))$ for $k > 0$. For $\zeta \in \mathbb{T}$ and $M > 1$ let

$$\Delta(\zeta, M) = \{z \in \mathbb{D} : |\zeta - z| < M(1 - |z|)\}$$

denote a Stolz angle at ζ.

Let $\mathcal{A}$ denote the class of analytic functions in $\mathbb{D}$, and $\mathcal{S}$ its subclass of univalent functions.

A function $f \in \mathcal{A}$ is said to have at $\zeta \in \mathbb{T}$:

- the *angular limit* $f_\angle(\zeta) \in \overline{\mathbb{C}}$ (also denoted as $\angle f(\zeta)$ or $\angle \lim_{z\to\zeta} f(z)$) if
$$f_\angle(\zeta) = \lim_{\Delta(\zeta,M)\ni z\to\zeta} f(z)$$
for every Stolz angle $\Delta(\zeta, M)$;
- the *asymptotic value* $v \in \overline{\mathbb{C}}$ if $v = \lim_{\gamma\ni z\to\zeta} f(z)$ for some curve $\gamma \subset \mathbb{D}$ ending at ζ; the curve γ will be called an *asymptotic path*;
- the *angular derivative* $f'_\angle(\zeta) \in \overline{\mathbb{C}}$ if the angular limit $f_\angle(\zeta)$ exists and is finite, and
$$f'_\angle(\zeta) = \angle \lim_{z\to\zeta} \frac{f(z) - f_\angle(\zeta)}{z - \zeta}.$$

THEOREM 2.1 ([P2, p. 79]). *A function $f \in \mathcal{A}$ has a finite angular derivative $f'_\angle(\zeta)$ at $\zeta \in \mathbb{T}$ if and only if f' has a finite angular limit at ζ. Moreover*

$$f'_\angle(\zeta) = \angle f'(\zeta).$$

3. Julia functions

3.1. Basic facts. Denote by $\mathcal{B}$ the class of functions $\omega \in \mathcal{A}$ with $|\omega(z)| < 1$ for $z \in \mathbb{D}$ and by $\mathcal{P}$ the class of functions $p \in \mathcal{A}$ with $\operatorname{Re} p(z) > 0$ for $z \in \mathbb{D}$.

DEFINITION 3.1. Let

$$\mathcal{B}_1 = \{\omega \in \mathcal{B} : \omega_\angle(1) = 1\}.$$

Let us call functions in the class $\mathcal{B}_1$ *Julia functions.* Similarly, let

$$\mathcal{P}_0 = \{p \in \mathcal{P} : p_\angle(1) = 0\}.$$

EXAMPLE 3.2. For a given $a \in \mathbb{H}$ let

$$h_a(z) = |a|^2 \frac{1-z}{\overline{a}+az}, \quad z \in \mathbb{D}.$$

We see that $h_a \in \mathcal{P}_0$, $h_a(0) = a$ and $h_a(\mathbb{D}) = \mathbb{H}$. Moreover

$$h_a^{-1}(z) = \frac{\overline{a}}{a} \frac{a-z}{\overline{a}+z}, \quad z \in \mathbb{H}.$$

PROPOSITION 3.3. *Fix $a \in \mathbb{H}$.*

1. *If $\omega \in \mathcal{B}_1$, then $p = h_a \circ \omega \in \mathcal{P}_0$.*
2. *If $p \in \mathcal{P}_0$, then $\omega = h_a^{-1} \circ p \in \mathcal{B}_1$.*

In particular, for $a = 1$:

3. *If $\omega \in \mathcal{B}_1$, then*

$$p = h_1 \circ \omega = \frac{1-\omega}{1+\omega} \in \mathcal{P}_0. \tag{3.1}$$

4. *If $p \in \mathcal{P}_0$, then*

$$\omega = h_1^{-1} \circ p = \frac{1-p}{1+p} \in \mathcal{B}_1. \tag{3.2}$$

For $\omega \in \mathcal{B}_1$ let

$$\Lambda(\omega) = \sup\left\{\frac{|1-\omega(z)|^2}{1-|\omega(z)|^2} \cdot \frac{1-|z|^2}{|1-z|^2} : z \in \mathbb{D}\right\}. \tag{3.3}$$

Recall now the Julia–Carathéodory–Wolff theorem (see [J], [P2, p. 82]), [Ca, p. 57], [Ga, p. 43]).

