Uniqueness of entire functions and their derivatives

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Abstract. We study the uniqueness of entire functions which share a value or a function with their first and second derivatives.

1. Introduction, definitions and results. Let \( f \) be a non-constant meromorphic function in the open complex plane \( \mathbb{C} \). A meromorphic function \( a = a(z) \) is called a small function of \( f \) if \( T(r,a) = S(r,f) \), where \( T(r,f) \) is the Nevanlinna characteristic function of \( f \) and \( S(r,f) = o\{T(r,f)\} \) as \( r \to \infty \) possibly outside a set of finite linear measure. Also we denote by \( E(a;f) \) the set of distinct zeros of \( f - a \).

The problem of uniqueness of meromorphic functions sharing values with their derivatives is a special case of the uniqueness theory of meromorphic functions. This problem was initiated by Rubel and Yang [4] with the following result.

**Theorem A** ([4]). Let \( f \) be a non-constant entire function. If \( f \) and \( f' \) share the values \( a \) and \( b \) counting multiplicities then \( f \equiv f' \).

Considering \( f = e^{ze^{-1}} \int_0^z e^{-e^t(1-e^t)} dt \) we see that \( f' - 1 = e^z(f - 1) \) and so the condition that \( f \) and \( f' \) share two values is essential for Theorem A. In 1986 Jank, Mues and Volkman [3] considered the problem of sharing a single value by the derivatives of an entire function and proved the following result.

**Theorem B** ([3]). Let \( f \) be a non-constant entire function and \( a \) (\( \neq 0 \)) be a finite number. If \( E(a;f) = E(a;f') \) and \( E(a;f) \subset E(a;f'') \) then \( f \equiv f' \).

In 2002 Chang and Fang [1] extended Theorem B and proved the following result.
Theorem C ([1]). Let \( f \) be a non-constant entire function. If \( E(z; f) = E(z; f') \) and \( E(z; f') \subset E(z; f'') \), then \( f \equiv f' \).

The purpose of the paper is to further extend Theorem C and prove the following theorem.

Theorem 1.1. Let \( f \) be a non-constant entire function and \( a(z) = \alpha z + \beta \), where \( \alpha \neq 0 \) and \( \beta \) are constants. If \( E(a; f) \subset E(a; f') \) and \( E(a; f') \subset E(a; f'') \), then either \( f = A \exp\{z\} \) or

\[
f = \alpha z + \beta + (\alpha z + \beta - 2\alpha) \exp\left\{\frac{\alpha z + \beta - 2\alpha}{\alpha}\right\},
\]

where \( A \) is a non-zero constant.

Corollary 1.1. If in Theorem 1.1 we assume \( E(a; f) = E(a; f') \), then \( f = A \exp\{z\} \), where \( A \) is a non-zero constant.

Let \( f, g, a \) and \( b \) be meromorphic functions in \( \mathbb{C} \). We denote by \( N(r, a; f | g \neq b) \) the integrated counting functions of those zeros of \( f - a \) (counted with multiplicities) which are not the zeros of \( g - b \).

For the standard definitions and notations of value distribution theory we refer the reader to [2].

2. Lemma. In this section we prove a lemma which is required to prove the theorem.

Lemma 2.1. Let \( f \) be a transcendental entire function and \( a = a(z) \) \((\neq 0, \infty)\) be a non-constant small function of \( f \) such that \( E(a; f) \subset E(a; f') \) and \( E(a; f') \subset E(a; f'') \). Then \( f = A \exp\{z\} \) if and only if \( m(r, 1/(f - a)) = S(r, f) \), where \( A \) is a non-zero constant.

Proof. Since the “only if” part easily follows from Nevanlinna’s three small functions theorem, we prove the “if” part.

