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Uniqueness of entire functions and their derivatives

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Abstract. We study the uniqueness of entire functions which share a value or a function with their first and second derivatives.

1. Introduction, definitions and results. Let f be a non-constant meromorphic function in the open complex plane $\mathbb{C}$. A meromorphic function $a = a(z)$ is called a *small function* of f if $T(r,a) = S(r,f)$, where $T(r,f)$ is the Nevanlinna characteristic function of f and $S(r,f) = o\{T(r,f)\}$ as $r \to \infty$ possibly outside a set of finite linear measure. Also we denote by $E(a;f)$ the set of distinct zeros of $f - a$.

The problem of uniqueness of meromorphic functions sharing values with their derivatives is a special case of the uniqueness theory of meromorphic functions. This problem was initiated by Rubel and Yang [4] with the following result.

Theorem A ([4]). *Let f be a non-constant entire function. If f and f' share the values a and b counting multiplicities then $f \equiv f'$.*

Considering $f = e^{e^z} \int_0^z e^{-e^t}(1-e^t)\,dt$ we see that $f' - 1 = e^z(f-1)$ and so the condition that f and f' share two values is essential for Theorem A. In 1986 Jank, Mues and Volkman [3] considered the problem of sharing a single value by the derivatives of an entire function and proved the following result.

Theorem B ([3]). *Let f be a non-constant entire function and a $(\neq 0)$ be a finite number. If $E(a;f) = E(a;f')$ and $E(a;f) \subset E(a;f'')$ then $f \equiv f'$.*

In 2002 Chang and Fang [1] extended Theorem B and proved the following result.

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Theorem C ([1]). *Let f be a non-constant entire function. If $E(z;f) = E(z;f')$ and $E(z;f') \subset E(z;f'')$, then $f \equiv f'$.*

The purpose of the paper is to further extend Theorem C and prove the following theorem.

Theorem 1.1. *Let f be a non-constant entire function and $a(z) = \alpha z + \beta$, where α $(\neq 0)$ and β are constants. If $E(a;f) \subset E(a;f')$ and $E(a;f') \subset E(a;f'')$, then either $f = A\exp\{z\}$ or*

$$f = \alpha z + \beta + (\alpha z + \beta - 2\alpha)\exp\left\{\frac{\alpha z + \beta - 2\alpha}{\alpha}\right\},$$

where A is a non-zero constant.

Corollary 1.1. *If in Theorem 1.1 we assume $E(a;f) = E(a;f')$, then $f = A\exp\{z\}$, where A is a non-zero constant.*

Let f, g, a and b be meromorphic functions in $\mathbb{C}$. We denote by $N(r,a;f \mid g \neq b)$ the integrated counting functions of those zeros of $f-a$ (counted with multiplicities) which are not the zeros of $g-b$.

For the standard definitions and notations of value distribution theory we refer the reader to [2].

2. Lemma. In this section we prove a lemma which is required to prove the theorem.

Lemma 2.1. *Let f be a transcendental entire function and $a = a(z)$ $(\not\equiv 0,\infty)$ be a non-constant small function of f such that $E(a;f) \subset E(a;f')$ and $E(a;f') \subset E(a;f'')$. Then $f = A\exp\{z\}$ if and only if $m(r,1/(f-a)) = S(r,f)$, where A is a non-zero constant.*

Proof. Since the "only if" part easily follows from Nevanlinna's three small functions theorem, we prove the "if" part.

We suppose that

$$m\left(r,\frac{1}{f-a}\right) = S(r,f). \tag{2.1}$$

Let

$$\phi = \frac{f''-f'}{f-a} \quad \text{and} \quad \psi = \frac{(a-a')f''-a(f'-a')}{f-a}.$$

Also set $E = \{z : (a(z)-a'(z))(a(z)-a''(z)) = 0\}$. Since a zero of $f-a$ which does not belong to E is a simple zero, it is not a pole of ϕ and ψ. Hence $N(r,\phi) = S(r,f)$ and $N(r,\psi) = S(r,f)$. Also for any positive integer

p we get, by (2.1),

$$\begin{aligned} m\left(r, \frac{f^{(p)}}{f-a}\right) &= m\left(r, \frac{f^{(p)}-a^{(p)}}{f-a} + \frac{a^{(p)}}{f-a}\right) \\ &\le m\left(r, \frac{f^{(p)}-a^{(p)}}{f-a}\right) + m\left(r, \frac{1}{f-a}\right) + m(r, a^{(p)}) + O(1) \\ &= S(r,f). \end{aligned}$$

Hence $m(r,\phi) = S(r,f)$ and $m(r,\psi) = S(r,f)$. Therefore $T(r,\phi) = S(r,f)$ and $T(r,\psi) = S(r,f)$. We now consider the following two cases.

Case I. Let $\phi \equiv 0$. Then $f' \equiv f''$ and so $f = A\exp\{z\} + B$, where A ($\neq 0$) and B are constants. Hence $f = f' + B$. By (2.1) there exists z_1 such that $a(z_1) \neq \infty$ and $a(z_1) = a(z_1) + B$ and so $B = 0$. Therefore $f = A\exp\{z\}$.

