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Łojasiewicz exponent of the gradient near the fiber

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Abstract. It is well-known that if r is a rational number from $[-1, 0)$, then there is no polynomial f in two complex variables and a fiber $f^{-1}(t_0)$ such that r is the Łojasiewicz exponent of $\operatorname{grad}(f)$ near the fiber $f^{-1}(t_0)$. We show that this does not remain true if we consider polynomials in real variables. More exactly, we give examples showing that any rational number can be the Łojasiewicz exponent near the fiber of the gradient of some polynomial in real variables. The second main result of the paper is the formula computing the Łojasiewicz exponent of the gradient near a fiber of a polynomial in two real variables. In particular, this gives, in the case of two real variables, a way to tell whether a given value is an asymptotic critical value or not.

1. Introduction. Let $g\colon \mathbb{K}^n \to \mathbb{K}^p$ be a polynomial mapping, $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. For an unbounded set $S \subset \mathbb{K}^n$, put

$$\mathcal{L}_\infty(g|_S) := \sup\{\nu \in \mathbb{R}\colon \exists C, R > 0,\ \forall x \in S\ (\|x\| \geq R \Rightarrow \|g(x)\| \geq C\|x\|^\nu)\}.$$

Let $f\colon \mathbb{K}^n \to \mathbb{K}$ be a polynomial function and $D_\delta = \{t\colon |t - t_0| \leq \delta\}$. Ha Huy Vui [H1] defined

$$\mathcal{L}_\infty(f, t_0) = \lim_{\delta \to 0} \mathcal{L}_\infty(\operatorname{grad} f|_{f^{-1}(D_\delta)}),$$

or equivalently ([C-K], [Sk])

$$\widetilde{\mathcal{L}}_\infty(f, t_0) = \inf_{\Phi} \frac{\deg \operatorname{grad} f \circ \Phi}{\deg \Phi},$$

where Φ runs over the set of meromorphic functions in a neighborhood of infinity such that $\deg \Phi > 0$, $\deg(f - t_0) \circ \Phi < 0$. Following [R-S], we call them the *Łojasiewicz exponent of the gradient of f near the fiber $f^{-1}(t_0)$*.

This exponent plays an important role in the study of polynomial mappings. According to a fundamental result of R. Thom [Th], there is a finite

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subset $B(f)$ of $\mathbb{K}$, called the *bifurcation set*, such that the mapping

$$f\colon \mathbb{K}^n \setminus f^{-1}(B(f)) \to \mathbb{K} \setminus B(f)$$

defines a C^∞ locally trivial fiber bundle. It is known that the set $\Sigma(f)$ of critical values of f is a subset of $B(f)$ and in general it is not equal to $B(f)$. The points of $B_\infty(f) = B(f) \setminus \Sigma(f)$ are usually called *critical values of singularities at infinity* of f. It is important to be able to decide whether a given value in $\mathbb{K}$ belongs to $B_\infty(f)$. Although this problem has attracted attention of many specialists in singularity theory and algebraic geometry during the last twenty years [C-K], [H1], [H-L], [P], it is still open. We know the answer only for a number of particular cases.

It is easy to see that for any n, for $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$, $\mathcal{L}_\infty(f, t_0) < 0$ is always a necessary condition for t_0 to be in $B_\infty(f)$.

Assume $\mathbb{K} = \mathbb{C}$. If $n = 2$, then a value t_0 belongs to $B_\infty(f)$ if and only if $\mathcal{L}_\infty(f, t_0)$ is negative. Moreover, if $\mathcal{L}_\infty(f, t_0) < 0$ for some t_0 then $\mathcal{L}_\infty(f, t_0) < -1$ ([H1], [C-K]). The same holds for $n \geq 3$ if we assume that the polynomial f has only isolated singularities at infinity [P].

In the general case, the above results are no longer true:

- There is a polynomial f in three complex variables such that $\mathcal{L}_\infty(f, t_0)$ < -1 but t_0 is not a critical value of singularities at infinity for f [P-Z].
- For $n \geq 3$, there is a polynomial f of n variables such that the set of all t with $\mathcal{L}_\infty(f, t) < 0$ is the whole $\mathbb{K}$, while the set of t where $\mathcal{L}_\infty(f, t) < -1$ is always finite.

