Real hypersurfaces with parallel induced almost contact structures

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Abstract. Real affine hypersurfaces of the complex space $C^{n+1}$ with a $J$-tangent transversal vector field and an induced almost contact structure $(\varphi, \xi, \eta)$ are studied. Some properties of hypersurfaces with $\varphi$ or $\eta$ parallel relative to an induced connection are proved. Also a local characterization of these hypersurfaces is given.

1. Introduction. We study real affine hypersurfaces of the complex space $C^{n+1}$ with a $J$-tangent transversal vector field $C$ and an induced almost contact structure $(\varphi, \xi, \eta)$. The main purpose of this paper is to investigate some properties of hypersurfaces with $\nabla \varphi = 0$ or $\nabla \eta = 0$, where $\nabla$ is an affine connection induced by a transversal vector field $C$.

In Section 2 we briefly recall basic formulas of affine differential geometry, we introduce the notion of a $J$-tangent transversal vector field and give a lemma relating to differential equations required in the next sections.

In Section 3 we recall some results obtained in [SS] for an induced almost contact structure and show how induced almost contact structures are related to each other in case the $J$-tangent transversal vector field changes.

Section 4 contains the main results of this paper. In particular, we prove some properties of induced objects under the condition $\nabla \varphi = 0$ as well as $\nabla \eta = 0$. Moreover, we prove that the existence of a $J$-tangent transversal vector field $\varphi$ with $\nabla \varphi = 0$ is equivalent to the existence of a $J$-tangent transversal vector field $\eta$ with $\nabla \eta = 0$. At the end we give a local characterization of such hypersurfaces.

Throughout the paper we write $\alpha \equiv 0$ if $\alpha(x) = 0$ for all $x \in M$, and $\alpha \neq 0$ if $\alpha(x) \neq 0$ for every $x \in M$ (i.e. $\alpha$ is a nowhere vanishing function on $M$).

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2. Preliminaries. We briefly recall the basic formulas of affine differential geometry. For more details, we refer to [NS]. Let \( f : M \to \mathbb{R}^{n+1} \) be an orientable, connected differentiable \( n \)-dimensional hypersurface immersed in the affine space \( \mathbb{R}^{n+1} \) equipped with its usual flat connection \( D \). Then for any transversal vector field \( C \) we have

\[
D_X f_* Y = f_*(\nabla_X Y) + h(X, Y)C, \\
D_X C = -f_*(S X) + \tau(X)C,
\]

(2.1) (2.2)

where \( X, Y \) are vector fields tangent to \( M \). For any transversal vector field, \( \nabla \) is a torsion-free connection, \( h \) is a symmetric bilinear form on \( M \), called the second fundamental form, \( S \) is a tensor of type \((1,1)\), called the shape operator, and \( \tau \) is a 1-form, called the transversal connection form.

We shall now consider the change of a transversal vector field for a given immersion \( f \).

**Theorem 2.1** ([NS]). Suppose we change a transversal vector field \( C \) to

\[
\bar{C} = \Phi C + f_*(Z),
\]

where \( Z \) is a tangent vector field on \( M \) and \( \Phi \) is a nowhere vanishing function on \( M \). Then the affine fundamental form, the induced connection, the transversal connection form, and the affine shape operator change as follows:

\[
\bar{h} = \frac{1}{\Phi} h, \\
\bar{\nabla}_X Y = \nabla_X Y - \frac{1}{\Phi} h(X, Y)Z, \\
\bar{\tau} = \tau + \frac{1}{\Phi} h(Z, \cdot) + d \ln |\Phi|, \\
\bar{S} = \Phi S - \nabla.Z + \bar{\tau}(\cdot)Z.
\]

If \( h \) is non-degenerate, then we say that the hypersurface or the hypersurface immersion is non-degenerate. We have the following

**Theorem 2.2** ([NS], §II.2, Theorem 2.1). For an arbitrary transversal vector field \( C \) the induced connection \( \nabla \), the second fundamental form \( h \), the shape operator \( S \), and the 1-form \( \tau \) satisfy the following equations:

\[
R(X,Y)Z = h(Y,Z)SX - h(X,Z)SY, \\
(\nabla_X h)(Y, Z) + \tau(X)h(Y, Z) = (\nabla_Y h)(X, Z) + \tau(Y)h(X, Z), \\
(\nabla_X S)(Y) - \tau(X)SY = (\nabla_Y S)(X) - \tau(Y)SX, \\
h(X, SY) - h(SX, Y) = 2d\tau(X, Y).
\]

Equations (2.3), (2.4), (2.5), and (2.6) are called, respectively, the equation of Gauss, Codazzi for \( h \), Codazzi for \( S \) and Ricci.