THEOREM 3.4. *Let $\omega \in \mathcal{B}_1$.*

1. *The angular derivative $\omega'_\angle(1)$ exists and*

$$\begin{aligned}0 < \omega'_\angle(1) &= \angle\lim_{z\to 1} \frac{1-\omega(z)}{1-z} \\ &= \lim_{r\to 1^-} \frac{1-\omega(r)}{1-r} = \lim_{r\to 1^-} \frac{1-|\omega(r)|}{1-r} = \Lambda(\omega) \le \infty.\end{aligned}$$

2. *If $\omega'_\angle(1) < \infty$, then*
   (i) *$\omega'_\angle(1) = \angle\omega'(1)$;*
   (ii) *$\omega(\mathbb{O}_k) \subset \mathbb{O}_{\Lambda(\omega)k}$ for every $k > 0$.*

Observe that 2(i) is a consequence of Theorem 2.1.

From Theorem 3.4 and Proposition 3.3 we have

THEOREM 3.5. *Let $p \in \mathcal{P}_0$.*

1. *The angular derivative $p'_\angle(1)$ exists and*

$$-\infty \le p'_\angle(1) = \angle\lim_{z\to 1} \frac{p(z)}{z-1} < 0.$$

2. *If* $p'_\angle(1) > -\infty$, *then*

$$p'_\angle(1) = \angle p'(1).$$

As in Theorem 3.4, part 2 follows from Theorem 2.1.

The classes of functions defined below were introduced first in [Le1, Definition 1.1] (see also [Le3, p. 21]).

DEFINITION 3.6. For $\lambda \in (0, \infty]$ let

$$\mathcal{B}_1(\lambda) = \{\omega \in \mathcal{B}_1 : \omega'_\angle(1) = \lambda\}, \qquad \mathcal{P}_0(\lambda) = \{p \in \mathcal{P}_0 : p'_\angle(1) = -\lambda/2\}.$$

Clearly,

$$\mathcal{B}_1 = \bigcup_{\lambda \in (0,\infty]} \mathcal{B}_1(\lambda), \qquad \mathcal{P}_0 = \bigcup_{\lambda \in (0,\infty]} \mathcal{P}_0(\lambda).$$

PROPOSITION 3.7. *Let* $a \in \mathbb{H}$ *and* $\lambda \in (0, \infty]$.

1. *If* $\omega \in \mathcal{B}_1(\lambda)$, *then* $p = h_a \circ \omega \in \mathcal{P}_0(\lambda|a|^2/\mathrm{Re}\, a)$.
2. *If* $p \in \mathcal{P}_0(\lambda)$, *then* $\omega = h_1^{-1} \circ p \in \mathcal{B}_1(\lambda\, \mathrm{Re}\, a/|a|^2)$.

In particular, for $a = 1$:

3. *If* $\omega \in \mathcal{B}_1(\lambda)$, *then* $p = h_1 \circ \omega \in \mathcal{P}_0(\lambda)$.
4. *If* $p \in \mathcal{P}_0(\lambda)$, *then* $\omega = h_1^{-1} \circ p \in \mathcal{B}_1(\lambda)$.

Observe that from Theorems 3.4 and 3.5 we have

THEOREM 3.8. *Let* $\lambda \in (0, \infty)$.

1. *If* $\omega \in \mathcal{B}_1(\lambda)$, *then, for every* $k > 0$,

$$\omega(\mathbb{O}_k) \subset \mathbb{O}_{\lambda k}. \tag{3.4}$$

2. *If* $\omega \in \mathcal{P}_0(\lambda)$, *then, for every* $k > 0$,

$$p(\mathbb{O}_k) \subset \mathbb{D}(\lambda k/2, \lambda k/2). \tag{3.5}$$

3.2. Julia functions and Schwarz functions. Now we describe some relations for functions which satisfy the Schwarz and the Julia lemmas. First we distinguish some sets of functions in $\mathcal{B}$.