These facts lead to the following question which seems to be still open:

Is the set

$$K_\infty(f) := \{t \in \mathbb{R} \colon \mathcal{L}_\infty(f, t) < -1\}$$

equal to

$$\tilde{K}_\infty(f) := \{t \in \mathbb{R} \colon \mathcal{L}_\infty(f, t) < 0\}$$

for any polynomial in two real variables?

We will show that for every rational number α there is a polynomial f in $n \geq 2$ variables such that $\alpha = \mathcal{L}_\infty(f, t)$ for some t (Theorem 2.1).

In particular, when $\mathbb{K} = \mathbb{R}$, the set $K_\infty(f)$ can be strictly included in $\tilde{K}_\infty(f)$, although, for $n = 2$, both are finite.

Any point of $K_\infty(f)$ is called an *asymptotic critical value* of f. This notion appears in many problems of mathematics [K-M-P]. For example, the fact that $K_\infty(f)$ is finite plays an important role in the proof of the *gradient conjecture* [K-M-P].

We are interested in finding a simple way to decide whether a given value belongs to the set of asymptotic critical values or not. For polynomials in several complex variables, Jelonek and Kurdyka [J-K] gave an algorithm to compute this set. It turns out that this can also be done for polynomials in two variables, by using the Puiseux expansions at infinity (in the complex case) or their real approximations (in the real case) for the polar curve (Theorems 3.4 and 3.7).

2. The Łojasiewicz exponent of the gradient near the fiber in the real case

THEOREM 2.1. *For every rational α, there exists $f \in \mathbb{R}[x_1, \ldots, x_n]$ for which $\mathcal{L}_\infty(f, 0) = \alpha$.*

Proof. We consider first the case $n = 2$. It will be shown that $\mathcal{L}_\infty(f, 0) = -2q/p + 2m$ for the polynomial

$$\begin{aligned} f(x,y) = {} & \frac{1}{2p+2m+1}\, x^{2p+2m+1} y^{2q} - \frac{2}{p+2m+1}\, x^{p+2m+1} y^{q} \\ & + \frac{1}{2m+1}\, x^{2m+1} + \frac{1}{2m+3}\, x^{2m+3} \\ & + \frac{1}{2p+1}\, x^{2p+1} y^{2q+2m} - \frac{2}{p+1}\, x^{p+1} y^{q+2m} + xy^{2m} + \frac{1}{3}\, x^3 y^{2m}, \end{aligned}$$

where $p \geq q > 0$ and $m \geq 0$ are integers. We see that

$$\begin{aligned} f_x(x,y) &= \frac{\partial f}{\partial x}(x,y) = [(x^p y^q - 1)^2 + x^2](x^{2m} + y^{2m}), \\ f_y(x,y) &= \frac{2q}{2p+2m+1}\, x^{2p+2m+1} y^{2q-1} - \frac{2q}{p+2m+1}\, x^{p+2m+1} y^{q-1} \\ & \quad + \frac{2q+2m}{2p+1}\, x^{2p+1} y^{2q+2m-1} - \frac{2(q+2m)}{p+1}\, x^{p+1} y^{q+2m-1} \\ & \quad + 2mxy^{2m-1} + \frac{2m}{3}\, x^3 y^{2m-1}. \end{aligned}$$

For the series $x = \varphi(y) = y^{-q/p}$, we have

$$f_x(\varphi(y), y) \sim y^{-2q/p+2m} \quad \text{and} \quad f_y(\varphi(y), y) \sim y^{\beta}$$

with $\beta \leq -2q/p + 2m$. Thus

$$\mathcal{L}_\infty(f, 0) \leq -2q/p + 2m. \tag{2.1}$$

For every $(x, y) \in \mathbb{R}^2$, $x^2 + y^2 = r^2 \to \infty$. If $x^p y^q - 1 \nrightarrow 0$ or $x \nrightarrow 0$ as $r \to \infty$ then

$$|f_x(x,y)| \geq Cr^{2m} \geq Cr^{-2q/p+2m}.$$

Assume now that

$$x^p y^q - 1 = \varepsilon(r) \to 0 \quad \text{and} \quad x \to 0 \quad \text{as } r \to \infty.$$