For a hypersurface immersion \( f : M \to \mathbb{R}^{n+1} \) a transversal vector field \( C \) is said to be equiaffine (resp. locally equiaffine) if \( \tau = 0 \) (resp. \( d\tau = 0 \)).
Let $\dim M = 2n + 1$ and $f: (M, g) \to (\mathbb{R}^{2n+2}, \tilde{g})$ be a non-degenerate (relative to the second fundamental form) isometric immersion, where $\tilde{g}$ is the standard inner product on $\mathbb{R}^{2n+2}$. We assume that $\mathbb{R}^{2n+2} \cong \mathbb{C}^{n+1}$ is endowed with the standard complex structure $J$,

$$J(x_1, \ldots, x_{n+1}, y_1, \ldots, y_{n+1}) = (-y_1, \ldots, -y_{n+1}, x_1, \ldots, x_{n+1}).$$

Let $C$ be a transversal vector field on $M$. We say that $C$ is $J$-tangent if $JC_x \in f_*(T_xM)$ for every $x \in M$. We also define a distribution $\mathcal{D}$ on $M$ as the biggest $J$-invariant distribution on $M$, that is,

$$\mathcal{D}_x = f_*^{-1}(f_*(T_xM) \cap J(f_*(T_xM)))$$

for every $x \in M$. It is clear that $\dim \mathcal{D} = 2n$. A vector field $X$ is called a $\mathcal{D}$-field if $X_x \in \mathcal{D}_x$ for every $x \in M$. We use the notation $X \in \mathcal{D}$ for vectors as well as for $\mathcal{D}$-fields. We say that the distribution $\mathcal{D}$ is non-degenerate if $h$ is non-degenerate on $\mathcal{D}$. To simplify the writing, we will omit $f_*$ in front of vector fields in most cases.

We conclude this section with the following useful lemma relating to differential equations (we also give the proof for completeness):

**Lemma 2.3 ([S]).** Let $F: I \to \mathbb{R}^{2n}$ be a smooth function on the interval $I$ and let $\alpha, \beta \in C^\infty(I, \mathbb{R})$ be such that $\alpha^2 + \beta^2 \neq 0$ on $I$. If $F$ satisfies the differential equation

$$F'(y) = -\alpha(y)JF(y) + \beta(y)F(y),$$

then $F$ is of the form

$$F(y) = Je^{\hat{\beta}(y)} \cos(\hat{\alpha}(y)) + ve^{\hat{\beta}(y)} \sin(\hat{\alpha}(y)),$$

where $v \in \mathbb{R}^{2n}$ and $\hat{\alpha}, \hat{\beta}$ are any integrals of $\alpha$ and $\beta$ on $I$, respectively.

**Proof.** It is easily seen that functions of the form (2.8) satisfy the differential equation (2.7). On the other hand, since (2.7) is a first order ordinary differential equation, the Picard–Lindelöf theorem implies that any solution of (2.7) must be of the form (2.8). \qed

**3. Almost contact structures.** A $(2n + 1)$-dimensional manifold $M$ is said to have an almost contact structure if there exist on $M$ a tensor field $\varphi$ of type $(1, 1)$, a vector field $\xi$ and a 1-form $\eta$ which satisfy 

$$\varphi^2(X) = -X + \eta(X)\xi, \quad \eta(\xi) = 1$$

for every $X \in TM$.

Let $f: M \to \mathbb{R}^{2n+2}$ be a hypersurface with a $J$-tangent transversal vector field $C$. Then we can define a vector field $\xi$, a 1-form $\eta$ and a tensor field $\varphi$ of type $(1, 1)$ as follows:

$$\xi := JC, \quad \eta|_\mathcal{D} = 0 \quad \text{and} \quad \eta(\xi) = 1, \quad \varphi|_\mathcal{D} = J|_\mathcal{D} \quad \text{and} \quad \varphi(\xi) = 0.$$
It is easy to see that \((\varphi, \xi, \eta)\) is an almost contact structure on \(M\); it is said to be induced by \(C\).

For an induced almost contact structure we have the following theorem:

**Theorem 3.1 ([SS]).** If \((\varphi, \xi, \eta)\) is an induced almost contact structure on \(M\) then

\[
\eta(\nabla_X Y) = -h(X, \varphi Y) + X(\eta(Y)) + \eta(Y)\tau(X),
\]

\[
\varphi(\nabla_X Y) = \nabla_X \varphi Y + \eta(Y)SX - h(X, Y)\xi,
\]

\[
\eta([X, Y]) = -h(X, \varphi Y) + h(Y, \varphi X) + X(\eta(Y)) - Y(\eta(X)) + \eta(Y)\tau(X) - \eta(X)\tau(Y),
\]

\[
\varphi([X, Y]) = \nabla_X \varphi Y - \nabla_Y \varphi X - \eta(X)SY + \eta(Y)SX,
\]

\[
\eta(\nabla_X \xi) = \tau(X),
\]

\[
\eta(SX) = h(X, \xi),
\]

for all \(X, Y \in \mathcal{X}(M)\).