DEFINITION 3.9. Let

$$\mathcal{B}_0 = \{\omega \in \mathcal{B} : \omega(0) = 0\}.$$

Functions in the class $\mathcal{B}_0$ are called *Schwarz functions*. For $n \in \mathbb{N}$ let

$$\mathcal{B}_0^{(n)} = \{\omega \in \mathcal{B}_0 : \omega'(0) = \cdots = \omega^{(n-1)}(0) = 0,\ \omega^{(n)}(0) \neq 0\}.$$

Let

$$\mathcal{B}_{0,1} = \mathcal{B}_0 \cap \mathcal{B}_1.$$

For each $n \in \mathbb{N}$ let

$$\mathcal{B}_{0,1}^{(n)} = \mathcal{B}_0^{(n)} \cap \mathcal{B}_1.$$

Let us recall the Schwarz lemma (see e.g. [Go, p. 87]).

LEMMA 3.10. *Let $n \in \mathbb{N}$. If $\omega \in \mathcal{B}_0^{(n)}$, then*

$$|\omega(z)| \le |z|^n, \quad z \in \mathbb{D}, \tag{3.6}$$

and

$$\frac{1}{n!}|\omega^{(n)}(0)| \le 1. \tag{3.7}$$

Equality in (3.6) *for some $z \neq 0$ or in* (3.7) *can occur only for $\omega(z) = \kappa z^n$, $z \in \mathbb{D}$, where $\kappa \in \mathbb{T}$.*

The following lemma can be found in [P2, p. 84].

LEMMA 3.11. *If $\omega \in \mathcal{B}$, then ω has at most one attractive fixed point in $\overline{\mathbb{D}}$, i.e. a point $\xi \in \overline{\mathbb{D}}$ such that*

- *$\omega(\xi) = \xi$ and $|\omega'(\xi)| < 1$ when $\xi \in \mathbb{D}$,*
- *$\omega_\angle(\xi) = \xi$ and $|\omega'_\angle(\xi)| < 1$ when $\xi \in \mathbb{T}$.*

The above lemma implies at once

LEMMA 3.12. *If $\omega \in \mathcal{B}_1(\lambda)$, $\lambda \in (0,1)$, then ω has no fixed point in $\mathbb{D}$.*

Lemma 3.11 also yields the following result for Schwarz functions in the class $\mathcal{B}_1$.

LEMMA 3.13. *If $\omega \in \mathcal{B}_{0,1}$, then $\omega \in \mathcal{B}_1(\lambda)$ for some $\lambda \ge 1$.*

The last result can be improved:

LEMMA 3.14. *Let $n \in \mathbb{N}$. If $\omega \in \mathcal{B}_{0,1}^{(n)}$, then*

$$n \le \omega'_\angle(1) \le \infty, \tag{3.8}$$

and $\omega \in \mathcal{B}_1(\lambda)$ for some $\lambda \ge n$.

Proof. Let $\omega \in \mathcal{B}_{0,1}^{(n)}$. Since $\omega \in \mathcal{B}_1$, by the Julia lemma $\omega \in \mathcal{B}_1(\lambda)$ for some $\lambda \in (0,\infty]$, where $\lambda = \omega'_\angle(1)$. Lemma 3.13 shows that $\lambda \ge 1$. When $\lambda = \infty$, the assertion of the lemma is obvious.