We have

$$|y|^q = \frac{1 + \varepsilon(r)}{|x|^p} \leq r^q.$$

Therefore

$$x^2 \geq Cr^{-2q/p}.$$

Thus

$$|f_x(x,y)| \geq Cr^{-2q/p+2m}.$$

Hence

$$\mathcal{L}_\infty(f,0) \geq -2q/p + 2m. \tag{2.2}$$

From (2.1) and (2.2), we get

$$\mathcal{L}_\infty(f,0) = -2q/p + 2m \quad \forall p \geq q > 0,\ m \geq 0. \tag{2.3}$$

Thus, for every rational $\alpha \geq -2$, there is an $f(x,y) \in \mathbb{R}[x,y]$ satisfying $\mathcal{L}_\infty(f,0) = \alpha$.

Consider now the polynomial

$$f(x,y) = x^p y^q - x,$$

where $p > 1, q > 0$ are integers $\mathcal{L}_\infty(f,0) = -\frac{p+q-1}{p-1}$. We have

$$f_x(x,y) = px^{p-1}y^q - 1 \quad \text{and} \quad f_y(x,y) = qx^p y^{q-1}.$$

Clearly

$$\mathcal{L}_\infty(f,0) = -\frac{p+q-1}{p-1}, \quad \forall p > 1,\ q > 0. \tag{2.4}$$

From (2.3) and (2.4), for every rational α, there is an $f(x,y) \in \mathbb{R}[x,y]$ satisfying $\mathcal{L}_\infty(f,0) = \alpha$.

Now we consider the general case. For every rational α, we put

$$F(x_1, x_2, \dots, x_n) = f(x_1, x_2) + x_3^k + \dots + x_n^k,$$

where $k > 0$ is integer, $k \geq \alpha + 1$ and $f \in \mathbb{R}[x,y]$ satisfies $\mathcal{L}_\infty(f,0) = \alpha$. It is clear that $\mathcal{L}_\infty(F,0) = \alpha$. ∎

Remark 2.2. A similar result (for the Łojasiewicz exponent at infinity) was given by E. A. Gorin (see [G]).

Let $f \in \mathbb{R}[x_1, \dots x_n]$ and let $g\colon \mathbb{R}^n \to \mathbb{R}^m$ be a polynomial mapping. Let $t_0 \in \mathbb{R}$. The authors of [R-S] define

$$\mathcal{L}_{\infty, f\to t_0}(g) = \sup\{\mathcal{L}_\infty(g|_{f^{-1}(U)})\colon U \subset \mathbb{R}^n \text{ is a neighborhood of } t_0\}.$$

Analogously to Theorem 2.1 we can show

Remark 2.3. For every $\alpha \in \mathbb{Q}$ there exist $f \in \mathbb{R}[x_1, \dots, x_n]$ and a polynomial mapping $g\colon \mathbb{R}^n \to \mathbb{R}^m$ such that $\mathcal{L}_{\infty, f\to 0}(g) = \alpha$.

3. The Łojasiewicz exponent of the gradient near the fiber in the two variables case. We begin by recalling the definition of the Newton polygon relative to an arc and the process of sliding which were introduced by Kuo and Parusiński in [K-P].

If $\varphi(\tau)$ is a series of the form

$$\varphi(\tau) = a_0\tau^{\alpha} + \text{terms of lower degree with } a_0 \neq 0,$$

then the number α is denoted by $\deg\varphi$.

Let $f\colon \mathbb{C}^2 \to \mathbb{C}$ be a polynomial. For a series

$$x = \varphi(y) = c_1 y^{n_1/N} + c_2 y^{n_2/N} + \cdots, \quad c_i \in \mathbb{C}^2,\ c_1 \neq 0,$$

we put

$$M(X,Y) = f(X + \varphi(1/Y), 1/Y) = \sum_{i,j} c_{ij} X^i Y^{j/N}.$$

For each $c_{ij} \neq 0$, let us plot a dot at $(i, j/N)$, called a Newton dot. The set of Newton dots is called the Newton diagram. They generate a convex hull, whose boundary is called the *Newton polygon of f relative to φ*, to be denoted by $\mathbb{P}(f,\varphi)$ or $\mathbb{P}(M)$.