**Lemma 3.2.** Let \(C\) be a \(J\)-tangent transversal vector field. Then any other \(J\)-tangent transversal vector field \(\bar{C}\) has the form

\[
\bar{C} = \phi C + f_\ast Z,
\]

where \(\phi \neq 0\) and \(Z \in \mathcal{D}\). Moreover, if \((\varphi, \xi, \eta)\) is the almost contact structure induced by \(C\), then \(\bar{C}\) induces the almost contact structure \((\bar{\varphi}, \bar{\xi}, \bar{\eta})\), where

\[
\bar{\xi} = \phi \xi + \varphi Z, \quad \bar{\eta} = \frac{1}{\phi} \eta, \quad \bar{\varphi} = \varphi + \eta(\cdot)\frac{1}{\phi} Z.
\]

**Proof.** Since \(Z \in \mathcal{D}\) and \(J = \varphi\) on \(\mathcal{D}\), we have

\[
\bar{\xi} = J\bar{C} = J(\phi C + f_\ast Z) = \phi JC + \varphi Z = \phi \xi + \varphi Z.
\]

Directly from the definition of \(\eta\) and \(\bar{\eta}\) we get \(\eta = \bar{\eta}\) on \(\mathcal{D}\) and

\[
\eta(\xi) = 1 = \bar{\eta}(\bar{\xi}) = \bar{\eta}(\phi \xi + \varphi Z) = \phi \bar{\eta}(\xi),
\]

thus

\[
\bar{\eta}(\xi) = \frac{1}{\phi} \eta(\xi),
\]

and finally \(\bar{\eta} = \frac{1}{\phi} \eta\). To prove the last equality of the statement, note that

\[
0 = \varphi(\xi) = \bar{\varphi}(\bar{\xi}) = \bar{\varphi}(\phi \xi + \varphi Z) = \phi \bar{\varphi}(\xi) + \varphi(\varphi Z).
\]

From the definition of \(\varphi\) and \(\bar{\varphi}\) we have \(\varphi = \bar{\varphi}\) on \(\mathcal{D}\), which implies that

\[
\bar{\varphi}(\xi) = \frac{1}{\phi} Z = \varphi(\xi) + \eta(\xi)\frac{1}{\phi} Z,
\]

since \(Z \in \mathcal{D}\). The last formula proves that

\[
\bar{\varphi}(X) = \varphi(X) + \eta(X)\frac{1}{\phi} Z.
\]
is valid for $X = \xi$. Clearly, it is also valid for every $X \in \mathcal{D}$, and thus for every $X \in TM$. ■

4. Parallel induced almost contact structures. In this section we always assume that $(\varphi, \xi, \eta)$ is an almost contact structure induced by a $J$-tangent transversal vector field $C$. It is important to note that we do not assume that the second fundamental form $h$ is non-degenerate.

**Lemma 4.1.** Let $(\varphi, \xi, \eta)$ be an induced almost contact structure such that $\nabla \varphi = 0$. Then

4.1. $h|_{\mathcal{D} \times \mathcal{D}} = 0$,
4.2. $h(\xi, X) = h(X, \xi) = 0$ for all $X \in \mathcal{D}$,
4.3. $S|_{\mathcal{D}} = 0$,
4.4. $S\xi = h(\xi, \xi)\xi$,
4.5. $d\tau = 0$.

**Proof.** From formula (3.2) we have

$$(\nabla_X \varphi)(Y) = -\eta(Y)SX + h(X, Y)\xi$$

for all $X, Y \in \mathcal{X}(M)$. Since $\nabla \varphi = 0$ we get $h(X, Y) = 0$ and $h(\xi, Y) = 0$ for all $X, Y \in \mathcal{D}$. Now, taking $X \in \mathcal{D}$ and $Y = \xi$ we have $SX = 0$. Taking $X = Y = \xi$ we easily get $S\xi = h(\xi, \xi)\xi$. The last equation follows immediately from the Ricci equation (2.6). ■

The above lemma implies that if $\nabla \varphi = 0$, then $C$ is a locally equiaffine transversal vector field, so locally we can find a nowhere vanishing function $\Phi$ such that $\bar{C} = \Phi C$ is an equiaffine $J$-tangent vector field. Now, using Theorem 2.1 and Lemma 3.2 we get the following corollary:

**Corollary 4.2.** Let $C$ be a $J$-tangent transversal vector field such that $\nabla \varphi = 0$ and let $\Phi$ be a nowhere vanishing function on $M$. Denote by $\bar{C}$ the transversal vector field $\Phi C$. Then $\bar{\nabla} \bar{\varphi} = 0$. Thus, parallelism of $\varphi$ relative to $\bar{\nabla}$ is the direction property. In particular, locally we can always choose $C$ equiaffine.