Assume that $\lambda < \infty$. Since in view of (3.6) we have

$$|\omega(r)| \le r^n, \quad r \in [0,1),$$

it follows that

$$\frac{1-|\omega(r)|}{1-r} \ge \frac{1-r^n}{1-r}.$$

Hence and by part 1 of Theorem 3.4 we obtain

$$\lambda = \omega'_\angle(1) = \lim_{r\to 1^-} \frac{1-|\omega(r)|}{1-r} \ge \lim_{r\to 1^-} \frac{1-r^n}{1-r} = n. \ \blacksquare$$

EXAMPLE 3.15. For $n \in \mathbb{N}$ let $\omega(z) = z^n$, $z \in \mathbb{D}$. Clearly, $\omega \in \mathcal{B}_{0,1}^{(n)}$. Moreover $\omega'_\angle(1) = \omega'(1) = n$. Thus $\omega \in \mathcal{B}(\lambda)$ with $\lambda = n$.

The next lemma follows directly from the Julia lemma. Under the stronger assumptions on f, namely, that f is continuous on $\overline{\mathbb{D}}_r$ and analytic in $\mathbb{D}_r \cup \{z_0\}$, it is known as the Jack lemma or the Clunie–Jack lemma (see e.g. [MM, Lemma 2.2a]). The Jack lemma is the fundamental result in the theory of differential subordinations (for a survey of this theory see [MM]).

LEMMA 3.16. *Let $n \in \mathbb{N}$. Let f be analytic in the disk $\mathbb{D}_r$, $r > 0$, with*

$$f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0, \quad f^{(n)}(0) \neq 0. \tag{3.9}$$

Assume that at $z_0 \in \partial\mathbb{D}_r$ the limit $f_\angle(z_0)$ exists and is finite with

$$|f_\angle(z_0)| = \sup\{|f(z)| : z \in \mathbb{D}_r\}. \tag{3.10}$$

Then $f'_\angle(z_0)$ exists and

$$\frac{z_0 f'_\angle(z_0)}{f_\angle(z_0)} = \lambda, \tag{3.11}$$

where

$$n \le \lambda \le \infty. \tag{3.12}$$

Proof. Since, in view of (3.9), $f \not\equiv 0$, (3.10) yields $f_\angle(z_0) \neq 0$. Thus the function

$$\mathbb{D} \ni z \mapsto \omega(z) = \frac{f(z_0 z)}{f_\angle(z_0)} \tag{3.13}$$

is well defined and analytic in $\mathbb{D}$. Moreover, the limit $\omega_\angle(1)$ exists and

$$\omega_\angle(1) = 1. \tag{3.14}$$

Observe that (3.10) yields

$$|\omega_\angle(1)| = \sup\{|\omega(z)| : z \in \mathbb{D}\} = 1.$$

Consequently, $\omega(\mathbb{D}) \subset \mathbb{D}$, i.e. $\omega \in \mathcal{B}$. Hence, taking into account (3.9) and (3.14), we deduce that $\omega \in \mathcal{B}^{(n)}_{0,1}$. Thus Lemma 3.14 shows that $\omega \in \mathcal{B}_1(\lambda)$ for $n \le \lambda = \omega'_\angle(1) \le \infty$. Hence and from (3.13) it follows that $f'_\angle(z_0)$ exists and

$$n \le \lambda = \omega'_\angle(1) = \frac{z_0 f'_\angle(z_0)}{f_\angle(z_0)} \le \infty. \ \blacksquare$$

REMARK 3.17. Assuming that f is continuous on $\overline{\mathbb{D}}_r$ and $f'(z_0)$ exists, the above lemma reduces to Lemma 2.2a, part (i), of [MM]. Then (3.11) has the form

$$\frac{z_0 f'(z_0)}{f(z_0)} = \lambda, \tag{3.15}$$

where

$$n \le \lambda < \infty. \tag{3.16}$$

THEOREM 3.18. *Let $q \in \mathcal{S}$ and assume that $q'(\zeta_0) \neq 0$ at $\zeta_0 \in \partial\mathbb{D}$ exists. Let $n \in \mathbb{N}$ and p be a function analytic in the disk $\mathbb{D}_r$, $r > 0$, with $p \not\equiv p(0)$,*

$$p'(0) = \cdots = p^{(n-1)}(0) = 0, \quad p^{(n)}(0) \neq 0,$$

such that $p(0) = q(0)$ and

$$p(\mathbb{D}_r) \subset q(\mathbb{D}).$$

Assume that at $z_0 \in \partial\mathbb{D}_r$ the limit

$$p(z_0) = \lim_{z \to z_0} p(z) \tag{3.17}$$

exists and is finite, and

$$p(z_0) = q(\zeta_0). \tag{3.18}$$

If $p'_\angle(z_0)$ exists and is finite, then

$$z_0 p'_\angle(z_0) = \lambda \zeta_0 q'(\zeta_0), \tag{3.19}$$

with λ satisfying (3.16).