Assume that $x = \varphi(y)$ is not a Puiseux root at infinity of $f = 0$. Then the Y-axis contains at least one dot of M. Let $(0, h_M)$ be the lowest Newton dot. We see that $h_M = -\deg f(\varphi(y), y)$.

By the *highest Newton edge H_M* of M we mean the edge of $\mathbb{P}(M)$ with one of extremities at $(0, h_M)$ and such that all Newton dots of M lie on or above the line containing H_M. Let $\theta_M = \tan\nu$, where ν is the angle between H_M and the X-axis. Note that if $(i, j/N)$ is a Newton dot of M then $\theta_M i + j/N \geq h_M$, and $(i, j/N) \in H_M$ if and only if $\theta_M i + j/N = h_M$. If $x = \varphi(y)$ is a Puiseux root at infinity of $f = 0$, we set $h_M = +\infty$ and $\theta_M = +\infty$.

We associate to H_M the polynomial $\varepsilon_M(x) := \varepsilon(x, 1)$, where

$$\varepsilon(X,Y) = \sum c_{ij} X^i Y^{j/N} \quad \text{with } (i, j/N) \in H_M.$$

Lemma 3.1 ([H-D, Lemma 2.1]). *Let $\widetilde{M}(X,Y) = M(X + cY^{\theta}, Y)$.*

(a) *If $\theta > \theta_M$, then $h_{\widetilde{M}} = h_M$ and $\theta_{\widetilde{M}} = \theta_M$.*
(b) *If $\theta = \theta_M$ and c is a non-zero root of $\varepsilon_M(x)$, then $h_{\widetilde{M}} > h_M$ and $\theta_{\widetilde{M}} > \theta_M$.*
(c) *If $\theta = \theta_M$ and $\varepsilon_M(c) \neq 0$, then $h_{\widetilde{M}} = h_M$ and $\theta_{\widetilde{M}} = \theta_M$.*

If c is a non-zero root of $\varepsilon_M(x)$, the series $\varphi_1(y) = \varphi(y) + cy^{-\theta_M}$ will be called the *sliding of $\varphi(y)$ along f*. A recursive sliding $\varphi \to \varphi_1 \to \cdots$ produces a limit, φ_∞, where $\varphi_\infty(y) = \varphi_i(y)$ if $f(\varphi_i(y), y) = 0$. The series φ_∞ is a Puiseux root at infinity of $f = 0$ (see [W]) and will be called a *final result of sliding φ along f*.

Lemma 3.2 ([H-D, Lemma 2.2]). *Let $f, g\colon \mathbb{C}^2 \to \mathbb{C}$ be two polynomials. For a series $x = \varphi(y)$, we put*

$$M(X,Y) = f(X + \varphi(1/Y), 1/Y),$$
$$N(X,Y) = g(X + \varphi(1/Y), 1/Y).$$

We have

(a) *if $\theta_M > \theta_N$, then* $\deg g(\varphi_\infty(y), y) = \deg g(\varphi(y), y)$;
(b) *if $\theta_M = \theta_N$, then* $\deg g(\varphi_\infty(y), y) \le \deg g(\varphi(y), y)$,

where $x = \varphi_\infty(y)$ is a final result of sliding φ along f.

Let us consider a series $x = \lambda(y)$ of the form

$$x = \lambda(y) = a_1 y^{\alpha_1} + a_2 y^{\alpha_2} + \cdots + a_{s-1} y^{\alpha_{s-1}} + a_s y^{\alpha_s} + \cdots$$

where $\alpha_1 > \alpha_2 > \cdots$.

If $a_1, \ldots, a_{s-1} \in \mathbb{R}$ and $a_s \notin \mathbb{R}$, following Kuo [K] we put

$$\lambda^{\mathbb{R}}(y) := a_1 y^{\alpha_1} + a_2 y^{\alpha_2} + \cdots + a_{s-1} y^{\alpha_{s-1}} + c y^{\alpha_s},$$

where c is a generic real number. We call $\lambda^{\mathbb{R}}(y)$ the *real approximation* of $\lambda(y)$.