We suppose that

\[
(2.1) \quad m\left(r, \frac{1}{f - a}\right) = S(r, f).
\]

Let

\[
\phi = \frac{f'' - f'}{f - a} \quad \text{and} \quad \psi = \frac{(a - a')(f' - a') - a(f'' - a'')}{f - a}.
\]

Also set \( E = \{z : (a(z) - a'(z))(a(z) - a''(z)) = 0\} \). Since a zero of \( f - a \) which does not belong to \( E \) is a simple zero, it is not a pole of \( \phi \) and \( \psi \). Hence \( N(r, \phi) = S(r, f) \) and \( N(r, \psi) = S(r, f) \). Also for any positive integer
From (2.2) and (2.3) we get
\[ \psi \]
Hence
\[ \phi \]
Hence
\[ z \]
Also in some neighbourhood of \( A \) we get, by (2.1),
\[ f = \exp \{ z \} + B, \]
where \( a_1 = 2a_2 = a(z_0) \) and \( 6a_3 = f^{(3)}(z_0) \).
So in some neighbourhood of \( z_0 \) we get
\[ f = a(z_0) + a_1(z - z_0) + a_2(z - z_0)^2 + a_3(z - z_0)^3 + O((z - z_0)^4), \]
\[ f' = a_1 + 2a_2(z - z_0) + 3a_3(z - z_0)^2 + O((z - z_0)^3), \]
\[ f'' = 2a_2 + 6a_3(z - z_0) + O((z - z_0)^2), \]
where \( a_1 = 2a_2 = a(z_0) \) and \( 6a_3 = f^{(3)}(z_0) \).

Case I. Let \( \phi \equiv 0 \). Then \( f' \equiv f'' \) and so \( f = A \exp \{ z \} + B \), where \( A \neq 0 \) and \( B \) are constants. Hence \( f = f' + B \). By (2.1) there exists \( z_1 \) such that \( a(z_1) \neq \infty \) and \( a(z_1) = a(z_1) + B \) and so \( B = 0 \). Therefore \( f = A \exp \{ z \} \).

Case II. Let \( \phi \neq 0 \). Let \( z_0 \) be a zero of \( f - a \) and \( z_0 \not\in E \). Then in some neighbourhood of \( z_0 \) we get
\[ f = a(z_0) + a_1(z - z_0) + a_2(z - z_0)^2 + a_3(z - z_0)^3 + O((z - z_0)^4), \]
\[ f' = a_1 + 2a_2(z - z_0) + 3a_3(z - z_0)^2 + O((z - z_0)^3), \]
\[ f'' = 2a_2 + 6a_3(z - z_0) + O((z - z_0)^2), \]
where \( a_1 = 2a_2 = a(z_0) \) and \( 6a_3 = f^{(3)}(z_0) \).
So in some neighbourhood of \( z_0 \) we obtain
\[ \phi = \frac{(6a_3 - 2a_2)(z - z_0) + O((z - z_0)^2)}{a(z_0) - a(z) + a_1(z - z_0) + O((z - z_0)^2)} \]
\[ = \frac{(6a_3 - 2a_2)(z - z_0) + O((z - z_0)^2)}{(a_1 - a'(z_0) + o(1))(z - z_0) + O((z - z_0)^2)} \]
\[ = \frac{6a_3 - 2a_2 + O(z - z_0)}{a_1 - a'(z_0) + o(1) + O(z - z_0)}. \]

Hence
\[ \phi(z_0) = \frac{f^{(3)}(z_0) - a(z_0)}{a(z_0) - a'(z_0) + o(1) + O(z - z_0)}. \]

Also in some neighbourhood of \( z_0 \) we get
\[ \psi = \frac{a'(z)a'(z_0) + f^{(3)}(z_0)(a(z) - a'(z)) - a(z)a(z_0) + O(z - z_0)}{a(z_0) - a'(z_0) + o(1) + O(z - z_0)}, \]
\[ \psi(z_0) = f^{(3)}(z_0) - a(z_0) - a'(z_0). \]

Hence
\[ \psi(z_0) = f^{(3)}(z_0) - a(z_0) - a'(z_0). \]

From (2.2) and (2.3) we get
\[ \{ a(z_0) - a'(z_0) \} \phi(z_0) - \psi(z_0) - a'(z_0) = 0. \]
If \((a - a')\phi - \psi - a' \neq 0\), then from (2.4) we get
\[
N\left( r, \frac{1}{f - a} \right) \leq N(r, 0; (a - a')\phi - \psi - a') + S(r, f) = S(r, f),
\]
which together with (2.1) implies \(T(r, f) = S(r, f)\), a contradiction. Therefore
\[
(a - a')\phi - \psi \equiv a'. \tag{2.5}
\]

First we suppose that \(\psi \equiv 0\). Then from (2.5) and the definitions of \(\phi\) and \(\psi\) we get
\[
(a - a') \frac{f'' - f'}{f - a} \equiv a' \tag{2.6}
\]
and
\[
(a - a')f'' \equiv a(f' - a'). \tag{2.7}
\]
From (2.6) and (2.7) we obtain \(f \equiv f'\) and so \(\phi \equiv 0\), which is a contradiction.