If $p'(z_0)$ exists, then

$$z_0 p'(z_0) = \lambda \zeta_0 q'(\zeta_0), \tag{3.20}$$

with λ satisfying (3.16).

Proof. From the univalence of q it follows that the function $f = q^{-1} \circ p$ is well defined and analytic in $\mathbb{D}_r$ with

$$f(\mathbb{D}_r) \subset \mathbb{D}. \tag{3.21}$$

Moreover $f(0) = q^{-1} \circ p(0) = q^{-1}(q(0)) = 0$ and

$$f^{(k)}(0) = p^{(k)}(0) = 0, \quad k = 1, \ldots, n-1, \quad f^{(n)}(0) \neq 0. \tag{3.22}$$

Since, in view of (3.18),

$$f(z_0) = q^{-1} \circ p(z_0) = q^{-1}(\zeta_0) = \zeta_0 \in \partial\mathbb{D}, \tag{3.23}$$

from (3.21) we see that $|f(z_0)| = \sup\{|f(z)| : z \in \mathbb{D}_r\} = 1$. Hence taking account (3.22) and Lemma 3.16 we deduce that $f'_\angle(z_0)$ exists and

$$\frac{z_0 f'_\angle(z_0)}{f(z_0)} = \lambda \tag{3.24}$$

with $n \leq \lambda \leq \infty$. Since $p'(z) = q'(f(z))f'(z)$ for $z \in \mathbb{D}_r$, and $q'(f(z_0)) = q'(\zeta_0) \neq 0$, from the assumption that $p'_\angle(z_0)$ is finite it follows that $f'_\angle(z_0)$ is finite. Hence and from (3.23) and (3.24) we have

$$z_0 p'_\angle(z_0) = \frac{z_0 f'_\angle(z_0)}{f(z_0)} f(z_0) q'(f(z_0)) = \lambda \zeta_0 q'(\zeta_0).$$

In this way we proved (3.19) and, consequently, (3.20). ∎

4. Boundary subordination

4.1. Basic properties. Now we introduce the notion of boundary subordination of two analytic functions. This concept appeared first in [Le3, p. 182].

Definition 4.1. Let $f, F \in \mathcal{A}$ and suppose that $f_\angle(1)$ and $F_\angle(1)$ exist with

$$f_\angle(1) = F_\angle(1). \tag{4.1}$$

- We say that f is *boundary subordinated* to F if there exists $\omega \in \mathcal{B}_1$ such that $f = F \circ \omega$ in $\mathbb{D}$. We then write $f \preccurlyeq F$.
- Let $\lambda \in (0, \infty]$. We say that f is λ-*boundary subordinated* to F if there exists $\omega \in \mathcal{B}_1(\lambda)$ such that $f = F \circ \omega$ in $\mathbb{D}$. We then write $f \preccurlyeq_\lambda F$.

As an immediate consequence of Definition 4.1 we have

Theorem 4.2. *Let $f, F \in \mathcal{A}$ satisfy* (4.1). *If $f \preccurlyeq F$, then $f(\mathbb{D}) \subset F(\mathbb{D})$.*

Theorem 4.3. *Let $\lambda \in (0, \infty)$ and f, $F \in \mathcal{A}$ satisfy* (4.1). *If $f \preccurlyeq_\lambda F$, then*

$$f(\mathbb{O}_k) \subset F(\mathbb{O}_{\lambda k}) \quad \text{for every } k > 0.$$

Proof. Since $f = F \circ \omega$ for some $\omega \in \mathcal{B}_1(\lambda)$, it suffices to apply Theorem 3.8. ∎

Observe that Proposition 3.7 implies

Proposition 4.4. *$p \in \mathcal{P}_0$ if and only if*

$$p(z) \preccurlyeq \frac{1-z}{1+z}, \quad z \in \mathbb{D}.$$

4.2. Boundary subordination of univalent functions. If F is univalent we can obtain additional properties of boundary subordination.