Lemma 3.3 ([H-D, Lemma 2.3]). *Let $f, g\colon \mathbb{R}^2 \to \mathbb{R}$ be polynomials. For a series $x = \varphi(y)$, we put*

$$M(X,Y) = f(X + \varphi(1/Y), 1/Y),$$
$$N(X,Y) = g(X + \varphi(1/Y), 1/Y).$$

Let $x = \varphi_\infty(y)$ be a final result of sliding φ along f and $\varphi_\infty^{\mathbb{R}}(y)$ be the real approximation of $\varphi_\infty(y)$. We have

(a) *if $\theta_M > \theta_N$, then* $\deg g(\varphi_\infty^{\mathbb{R}}(y), y) = \deg g(\varphi(y), y)$;
(b) *if $\theta_M = \theta_N$, then* $\deg g(\varphi_\infty^{\mathbb{R}}(y), y) \le \deg g(\varphi(y), y)$.

In particular for $g = f$, we have $\deg f(\varphi_\infty^{\mathbb{R}}(y), y) \le \deg f(\varphi(y), y)$.

Theorem 3.4. *Let f be a polynomial in two complex variables (x, y). Assume that f is monic in x, and $t_0 \in \mathbb{C}$. Let $x = x_i(y)$, $i = 1, \ldots, d-1$, be the Puiseux expansions at infinity of $f_x(x, y) = 0$. Put*

$$V(f, t_0) = \{x_i(y)\colon \deg(f(x_i(y), y) - t_0) < 0\}.$$

If $\mathcal{L}_{\infty, f \to t_0}(\operatorname{grad} f) < 0$, then $V(f, t_0) \neq \emptyset$ and

$$\mathcal{L}_{\infty, f \to t_0}(\operatorname{grad} f) = \min_i \{\deg(f(x_i(y), y) - t_0) - 1\colon x_i(y) \in V(f, t_0)\}.$$

Proof. Let $x = \varphi(y)$ be any series satisfying

$$\deg(f(\varphi(y), y) - t_0) < 0, \qquad \deg \operatorname{grad} f(\varphi(y), y) < 0.$$

We put

$$\begin{aligned} M(X,Y) &= f(X+\varphi(1/Y),1/Y)-t_0=\sum_{i,j} c_{ij}X^iY^{j/N},\\ P(X,Y) &= f_x(X+\varphi(1/Y),1/Y),\\ Q(X,Y) &= f_x(X+\varphi(1/Y),1/Y)+cf_y(X+\varphi(1/Y),1/Y),\\ R(X,Y) &= f_x(X+\varphi(1/Y),1/Y)+c(f(X+\varphi(1/Y),1/Y)-t_0),\end{aligned}$$

where c is a generic number. Then

$$\begin{aligned} P(X,Y) &= \frac{\partial M}{\partial X}(X,Y),\\ Q(X,Y) &= \frac{\partial M}{\partial X}(X,Y)-c\biggl(\varphi'\Bigl(\frac{1}{Y}\Bigr)\frac{\partial M}{\partial X}(X,Y)+Y^2\frac{\partial M}{\partial Y}(X,Y)\biggr),\\ R(X,Y) &= \frac{\partial M}{\partial X}(X,Y)-cM(X,Y).\end{aligned}$$

Claim. *$\theta_P>0$, $\theta_P\ge\theta_Q$ and $\theta_P\ge\theta_R$.*

Note that if $(i,j/N)$ is a Newton dot of P then $\theta_Pi+j/N\ge h_P$. Since the point $(d-1,0)$ is a Newton dot of P, $\theta_P(d-1)+0\ge h_P$. Hence $\theta_P>0$.

Let (α,β) be the second extremity of H_Q. We will show that $\theta_P\alpha+\beta\ge h_P$. In fact:

- If (α,β) is a Newton dot of P, then $\theta_P\alpha+\beta\ge h_P$.
- If (α,β) is a Newton dot of $\varphi'(1/Y)\frac{\partial M}{\partial X}(X,Y)$, then $(\alpha,\beta-s)$ is a dot of P, where $s\ge 0$ since $\deg\varphi\le 1$. Therefore

$$\theta_P\alpha+(\beta-s)\ge h_P.$$

  Thus

$$\theta_P\alpha+\beta\ge h_P.$$

- If (α,β) is a Newton dot of $Y^2\frac{\partial M}{\partial Y}(X,Y)$, then $(\alpha-1,\beta-1)$ is a dot of P. Therefore

$$\theta_P(\alpha-1)+(\beta-1)\ge h_P.$$

  Thus $\theta_P\alpha+\beta\ge h_P$ since $\theta_P>0$.