Next we suppose that \(\psi \not\equiv 0\). Then from (2.5) and the definitions of \(\phi\) and \(\psi\) we get
\[
(a - a') \frac{f'' - f'}{f - a} - \frac{(a - a')f'' - a(f' - a')}{f - a} \equiv a' \tag{2.5}
\]
This implies \(f \equiv f'\) and so \(\phi \equiv 0\), which is a contradiction. This proves the lemma.

3. Proofs of the theorem and corollary. In this section we prove the main result of the paper.

Proof of Theorem 1.1. First we suppose that \(f\) is a polynomial and consider the following cases.

Case I. Let \(f = Az + B\), where \(A \neq 0\) and \(B\) are constants. If \(z_0\) is a zero of \(f - a\), then by the hypotheses \(z_0\) is also a zero of \(f' - a\) and \(f'' - a\). Hence \(A = a(z_0) = 0\), a contradiction.

Case II. Let \(f = Az^2 + Bz + C\), where \(A \neq 0\), \(B\) and \(C\) are constants. If \(f(z) - a(z) = 0\) has two distinct roots, then \(E(a; f) \subset E(a; f')\) implies that \(f'(z) \equiv a(z)\). Again since \(E(a; f') \subset E(a; f'')\), we arrive at a contradiction. So \(f(z) - a(z) = 0\) has only one double root. Also \(E(a; f) \subset E(a; f')\) implies that if this root is \(z_0\) then \(a(z_0) = a'(z_0)\) and so \(z_0 = (\alpha - \beta)/\alpha\). Since \(f''(z_0) = a(z_0)\), we get \(\alpha = 2A\). Also \(f'(z_0) = a(z_0)\) implies \(B = \beta\) and \(f(z_0) = a(z_0)\) implies \(C = (\alpha^2 + \beta^2)/2\alpha\). Therefore
\[
f(z) = \frac{\alpha}{2} z^2 + \beta z + \frac{\alpha^2 + \beta^2}{2\alpha}
\]
and so \(f'(z) \equiv a(z)\). Since \(E(a; f') \subset E(a; f'')\), we arrive at a contradiction.
Hence simple roots of \( f \) \( \subset \mathbb{Q} \) and \( p \) is impossible. So \( z \) is greater than two, then it is a root of \( \mathbb{Q} \) and \( \deg \) must be a root of all be simple. By the hypotheses we see that a multiple root of \( \mathbb{Q} \) arrive at a contradiction. Therefore \( \psi \equiv 0 \).

Also by the hypotheses

\[
f'(z) = a(z) + B(z - z_1)^{q_1} \cdots (z - z_n)^{q_n} Q(z)
\]

and

\[
f''(z) = a(z) + C(z - z_1)^{r_1} \cdots (z - z_n)^{r_n} Q(z) R(z),
\]

where \( Q, R \) are polynomials such that \( q_1 + \cdots + q_n + \deg Q = d - 1, r_1 + \cdots + r_n + \deg Q + \deg R = d - 2 \) and \( B \neq 0 \), \( C \) are constants.

First we suppose that \( f(z) = 0 \) for \( \mathbb{Q} = \mathbb{C} \). Hence \( \psi = 0 \) if \( \deg \psi = 1 \)…

\[
f''(z) = \frac{\alpha^2}{6} z^3 + \beta \frac{1}{2} z^2 + \gamma z + \delta
\]

and

\[
f'(z) = \frac{\alpha}{2} z^2 + \beta z + \gamma,
\]

where \( \gamma, \delta \) are constants. Since \( E(a; f) \subset E(a; f') \), we see that \( f(z) - a(z) = 0 \) must have one multiple root, say \( z_0 \). If its multiplicity is three, then by the hypotheses we have \( a(z_0) = a'(z_0) = a''(z_0) \), which is impossible because \( \alpha \neq 0 \). So \( f(z) - a(z) = 0 \) has one double root and it is a root of \( a(z) - a'(z) = 0 \). Hence \( z = (\alpha - \beta)/\alpha \) is a multiple root of \( f(z) - a(z) = 0 \). Also it is a root of \( f'(z) - a(z) = 0 \) and so \( \gamma = (\alpha^2 + \beta^2)/2\alpha \).

\[
f'(z) - a(z) = \frac{\alpha}{2} \left( z - \frac{\alpha - \beta}{\alpha} \right)^2.
\]

Since \( E(a; f) \subset E(a; f') \) and \( f(z) - a(z) = 0 \) has two distinct roots, we arrive at a contradiction. Therefore \( C \neq 0 \).