Theorem 4.5. *Let $\lambda \in (0, 1)$, $f \in \mathcal{A}$ and $F \in \mathcal{S}$. If $f \preccurlyeq_\lambda F$, then $f(z) \neq F(z)$ for every $z \in \mathbb{D}$.*

Proof. Suppose that, on the contrary, $f(\xi) = F(\xi)$ for some $\xi \in \mathbb{D}$. Since $f = F \circ \omega$ for some $\omega \in \mathcal{B}_1(\lambda)$, $\lambda \in (0, 1)$, we have

$$F(\xi) = f(\xi) = F(\omega(\xi)),$$

so $\omega(\xi) = \xi$ by the univalence of F in $\mathbb{D}$. This contradicts Lemma 3.12. ∎

Remark 4.6. Assume now that F is normal, the angular limit $f_\angle(1)$ exists and $f = F \circ \omega$ for some $\omega \in \mathcal{B}_1(\lambda)$, $\lambda \in (0, \infty]$. Let $I = [0, 1)$. Since ω has radial limit 1 at 1, the curve $\omega(I)$ ends at 1. As $f(I) = F(\omega(I))$, the curve $F(\omega(I))$ ends at $f_\angle(1)$, which means that F has asymptotic value $f_\angle(1)$ along $\omega(I)$. Since F is normal, applying [P1, Theorem 9.3], we deduce that $F_\angle(1)$ exists and (4.1) holds. Thus, if F is normal, then in Definition 4.1, instead of (4.1), it is enough to assume that the angular limit of f at 1 exists. Since every univalent function in $\mathbb{D}$ is normal, this is true when F is univalent.

THEOREM 4.7. *Let $f \in \mathcal{A}$ and $F \in \mathcal{S}$ satisfy* (4.1). *If $F(\mathbb{D})$ is a Jordan domain and $f(\mathbb{D}) \subset F(\mathbb{D})$, then $f \preccurlyeq_\lambda F$ for some $\lambda \in (0, \infty]$.*

Proof. By the univalence of F and by the fact that $f(\mathbb{D}) \subset F(\mathbb{D})$, we see that $\omega = F^{-1} \circ f$ is well defined in $\mathbb{D}$ and $\omega \in \mathcal{B}$.

Let $I = [0, 1)$ and $\Gamma = f(I)$. Since $f_\angle(1)$ exists, Γ is a curve in $f(\mathbb{D})$ and hence in $F(\mathbb{D})$ ending at $f_\angle(1)$. Thus, by [P2, Proposition 2.14], $\omega(I) = F^{-1}(\Gamma)$ is a curve in $\mathbb{D}$ ending at some $\zeta_0 \in \mathbb{T}$. Since F has a homeomorphic extension on $\overline{\mathbb{D}}$, we see from (4.1) that $\zeta_0 = 1$. Consequently, ω has a radial limit, and, by the Lehto–Virtanen Theorem [P2, p. 71], an angular limit 1 at 1. Thus $\omega \in \mathcal{B}_1$. By Theorem 3.4 the angular derivative $\omega'_\angle(1)$ exists with $0 < \omega'_\angle(1) = \Lambda(\omega) \le +\infty$. This implies that $\omega \in \mathcal{B}_1(\lambda)$ with $\lambda = \Lambda(\omega)$. ∎

REMARK. The inclusion $f(\mathbb{D}) \subset F(\mathbb{D})$, where f and F are univalent in $\mathbb{D}$ and satisfy (4.1), is not sufficient for $f \preccurlyeq_\lambda F$ with a finite λ. As an example, take $f(z) = z$ and $F(z) = 4z/(1 + z)^2$ for $z \in \mathbb{D}$. Clearly, $f(\mathbb{D}) \subset F(\mathbb{D})$ and $f_\angle(1) = F_\angle(1) = 1$. But $\omega'_\angle(1) = \infty$, where $\omega = F^{-1} \circ f$. This happens when $F'_\angle(1) = 0$ and $f'_\angle(1) \neq 0$.