We shall prove

**Lemma 4.3.** Let $(\varphi, \xi, \eta)$ be an induced almost contact structure such that $\nabla \eta = 0$. Then

4.6. $h|_{\mathcal{D} \times \mathcal{D}} = 0$,
4.7. $h(\xi, X) = h(X, \xi) = 0$ for every $X \in \mathcal{D}$,
\( \tau = 0, \)
\( \nabla_X Y \in \mathcal{D} \quad \text{for all } X, Y \in \mathcal{D}, \)
\( \nabla_X \xi \in \mathcal{D} \quad \text{for every } X \in \mathcal{X}(M), \)
\( \nabla_{\xi} X \in \mathcal{D} \quad \text{for every } X \in \mathcal{D}, \)
\( X(h(\xi, \xi)) = 0 \quad \text{for every } X \in \mathcal{D}. \)

\textbf{Proof.} Since \( \nabla \eta = 0 \) we have
\( \eta(\nabla_X Y) = X(\eta(Y)) \)
for all \( X, Y \in \mathcal{X}(M) \). Now, using formula (3.1) we get
\( h(X, \varphi Y) = \eta(Y) \tau(X) \)
for all \( X, Y \in \mathcal{X}(M) \). Hence, if \( X, Y \in \mathcal{D} \), then \( h(X, \varphi Y) = 0 \), which proves (4.6). Taking \( X = \xi \) and \( Y \in \mathcal{D} \) in (4.14) we easily get (4.7). On the other hand, taking \( Y = \xi \) we have \( \tau(X) = 0 \), that is, (4.8). Formulas (4.9)–(4.11) can be obtained directly from (4.13). To prove (4.12) note that from the Codazzi equation (2.4) for \( h \) (and using (4.8)) we have
\( (\nabla_X h)(\xi, \xi) = (\nabla_\xi h)(X, \xi) = \xi(h(X, \xi)) - h(\nabla_\xi X, \xi) - h(X, \nabla_\xi \xi). \)
Now, if we take \( X \in \mathcal{D} \) then because of (4.6)–(4.7) we get \( h(X, \xi) = 0 \) and \( h(X, \nabla_\xi \xi) = 0 \), whereas (4.11) implies that also \( h(\nabla_\xi X, \xi) = 0 \). Thus, we obtain
\[ 0 = (\nabla_X h)(\xi, \xi) = X(h(\xi, \xi)) - 2h(\nabla_X \xi, \xi) \]
for every \( X \in \mathcal{D} \). Now, using (4.10) in the above formula leads to
\[ X(h(\xi, \xi)) = 0 \]
for every \( X \in \mathcal{D} \). This finishes the proof of (4.12). \( \blacksquare \)

Denote by \( N \) the metric normal field for \( f : M \to \mathbb{R}^{2n+2} \) (relative to the standard inner product on \( \mathbb{R}^{2n+2} \)). The metric normal field induces objects \( \hat{\nabla}, \hat{h} \) and \( \hat{S} \) as the transversal vector field on \( M \). Recall that the induced connection \( \hat{\nabla} \) is the Levi-Civita connection of the induced Riemannian metric \( g \). It is clear that \( N \) is \( J \)-tangent, thus induces an almost contact structure \( (\hat{\varphi}, \hat{N}, \hat{\eta}) \) on \( M \).

\textbf{Theorem 4.4.} Let \( f : M \to \mathbb{R}^{2n+2} \) be an affine immersion. Then the following conditions are equivalent:

(1) For every point on \( M \) there exist a neighborhood \( U \) and a \( J \)-tangent transversal vector field \( C \) defined on \( U \) such that \( \nabla \varphi = 0 \).
(2) For every point on \( M \) there exist a neighborhood \( U \) and a \( J \)-tangent transversal vector field \( C \) defined on \( U \) such that \( \nabla \eta = 0 \).
(3) An induced almost contact structure \( (\hat{\varphi}, \hat{N}, \hat{\eta}) \) is \( \hat{\nabla} \)-parallel.
Proof. Let $x$ be any point on $M$. Assume that in some neighborhood $U$ of $x$ there exists a $J$-tangent transversal vector field $C$ such that $\nabla \phi = 0$. Then, by virtue of Corollary 4.2 we can assume (possibly shrinking $U$) that $C$ is equiaffine. Now, by Theorem 3.1 (formula (3.1)) we get

$$(\nabla_X \eta)(Y) = h(X, \phi Y) - \eta(Y)\tau(X) = h(X, \phi Y)$$

for all $X, Y \in \mathcal{X}(U)$. Using the first two formulas from Lemma 4.1 we get

$$\nabla \eta \equiv 0,$$

which proves the implication (1)$\Rightarrow$(2).