Since the point $(0,h_P)$ is a Newton dot of Q, $h_P\ge h_Q$. Therefore

$$\theta_P\alpha+\beta\ge h_P\ge h_Q=\theta_Q\alpha+\beta.$$

Hence $\theta_P\ge\theta_Q$.

Analogously, we can show that $\theta_P\ge\theta_R$.

Now, using the claim and Lemma 3.2, we get

$$\deg(f_x(\varphi_\infty(y),y)+cf_y(\varphi_\infty(y),y))\le\deg(f_x(\varphi(y),y)+cf_y(\varphi(y),y))$$

and

$$\deg(f_x(\varphi_\infty(y),y)+c(f(\varphi_\infty(y),y)-t_0)) \\ \leq \deg(f_x(\varphi(y),y)+c(f(\varphi(y),y)-t_0)),$$

where $\varphi_\infty(y)$ is a final result of sliding $\varphi(y)$ along f_x. Therefore

$$\deg(f(\varphi_\infty(y),y)-t_0)-1=\deg\frac{df(\varphi_\infty(y),y)}{dy}=\deg f_y(\varphi_\infty(y),y) \\ \leq \deg(\operatorname{grad} f(\varphi(y),y))$$

and

$$\deg(f(\varphi_\infty(y),y)-t_0)\leq \deg(f_x(\varphi(y),y)+cf(\varphi(y),y))<0.$$

Thus $V(f,t_0)\neq\emptyset$ and

$$\mathcal{L}_{\infty,f\to t_0}(\operatorname{grad} f)\geq \min_i\{\deg(f(x_i(y),y)-t_0)-1\colon x_i(y)\in V(f,t_0)\}.$$

On the other hand, the inequality

$$\mathcal{L}_{\infty,f\to t_0}(\operatorname{grad} f)\leq \min_i\{\deg(f(x_i(y),y)-t_0)-1\colon x_i(y)\in V(f,t_0)\}$$

is always satisfied. Hence

$$\mathcal{L}_{\infty,f\to t_0}(\operatorname{grad} f)= \min_i\{\deg(f(x_i(y),y)-t_0)-1\colon x_i(y)\in V(f,t_0)\}. \blacksquare$$

Remark 3.5. This result is implicitly contained in [K-P] (see also [H2] for a different proof).

Corollary 3.6 ([G-S, Theorem 3.1]). *Let $f(x,y)$ be a polynomial in two complex variables and $d=\deg f>2$. If $\mathcal{L}_{\infty,f\to t_0}(\operatorname{grad} f)<0$, then*

$$\mathcal{L}_{\infty,f\to t_0}(\operatorname{grad} f)\leq -1-\frac{1}{d-2}.$$

Proof. The proof goes along the same lines as in [G-S]. Let $x=x_i(y)$, $i=1,\dots,d-1$, be the Puiseux expansions at infinity of $f_x(x,y)=0$. Assume that $\mathcal{L}_{\infty,f\to t_0}(\operatorname{grad} f)<0$. By Theorem 3.4 there exists $i_0\in\{1,\dots,d-1\}$ such that

$$\mathcal{L}_{\infty,f\to t_0}(\operatorname{grad} f)=\deg(f(x_{i_0}(y),y)-t_0)-1$$

and

$$\deg(f(x_{i_0}(y),y)-t_0)<0. \tag{3.1}$$

Let $Q\in\mathbb{C}[\tau,y]$ be the resultant $Q(\tau,y)=\operatorname{Res}_x(f-\tau,f_x)$. Denote by $\mathbb{P}$ the Newton polygon of $Q(\tau,y)$. Then

$$\mathbb{P}\subset\operatorname{conv}\{(0,0);(d-1,0);(0,d(d-1))\} \tag{3.2}$$

and

$$Q(\tau,y)=\prod_{i=1}^{d-1}(f(x_i(y),y)-\tau).$$

From (3.1), analogously to [G-S, Lemma 3.3] we obtain

$$\deg f(x_{i_0}(y),y) = -\frac{j_2-j_1}{i_2-i_1},$$

where the segment $[(i_1,j_1),(i_1,j_2)] \subset \mathbb{P}$ is such that $i_1,i_2,j_1,j_2 \in \mathbb{Z}$, $0 \le i_1 < i_2$, $0 \le j_1 < j_2$. By (3.2) we see $i_2 - i_1 \le d-2$. Therefore

$$\deg f(x_{i_0}(y),y) \le -\frac{1}{d-2}.$$

Thus

$$\mathcal{L}_{\infty,f\to t_0}(\mathrm{grad} f) \le -1 - \frac{1}{d-2}. \ \blacksquare$$