The theorem below is due to Carathéodory and Lelong-Ferrand (see [P1, pp. 307–308]).

THEOREM 4.8. *Let f, $F \in \mathcal{A}$. Let F' be normal in $\mathbb{D}$ and let $f \preccurlyeq_\lambda F$ for some $\lambda \in (0, \infty)$.*

- *If $f'_\angle(1)$ exists and is finite, then $F'_\angle(1)$ exists and is finite.*
- *If $f'_\angle(1) = 0$, then $F'_\angle(1) = 0$.*

Proof. We have $f = F \circ \omega$ for some $\omega \in \mathcal{B}_1(\lambda)$ with $\lambda = \omega'_\angle(1) \in (0, \infty)$. Since $f'_\angle(1)$ is finite and $f'(z) = F'(\omega(z))\omega'(z)$ for $z \in \mathbb{D}$, we deduce that the limit

$$\lim_{r \to 1^-} F'(\omega(r)) = \lim_{r \to 1^-} \frac{f'(r)}{\omega'(r)} = \frac{f'_\angle(1)}{\omega'_\angle(1)}$$

exists and is finite. Hence F' has asymptotic value $f'_\angle(1)/\omega'_\angle(1)$ along the curve $\omega([0, 1))$, which is also its angular limit at 1 as F' is a normal function.

Moreover, $f'_\angle(1) = 0$ implies that $F'_\angle(1) = 0$. ∎

4.3. Applications. Now we apply the notion of boundary subordination to three classes of univalent functions.

4.3.1. *Functions starlike with respect to a boundary point.* The class of functions starlike with respect to a boundary point, denoted here by $\mathcal{S}_0^*$, was introduced by Robertson [R2]. He proposed an analytic formula for the class

$\mathcal{S}_0^*$ and proved it partially. Lyzzaik [Ly] completed the proof. The analytic characterization of the class $\mathcal{S}_0^*$ alternative to Robertson's formula was given in [Le1] and proved there partially. The proof was completed in [LL].

DEFINITION 4.9. A simply connected domain $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$, with $0 \in \partial\Omega$ is called *starlike with respect to the boundary point* (at the origin) if for every $w \in \Omega$,

$$(0, w] = \{tw : t \in (0,1]\} \subset \Omega.$$

The class of all such domains will be denoted by $\mathcal{Z}^*$.

Let $\mathcal{S}_0^* \subset \mathcal{S}$ be the class of all functions f such that $0 \in \partial f(\mathbb{D})$ and $f(\mathbb{D}) \in \mathcal{Z}_0^*$. Functions belonging to $\mathcal{S}_0^*$ will be called *starlike with respect to the boundary point.*

Using the notion of boundary subordination, the characterization of the class $\mathcal{S}_0^*$ proved in [Le1] and [LL] (see also [Le3, Chapter VII]) can be written as follows.

THEOREM 4.10. *Let $f \in \mathcal{A}$. Then the following conditions are equivalent:*

(1) *$f \in \mathcal{S}_0^*$ and $f_\angle(1) = 0$.*
(2) *There exists $\lambda \in (0,1]$ such that*

$$-(1-z)^2 \frac{f'(z)}{f(z)} \preccurlyeq_\lambda 4\frac{1-z}{1+z}, \quad z \in \mathbb{D}. \tag{4.2}$$

Multivalent starlike functions with respect to a boundary point were considered in [ESZ1], where the main result can be formulated now as follows.