Since \( E(a; f) \subset E(a; f') \), we see that the roots of \( f(z) - a(z) = 0 \) cannot all be simple. By the hypotheses we see that a multiple root of \( f(z) - a(z) = 0 \) must be a root of \( a(z) - a'(z) = 0 \) and so it is \( (\alpha - \beta)/\alpha \). If its multiplicity is greater than two, then it is a root of \( a(z) - a''(z) = 0 \) and so \( \alpha = 0 \), which is impossible. So \( z = (\alpha - \beta)/\alpha \) is a double root of \( f(z) - a(z) = 0 \). Without loss of generality we put \( z_1 = (\alpha - \beta)/\alpha \) and \( p_1 = 2 \). Then \( z_2, \ldots, z_n \) are all simple roots of \( f(z) - a(z) = 0 \). Therefore \( d = n + 1 \) and so \( q_1 = \cdots = q_n = 1 \) and \( \deg Q = 0 \). Since \( E(a; f') \subset E(a; f'') \), we get \( r_j \geq 1 \) for \( j = 1, \ldots, n \). Hence \( n + \deg R \leq r_1 + \cdots + r_n + \deg R = n - 1 \), which is a contradiction.

Therefore \( f \) is a transcendental entire function. Let

\[
\psi = \frac{(a - a') f'' - a (f' - a')}{f - a}.
\]

If \( \psi \equiv 0 \), then

\[
\frac{f''}{f' - \alpha} \equiv 1 + \frac{\alpha}{\alpha z + \beta - \alpha}.
\]
This gives on integration \( f' = \alpha + A(\alpha z + \beta - \alpha) \exp \{z\} \) and \( f = \alpha z + A(\alpha z + \beta - 2\alpha) \exp \{z\} + B \), where \( A \neq 0 \) and \( B \) are constants. Also \( f'' = A(\alpha z + \beta) \exp \{z\} \). Since \( E(a; f) \subset E(a; f') \) and \( E(a; f') \subset E(a; f''), \)
we see that \( f(z) - a(z) = 0 \) has the unique solution \( z_0 = (2\alpha - B)/\alpha \). Also
\( f(z) - (\alpha z + B) = 0 \) has only one solution \( z_1 = (2\alpha - \beta)/\alpha \). Hence by
Nevanlinna’s three small functions theorem we get \( B = \beta \). So
\[
    f = \alpha z + \beta + A(\alpha z + \beta - 2\alpha) \exp \{z\}.
\]
Also since \( E(a; f) \subset E(a; f') \), it follows that \( \alpha + A(\alpha z_0 + \beta - \alpha) \exp \{z_0\} = \alpha z_0 + \beta \) and so \( A = \exp \{\beta - 2\alpha/\alpha\} \). Therefore
\[
    f = \alpha z + \beta + (\alpha z + \beta - 2\alpha) \exp \left\{ \frac{\alpha z + \beta - 2\alpha}{\alpha} \right\}.
\]
Now we suppose that \( \psi \neq 0 \). Then
\[
    f - a \equiv \frac{1}{\psi} [(a - a') f'' - a(f' - a')]
\]
and so
\[
    (3.1) \quad \left[ 1 + a \left( \frac{1}{\psi} \right)' + \frac{a'}{\psi} \right] (f' - a) \equiv (a' - a) \left[ 1 + \left( \frac{1}{\psi} \right)' (a - a') + \frac{2a'}{\psi} \right] \\
    \quad \quad + (a' - a) \left[ \frac{1}{\psi} - \left( \frac{1}{\psi} \right)' \right] (f'' - a') - (a' - a) \frac{f'''}{\psi}.
\]
Let
\[
    \Delta = 1 + \left( \frac{1}{\psi} \right)' (a - a') + \frac{2a'}{\psi} \equiv 0.
\]
Then
\[
    (3.2) \quad \psi^2 + 2\alpha \psi \equiv \psi'(\alpha z + \beta - \alpha).
\]
If \( \psi \) is transcendental, then from (3.2) we get
\[
    T(r, \psi) = m(r, \psi) + N(r, \psi) \leq m(r, \psi' / \psi) + O(\log r) = S(r, \psi),
\]
a contradiction.