Theorem 3.7. *Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a monic polynomial in x. Let $t_0 \in \mathbb{R}$. Let $x = x_i(y)$, $i = 1,\dots,d-1$, be the Puiseux expansions at infinity of $f_x(x,y) = 0$ and $x_i^{\mathbb{R}}(y)$ be the real approximation of $x_i(y)$. Put*

$$V_{\mathbb{R}}(f,t_0) = \{x_i(y)\colon\ \deg(f(x_i^{\mathbb{R}}(y),y) - t_0) < 0\}.$$

If $\mathcal{L}_{\infty,f\to t_0}(\mathrm{grad} f) < 0$, then $V_{\mathbb{R}}(f,t_0) \ne \emptyset$ and

$$\mathcal{L}_{\infty,f\to t_0}(\mathrm{grad} f) = \min_i\{\deg \mathrm{grad} f(x_i^{\mathbb{R}}(y),y)\colon x_i(y) \in V_{\mathbb{R}}(f,t_0)\}.$$

Proof. Let $x = \varphi(y)$ be any real series satisfying

$$\deg(f(\varphi(y),y) - t_0) < 0, \qquad \deg \mathrm{grad}\, f(\varphi(y),y) < 0.$$

We put

$$\begin{aligned}
M(X,Y) &= f(X+\varphi(1/Y),1/Y) - t_0 = \sum_{i,j} c_{ij}X^iY^{j/N},\\
P(X,Y) &= f_x(X+\varphi(1/Y),1/Y),\\
Q(X,Y) &= f_x(X+\varphi(1/Y),1/Y) + cf_y(X+\varphi(1/Y),1/Y),\\
R(X,Y) &= f_x(X+\varphi(1/Y),1/Y) + c(f(X+\varphi(1/Y),1/Y) - t_0),
\end{aligned}$$

where c is a generic number.

Using the Claim in the proof of Theorem 3.4, we get $\theta_P \ge \theta_Q$ and $\theta_P \ge \theta_R$.

Now, the proof repeats that of Theorem 3.4 with the only exception that instead of Lemma 3.2 we use Lemma 3.3. ■

Remark 3.8. The following conditions on the behavior at infinity of the gradient of maps appear in many problems. Let $f\colon \mathbb{K}^n \to \mathbb{K}$ be a polynomial and $t_0 \in \mathbb{K}$.

(1) We say that t_0 satisfies the *Fedoryuk condition* (F) if $\mathcal{L}_{\infty,f\to t_0}(\mathrm{grad}\, f) \ge 0$, i.e. for every sequence $\{z_m\} \subset \mathbb{K}^n$ with $z_m \to \infty$ and $f(z_m) \to t_0$ we have $\mathrm{grad}\, f(z_m) \nrightarrow 0$.

(2) We say t_0 satisfies the *Malgrange condition* (M) if $\mathcal{L}_{\infty,f\to t_0}(\mathrm{grad}\, f) \ge -1$, i.e. for every sequence $\{z_m\} \subset \mathbb{K}^n$ with $z_m \to \infty$ and $f(z_m) \to t_0$ we have $\|z_m\|\,\|\mathrm{grad}\, f(z_m)\| \nrightarrow 0$.

(3) We say that t_0 is an *asymptotic critical value* if it does not satisfy the Malgrange condition.

Theorem 3.7 gives us a simple way to check whether a given value $t_0 \in \mathbb{R}$ satisfies (F) or (M) or not. In fact, with the notation of Theorem 3.7, it is enough to compute

$$\min_i \{\deg \operatorname{grad} f(x_i^{\mathbb{R}}(y), y) \colon x_i(y) \in V_{\mathbb{R}}(f, t_0)\},$$

and to compare it with 0 or -1.

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