To prove (2)$\Rightarrow$(3) note that if $(\phi, \xi, \eta)$ is an almost contact structure induced by a $J$-tangent transversal vector field $C$ defined on some neighborhood $U$ of $x$ and such that $\nabla \eta = 0$ then

$$\hat{\nabla}|_U = \Phi \xi + \phi Z,$$

where $Z \in \mathcal{D}$ and $\Phi = \text{const}$. Also note that the condition $\nabla \eta = 0$ is invariant under scaling the field $C$ by a constant. Therefore, we can later assume that $C$ is chosen in such a way that

$$\hat{\nabla}|_U = \xi + \phi Z.$$

By Theorem 2.1 and Lemma 3.2 we obtain $\hat{h} = h$ and $\hat{\eta} = \eta$. Since $N$ is the metric normal field we see that $g, \hat{h} = h$ and $\hat{S}$ are related by the formula

$$h(X, Y) = g(\hat{S}X, Y)$$

for all $X, Y \in \mathcal{X}(U)$. The above equality and Lemma 4.3 imply

$$\hat{S}X = h(\hat{N}, X)\hat{N}$$

for every $X \in \mathcal{X}(U)$. Now, using (3.2) and (3.5) for the structure $(\hat{\phi}, \hat{N}, \hat{\eta})$ we easily get

$$\hat{\phi}(\hat{\nabla}_X \hat{N}) = \hat{S}X - h(\hat{N}, X)\hat{N} = 0 \quad \text{and} \quad \hat{\eta}(\hat{\nabla}_X \hat{N}) = 0$$

for every $X \in \mathcal{X}(U)$, that is, $\hat{\nabla}_X \hat{N} = 0$ for every $X \in \mathcal{X}(U)$. Lemma 4.3 implies that

$$(\hat{\nabla}_X \hat{\phi})(Y) = \hat{h}(X, Y)\hat{N} - \hat{\eta}(Y)\hat{S}X = h(X, Y)\hat{N} - \eta(Y)h(\hat{N}, X)\hat{N}$$

$$= (h(X, Y) - \eta(Y)h(\xi, X))\hat{N} = 0$$

for all $X, Y \in \mathcal{X}(U)$. Arbitrariness of $x \in M$ gives $\hat{\nabla}\hat{N} = 0$ and $\hat{\nabla}\hat{\phi} = 0$ on the whole $M$. The condition $\hat{\nabla}\hat{\eta} = 0$ can easily be obtained from the equality $\hat{\nabla}\hat{\phi} = 0$, the fact that $N$ is equiaffine and the proof of (1)$\Rightarrow$(2).

To prove (3)$\Rightarrow$(1) it is sufficient to take $C := N$. □
From the proof of Theorem 4.4 it follows that if there exists an equiaffine $J$-tangent transversal vector field $C$ with $\nabla \varphi = 0$, then we also have $\nabla \eta = 0$ for $C$. Moreover, condition (3) in the above theorem is equivalent to the global versions of conditions (1) and (2), that is,

(1') There exists a $J$-tangent transversal vector field $C$ on $M$ such that $\nabla \varphi = 0$.

(2') There exists a $J$-tangent transversal vector field $C$ on $M$ such that $\nabla \eta = 0$.

It follows from Lemmas 4.1 and 4.3 that rank $f \leq 1$. However, the converse is not true in general since we have the following

**Example 4.5.** Let us consider an affine immersion defined as follows:

$$ f: \mathbb{R}^3 \ni (x, y, z) \mapsto \begin{bmatrix} x \\ y \\ z \\ e^z \end{bmatrix} \in \mathbb{R}^4. $$

Of course rank $f = 1$. Let $\{\partial_1, \partial_2, \partial_3\}$ be the canonical basis on $\mathbb{R}^3$ generated by the coordinate system $(x, y, z)$ on $\mathbb{R}^3$. It is not difficult to see that

$$ N: \mathbb{R}^3 \ni (x, y, z) \mapsto \begin{bmatrix} 0 \\ 0 \\ -\frac{e^z}{\sqrt{e^{2z} + 1}} \\ \frac{1}{\sqrt{e^{2z} + 1}} \end{bmatrix} \in \mathbb{R}^4 $$

is the metric normal field for $f$. Now,

$$ \hat{N} = \begin{bmatrix} e^z \\ \frac{1}{\sqrt{e^{2z} + 1}} \\ 0 \\ 0 \end{bmatrix}. $$

The above implies that

$$ f_*(\partial_3) = f_z = \begin{bmatrix} 0 \\ 0 \\ 1 \\ e^z \end{bmatrix} $$
is orthogonal to $\tilde{N}$, thus it belongs to the distribution $\mathcal{D}$. We will show that $(\tilde{\nabla}_{\partial_3} \tilde{\varphi})(\partial_3) \neq 0$. By straightforward computations we get

$$\tilde{\nabla}_{\partial_2} \partial_3 = \frac{e^{2z}}{e^{2z} + 1} \partial_3 \quad \text{and} \quad \tilde{\varphi}(\partial_3) = -\partial_1 - e^z \partial_2.$$ 