Hence \( \psi \) is a rational function. If \( \psi \) has a pole, then by the hypotheses
we see that \( z = (\alpha - \beta)/\alpha \) is the only pole of \( \psi \). If \( p \) is its multiplicity, then
from (3.2) we get \( 2p = p \). So \( \psi \) has no pole at all. If \( n \) is the degree of \( \psi \),
then from (3.2) we get \( 2n = n \) and so \( n = 0 \). Hence \( \psi \) is a constant and from
(3.2) we get \( \psi = -2\alpha \). Therefore
\[
    (\alpha z + \beta - \alpha) f'' - (\alpha z + \beta)(f' - \alpha) + 2\alpha (f - \alpha z - \beta) \equiv 0.
\]
Differentiating twice we get
\[
    \frac{f^{(4)}}{f^{(3)}} = 1 - \frac{\alpha}{\alpha z + \beta - \alpha}.
\]
On integration we obtain
\[ f^{(3)} = \frac{A}{\alpha z + \beta - \alpha} \exp\{z\}, \]
where \( A (\neq 0) \) is a constant. This is impossible because \( f \) is entire. Therefore \( \Delta \neq 0 \) and so from (3.1) we get
\[ \frac{1}{f' - a} = \frac{1 + a\left(\frac{1}{\psi}\right)' + a'\frac{1}{\psi} - \frac{(1/\psi)'}{\Delta}}{(a' - a)\Delta} - \frac{f'' - a'}{f' - a} + \frac{1}{\psi\Delta} \cdot \frac{f'''}{f' - a}. \]
Since \( T(r, \psi) = S(r, f) \) and \( f \) is transcendental, we get
\[ (3.3) \quad m\left(r, \frac{1}{f' - a}\right) = S(r, f). \]
By the hypotheses we see that \( z = (\alpha - \beta)/\alpha \) is the only possible multiple (actually double) zero of \( f' - a \). So \( N(r, a; f' \mid f \neq a) \leq N(r, 0; \psi) + O(\log r) = S(r, f) \). Therefore
\[ (3.4) \quad N(r, a; f') = N(r, a; f) + N(r, a; f' \mid f \neq a) + O(\log r) = N(r, a; f) + S(r, f). \]
Again since \( f \) is entire and
\[ f = a + \frac{f' - a'}{\psi} \left[ (a - a') \cdot \frac{f''}{f' - a'} - a \right], \]
we get
\[ T(r, f) = m(r, f) \leq m(r, f' - a') + S(r, f) \leq m(r, f') + S(r, f) = T(r, f') + S(r, f). \]
Also
\[ T(r, f') = m(r, f') \leq m(r, f) + m(r, f' / f) = T(r, f) + S(r, f). \]
Therefore
\[ (3.5) \quad T(r, f) = T(r, f') + S(r, f). \]
From (3.3)–(3.5) we get
\[ m\left(r, \frac{1}{f - a}\right) = T(r, f) - N\left(r, \frac{1}{f - a}\right) + S(r, f) = T(r, f') - N\left(r, \frac{1}{f' - a}\right) + S(r, f) = N\left(r, \frac{1}{f' - a}\right) - N\left(r, \frac{1}{f - a}\right) + S(r, f) = S(r, f). \]
Therefore by Lemma 2.1 we get \( f = A \exp\{z\} \). This proves the theorem. \( \blacksquare \)
Proof of Corollary 1.1. If
\[ f = (\alpha z + \beta) + (\alpha z + \beta - 2\alpha) \exp \left\{ \frac{\alpha z + \beta - 2\alpha}{\alpha} \right\}, \]
then we see that \( E(a; f) \) contains only one element but \( E(a; f') \) contains infinitely many elements. This contradicts the hypothesis \( E(a; f) = E(a; f') \). Therefore by Theorem 1.1 we get \( f = A \exp\{z\} \). This proves the corollary. ■

References


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