THEOREM 4.11. *Let $\lambda \in (0, \infty)$. If $f \in \mathcal{A}$ satisfies* (4.2)*, then*

(1) $f(\mathbb{D}) \in \mathcal{Z}^*$;
(2) *f is p-valent function if and only if $p - 1 < \lambda \leq p$.*

4.3.2. *Functions convex in the positive direction of the real axis.* The class of functions convex in the direction of the imaginary axis was introduced by Robertson [R1]. The classes of functions convex in the positive (negative) direction of the imaginary (real) axis as the subclasses of Robertson's class were distinguished by Hengartner and Schober [HS] who proposed analytic formulas without detailed proofs. These were given by Ciozda [Ci1]–[Ci3].

Alternative proofs of an analytic characterization of functions convex in the positive (resp. negative) direction of the imaginary (resp. real) axis were found in [Le2], [Le3, Chapter VI], [Le4] and [ES].

DEFINITION 4.12. A simply connected domain $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$, is called *convex in the positive direction of the real axis* if

$$\{w + t : t \geq 0\} \subset \Omega \quad \text{for every } w \in \Omega.$$

Functions $f \in \mathcal{S}$ mapping $\mathbb{D}$ onto such domains are also called *convex in the positive direction of the real axis.*

Let $\mathcal{CR}^+ \subset \mathcal{S}$ denote the class of all those functions normalized by

$$f(0) = 0, \quad \lim_{t\to\infty} f^{-1}(f(z) + t) = 1, \quad z \in \mathbb{D}. \tag{4.3}$$

Using the notion of boundary subordination we can rewrite the analytic characterization of the class $\mathcal{CR}^+$ basing on results due to [ES] and [BL] as follows.

THEOREM 4.13. *Let $f \in \mathcal{A}$. Then the following conditions are equivalent:*

(1) $f \in \mathcal{CR}^+$.

(2) *There exists $\lambda \in (0, \infty]$ such that*

$$(1-z)^2 f'(z) \preccurlyeq_\lambda 4\frac{1-z}{1+z}, \quad z \in \mathbb{D}. \tag{4.4}$$

Elin and Shoikhet [ES] studied the problem of finding the horizontal strip of minimal width containing $f(\mathbb{D})$ and of maximal width lying in $f(\mathbb{D})$ when $f \in \mathcal{CR}^+$. They introduced the following subclasses in $\mathcal{CR}^+$.

DEFINITION 4.14. Let $0 < \nu \leq \lambda \leq \infty$, with $\nu < \lambda$ when $\lambda = \infty$. A function $f \in \mathcal{CR}^+$ belongs to $\mathcal{CR}^+(\lambda, \nu)$ if

- $f(\mathbb{D})$ lies in a horizontal strip of minimal width $2\lambda\pi$;
- $f(\mathbb{D})$ contains a horizontal strip of maximal width $2\nu\pi$.

Some properties of functions in $\mathcal{CR}^+(\lambda, \nu)$ were demonstrated in [ES]. An analytical characterization of $\mathcal{CR}^+(\lambda, \nu)$ was given in [BL]. We rewrite Theorem 2 of [BL] using boundary subordination.

THEOREM 4.15. *Let $0 < \nu \leq \lambda < \infty$ and let $f \in \mathcal{A}$. Then $f \in \mathcal{CR}^+(\lambda, \nu)$ if and only if* (4.4) *holds and there exists $\zeta = e^{i\phi} \neq 1$ such that*

$$\angle \lim_{z\to\zeta} \frac{1-\overline{\zeta}z}{1+\omega(z)} = \nu \sin^2 \frac{\phi}{2}.$$

REMARK. The class $\mathcal{CR}^+(\lambda, \nu)$ is related to the class of functions starlike with respect to a boundary point with the image containing a wedge of angle of size $2\nu\pi$ and contained in a wedge of angle of size $2\lambda\pi$. The details can be found e.g. in [ES] and [ESZ2].

4.3.3. *Functions spirallike with respect to a boundary point.* The class of spirallike functions with respect to a boundary point introduced in [AES] is another example of a class of univalent functions which can be analytically characterized in terms of boundary subordination. For details see e.g. [ES], [ESZ1], [ESZ2], [Le3, Chapter VIII], [Le5, Theorems 3.4–3.5].

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