Now

$$(\tilde{\nabla}_{\partial_3} \tilde{\varphi})(\partial_3) = \tilde{\nabla}_{\partial_3}(\tilde{\varphi}(\partial_3)) - \tilde{\varphi}(\tilde{\nabla}_{\partial_3} \partial_3)$$

$$= \tilde{\nabla}_{\partial_3}(-\partial_1 - e^z \partial_2) - \tilde{\varphi}\left(\frac{e^{2z}}{e^{2z} + 1} \partial_3\right)$$

$$= -\tilde{\nabla}_{\partial_3} \partial_1 - e^z \tilde{\nabla}_{\partial_3} \partial_2 - \partial_3(e^z) \partial_2 + \frac{e^{2z}}{e^{2z} + 1} \partial_1 + \frac{e^{3z}}{e^{2z} + 1} \partial_2$$

$$= \frac{e^{2z}}{e^{2z} + 1} \partial_1 + \left(\frac{e^{3z}}{e^{2z} + 1} - e^z\right) \partial_2 \neq 0,$$

since $\tilde{\nabla}_{\partial_3} \partial_1 = \tilde{\nabla}_{\partial_3} \partial_2 = 0$ and $\partial_1, \partial_2$ are linearly independent.

In later parts of this paper we will give a local characterization of affine hypersurfaces satisfying any (thus all) of the conditions from Theorem 4.4. We need the following lemma:

**Lemma 4.6.** Let $f : M \to \mathbb{R}^{2n+2}$ be a hypersurface with a metric normal field $N$. Assume that an almost contact structure $(\tilde{\phi}, \tilde{N}, \tilde{\eta})$ induced by $N$ is $\tilde{\nabla}$-parallel. Then, for every point $x$ of $M$ and for any nowhere vanishing smooth function $\alpha$ defined in some neighborhood of $x$ and constant in the direction of $\mathcal{D}$ (i.e. $X(\alpha) = 0$ for every $X \in \mathcal{D}$), there exist a neighborhood of $x$ and a map $\psi(y, x_1, \ldots, x_{2n})$ defined on this neighborhood such that the vector fields $\partial/\partial y, \partial/\partial x_1, \ldots, \partial/\partial x_{2n}$ satisfy

$$\frac{\partial}{\partial y} = \alpha \tilde{N} \quad \text{and} \quad \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{2n}} \in \mathcal{D}.$$ 

**Proof.** Since $(\tilde{\varphi}, \tilde{N}, \tilde{\eta})$ is $\tilde{\nabla}$-parallel, in particular (by (4.9)) the distribution $\mathcal{D}$ is involutive. Let $x$ be any point on $M$ and let $\alpha$ be a nowhere vanishing smooth function defined in some neighborhood of $x$ and constant in the direction of $\mathcal{D}$. The Frobenius theorem implies that for $x$ there exist an open neighborhood $U \subset M$ and linearly independent vector fields $X_1, \ldots, X_{2n}, X_{2n+1} = \alpha \tilde{N} \in \mathcal{X}(U)$ such that $[X_i, X_j] = 0$ for $i, j = 1, \ldots, 2n + 1$. For every $i = 1, \ldots, 2n$ we have

$$X_i = D_i + \alpha_i \tilde{N},$$

where $D_i \in \mathcal{D}$ and $\alpha_i \in C^\infty(U)$. Thus

$$0 = [X_i, X_{2n+1}] = [D_i, X_{2n+1}] - X_{2n+1}(\alpha_i) \tilde{N}. \tag{4.15}$$

From (4.10) and (4.11) it is clear that $[D_i, \tilde{N}] \in \mathcal{D}$. Since $D_i(\alpha) = 0$ we also
have
\[ [D_i, X_{2n+1}] = \alpha [D_i, \hat{N}] + D_i(\alpha) \hat{N} = \alpha [D_i, \hat{N}] \in D. \]

Now (4.15) implies that \([D_i, X_{2n+1}] = 0\) and \(X_{2n+1}(\alpha_i) = 0\) for \(i = 1, \ldots, 2n\).
Moreover, for all \(i, j = 1, \ldots, 2n\) we have
\[ [D_i, D_j] = [X_i, X_j] - [\alpha_i \hat{N}, X_j] - [X_i, \alpha_j \hat{N}] + [\alpha_i \hat{N}, \alpha_j \hat{N}]. \]

Since \([X_i, X_j] = 0\), \(D\) is involutive and the last three terms in the above equality are proportional to \(\hat{N}\), we obtain
\[ [D_i, D_j] = 0 \]
for all \(i, j = 1, \ldots, 2n\). Of course the vector fields \(D_1, \ldots, D_{2n}, X_{2n+1}\) are linearly independent over \(C^\infty(U)\), so we can find a map \(\psi(y, x_1, \ldots, x_{2n})\) on \(U\) such that \(\partial/\partial y = X_{2n+1}\) and \(\partial/\partial x_i = D_i\) for \(i = 1, \ldots, 2n\).

In the next two theorems we give a local characterization of hypersurfaces for which there exists a \(J\)-tangent transversal vector field inducing an almost contact structure \((\varphi, \xi, \eta)\) such that \(\nabla \varphi = 0\) or \(\nabla \eta = 0\).

**Theorem 4.7.** Let \(f : M \to \mathbb{R}^{2n+2}\) be a hypersurface such that the almost contact structure \((\widehat{\varphi}, \widehat{\omega}, \widehat{\eta})\) is \(\widehat{\omega}\)-parallel. Let \(U\) be a non-empty open subset of \(M\). If \(\text{rank } f = 0\) on \(U\) then \(f(U)\) is a piece of a hyperplane.

**Proof.** Since \(\text{rank } \hat{h} = 0\) and \(\widehat{\omega} = 0\) on \(U\), Lemma 4.1 implies
\[ D_X N = -\hat{S} X = 0 \]
for every \(X \in \mathcal{X}(U)\). It follows that a metric normal field \(N\) is constant on \(U\), thus \(f(U)\) is a hyperplane in \(\mathbb{R}^{2n+2}\).

**Theorem 4.8.** Let \(f : M \to \mathbb{R}^{2n+2}\) be a hypersurface such that the almost contact structure \((\widehat{\varphi}, \widehat{\omega}, \widehat{\eta})\) is \(\widehat{\omega}\)-parallel. Let \(x\) be a point on \(M\) such that \(\text{rank } f = 1\) at \(x\). Then there exists an open neighborhood \(U\) of \(x\) such that \(f\) can be expressed on \(U\) in the form
\[ f(x_1, \ldots, x_{2n}, y) = x_1 b_1 + \cdots + x_{2n} b_{2n} - v \int \alpha(y) \cos y dy + Jv \int \alpha(y) \sin y dy, \]
where \(v \in \mathbb{R}^{2n+2}\), \(\|v\| = 1\), \(\alpha\) is some nowhere vanishing smooth function on \(U\) and \(b_1, \ldots, b_{2n} \in \mathbb{R}^{2n+2}\) are linearly independent vectors from \(\{v, Jv\}^\perp\). Moreover, every hypersurface (4.16) has a \(\widehat{\omega}\)-parallel almost contact structure \((\widehat{\varphi}, \widehat{\omega}, \widehat{\eta})\).

**Proof.** First, note that since \(\text{rank } \hat{h}_x = 1\), we have \(\hat{h}_x(\hat{N}_x, \hat{N}_x) \neq 0\). Since \(\hat{h}(\hat{N}, \hat{N})\) is smooth we can find a neighborhood \(U\) of \(x\) such that \(\hat{h}(\hat{N}, \hat{N}) \neq 0\) on \(U\), thus \(\text{rank } \hat{h} = 1\) on \(U\). Moreover, by (4.12) the function \(\hat{h}(\hat{N}, \hat{N})\) is constant in a direction of the distribution \(D\).
Let us define a new function on $U$, 
$$
\alpha := \frac{1}{\hat{h}(\hat{N}, \hat{N})}.
$$
It is clear that $\alpha \neq 0$ and $\alpha$ is constant in a direction of $\mathcal{D}$. Using Lemma 4.6 and possibly shrinking $U$ we deduce that there exists a map $\psi$ on $U$ such that 
$$
\frac{\partial}{\partial y} = \alpha \hat{N} \quad \text{and} \quad \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{2n}} \in \mathcal{D}.
$$
By the Weingarten formula (2.2) and formulas (4.3), (4.4) we get 
$$
D_{\partial/\partial x_i} N = -\hat{S} \left( \frac{\partial}{\partial x_i} \right) = 0
$$
for $i = 1, \ldots, 2n$ and 
$$
D_{\partial/\partial y} N = -\hat{S} \left( \frac{\partial}{\partial y} \right) = -\alpha \hat{S}(\hat{N}) = -\alpha \hat{h}(\hat{N}, \hat{N}) \hat{N} = -\hat{N} = -JN,
$$
thus $N_{x_i} = 0$ for $i = 1, \ldots, 2n$ and $N_y = -JN$. Now, Lemma 2.3 implies that 
$$
N = Jv \cos y + v \sin y,
$$
where $v \in \mathbb{R}^{2n+2}$. Since $N$ is a metric normal field, we see that 
$$
1 = ||N|| = ||Jv \cos y + v \sin y|| = ||v||.
$$
Let $b_1, \ldots, b_{2n}$ be any linearly independent vectors from $\mathbb{R}^{2n+2}$ such that $b_i \in \{v, Jv\}^\perp$. We have 
$$
N \cdot b_i = 0 \quad \text{and} \quad \hat{N} \cdot b_i = 0,
$$
for every $i = 1, \ldots, 2n$, therefore the vectors $b_1, \ldots, b_{2n}$ span $f_*(\mathcal{D})$. Let $\partial_1, \ldots, \partial_{2n}$ be vector fields on $U$ such that $f_*(\partial_i) = b_i$ for $i = 1, \ldots, 2n$. Of course $\partial_1, \ldots, \partial_{2n}$ are linearly independent and span the distribution $\mathcal{D}$. For every $X \in TU$ and for every $i = 1, \ldots, 2n$ we have 
$$
D_X f_* \partial_i = D_X b_i = 0.
$$
On the other hand by the Gauss formula (2.1) and due to the fact that $\hat{h}|_{\mathcal{D} \times \mathcal{D}} = 0$ we obtain 
$$
D_X f_* \partial_i = f_* (\hat{\nabla}_X \partial_i),
$$
thus 
$$
\hat{\nabla}_X \partial_i = 0
$$
for every $X \in TU$. In particular, we have 
$$
\hat{\nabla}_{\partial_i} \partial_j = 0
$$
for all $i, j \in \{1, \ldots, 2n\}$ and 
$$
\hat{\nabla}_{\partial_i \partial_j} \partial_i = 0.
for \( i = 1, \ldots, 2n \). Moreover
\[
\hat{\nabla}_{\partial_i} \frac{\partial}{\partial y} = \hat{\nabla}_{\partial_i}(\alpha \hat{N}) = \partial_i(\alpha) \hat{N} + \alpha \hat{\nabla}_{\partial_i} \hat{N} = 0,
\]
since \( \alpha \) is constant in a direction of \( \mathcal{D} \) and \( \hat{\nabla} \hat{N} = 0 \). To sum up, the vector fields
\[
\partial_1, \ldots, \partial_{2n}, \frac{\partial}{\partial y}
\]
are associated with some map \( \tilde{\psi} \). Denoting again \( \partial_1, \ldots, \partial_{2n} \) by \( \partial/\partial x_1, \ldots, \partial/\partial x_{2n} \) we see that the immersion \( f \) satisfies the differential equations
\[
f_{x_i} = b_i
\]
for \( i = 1, \ldots, 2n \) and
\[
f_y = \alpha(y) \hat{N} = \alpha(y)(-v \cos y + Jv \sin y).
\]
Solving the above we get a local form of \( f \) as follows:
\[
f(x_1, \ldots, x_{2n}, y) = x_1 b_1 + \cdots + x_{2n} b_{2n} - v \int \alpha(y) \cos y \, dy + Jv \int \alpha(y) \sin y \, dy.
\]

To prove the second part of the theorem note that the function described by (4.16) is an immersion, since \( b_1, \ldots, b_{2n} \) and \(-v\alpha(y) \cos y + Jv\alpha(y) \sin y\) are linearly independent. Now, it is enough to show that \( \hat{\nabla} \hat{\eta} = 0 \). It is not difficult to see that \( N = Jv \cos y + v \sin y \), thus
\[
\hat{N} = -v \cos y + Jv \sin y;
\]
moreover \( \partial/\partial x_i \in \mathcal{D} \) and \( \hat{\nabla}_X(\partial/\partial x_i) = 0 \) for \( i = 1, \ldots, 2n \), which imply \( (\hat{\nabla}_X \hat{\eta})(Y) = 0 \) for all \( X \in TM \) and \( Y \in \mathcal{D} \). To complete the proof note that
\[
(\hat{\nabla}_X \hat{\eta})(\hat{N}) = X(\hat{\eta}(\hat{N})) - \hat{\eta}(\hat{\nabla}_X \hat{N}) = -\hat{\eta}(\hat{\nabla}_X \hat{N})
\]
for every \( X \in TM \). If \( X \in \mathcal{D} \) then \( \hat{\nabla}_X \hat{N} = 0 \), because
\[
D_X \hat{N} = D_X(-v \cos y + Jv \sin y) = 0
\]
for every \( X \in \mathcal{D} \). If \( X = \partial/\partial y \) then
\[
D_{\partial/\partial y} \hat{N} = Jv \cos y + v \sin y = N,
\]
thus \( \hat{\nabla}_{\partial/\partial y} \hat{N} = 0 \). Summarizing, we have shown that \( \hat{\nabla} \hat{\eta} = 0 \), which completes the proof. \( \blacksquare \)

References


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