Case II. Let $\phi \not\equiv 0$. Let z_0 be a zero of $f - a$ and $z_0 \notin E$. Then in some neighbourhood of z_0 we get

$$\begin{aligned} f &= a(z_0) + a_1(z-z_0) + a_2(z-z_0)^2 + a_3(z-z_0)^3 + O((z-z_0)^4), \\ f' &= a_1 + 2a_2(z-z_0) + 3a_3(z -' z_0)^2 + O((z-z_0)^3), \\ f'' &= 2a_2 + 6a_3(z-z_0) + O((z-z_0)^2), \end{aligned}$$

where $a_1 = 2a_2 = a(z_0)$ and $6a_3 = f^{(3)}(z_0)$.

So in some neighbourhood of z_0 we obtain

$$\begin{aligned} \phi &= \frac{(6a_3 - 2a_2)(z-z_0) + O((z-z_0)^2)}{a(z_0) - a(z) + a_1(z - z_0 + O((z-z_0)^2)} \\ &= \frac{(6a_3 - 2a_2)(z-z_0) + O((z-z_0)^2)}{(a_1 - a'(z_0) + o(1))(z-z_0) + O((z-z_0)^2)} \\ &= \frac{6a_3 - 2a_2 + O(z-z_0)}{a_1 - a'(z_0) + o(1) + O(z-z_0)}. \end{aligned}$$

Hence

$$\phi(z_0) = \frac{f^{(3)}(z_0) - a(z_0)}{a(z_0) - a'(z_0)}. \tag{2.2}$$

Also in some neighbourhood of z_0 we get

$$\psi = \frac{a'(z)a'(z_0) + f^{(3)}(z_0)(a(z) - a'(z)) - a(z)a(z_0) + O(z-z_0)}{a(z_0) - a'(z_0) + o(1) + O(z-z_0)}.$$

Hence

$$\psi(z_0) = f^{(3)}(z_0) - a(z_0) - a'(z_0). \tag{2.3}$$

From (2.2) and (2.3) we get

$$\{a(z_0) - a'(z_0)\}\phi(z_0) - \psi(z_0) - a'(z_0) = 0. \tag{2.4}$$

If $(a-a')\phi-\psi-a'\not\equiv 0$, then from (2.4) we get

$$N\left(r,\frac{1}{f-a}\right)\le N(r,0;(a-a')\phi-\psi-a')+S(r,f)=S(r,f),$$

which together with (2.1) implies $T(r,f)=S(r,f)$, a contradiction. Therefore

$$(a-a')\phi-\psi\equiv a'. \tag{2.5}$$

First we suppose that $\psi\equiv 0$. Then from (2.5) and the definitions of ϕ and ψ we get

$$(a-a')\,\frac{f''-f'}{f-a}\equiv a' \tag{2.6}$$

and

$$(a-a')f''\equiv a(f'-a'). \tag{2.7}$$

From (2.6) and (2.7) we obtain $f\equiv f'$ and so $\phi\equiv 0$, which is a contradiction.

Next we suppose that $\psi\not\equiv 0$. Then from (2.5) and the definitions of ϕ and ψ we get

$$(a-a')\,\frac{f''-f'}{f-a}-\frac{(a-a')f''-a(f'-a')}{f-a}\equiv a'.$$

This implies $f\equiv f'$ and so $\phi\equiv 0$, which is a contradiction. This proves the lemma. ∎

3. Proofs of the theorem and corollary. In this section we prove the main result of the paper.

Proof of Theorem 1.1. First we suppose that f is a polynomial and consider the following cases.

Case I. Let $f=Az+B$, where A $(\neq 0)$ and B are constants. If z_0 is a zero of $f-a$, then by the hypotheses z_0 is also a zero of $f'-a$ and $f''-a$. Hence $A=a(z_0)=0$, a contradiction.

Case II. Let $f=Az^2+Bz+C$, where A $(\neq 0)$, B and C are constants. If $f(z)-a(z)=0$ has two distinct roots, then $E(a;f)\subset E(a;f')$ implies that $f'(z)\equiv a(z)$. Again since $E(a;f')\subset E(a;f'')$, we arrive at a contradiction. So $f(z)-a(z)=0$ has only one double root. Also $E(a;f)\subset E(a;f')$ implies that if this root is z_0 then $a(z_0)=a'(z_0)$ and so $z_0=(\alpha-\beta)/\alpha$. Since $f''(z_0)=a(z_0)$, we get $\alpha=2A$. Also $f'(z_0)=a(z_0)$ implies $B=\beta$ and $f(z_0)=a(z_0)$ implies $C=(\alpha^2+\beta^2)/2\alpha$. Therefore

$$f(z)=\frac{\alpha}{2}\,z^2+\beta z+\frac{\alpha^2+\beta^2}{2\alpha}$$

and so $f'(z)\equiv a(z)$. Since $E(a;f')\subset E(a;f'')$, we arrive at a contradiction.

Case III. Let f be a polynomial of degree d (≥ 3). If $z_1, \ldots, z_n$ are the roots of the equation $f(z) - a(z) = 0$, we can write

$$f(z) = a(z) + A(z - z_1)^{p_1} \cdots (z - z_n)^{p_n},$$

where $p_1 + \cdots + p_n = d$ and A ($\neq 0$) is a constant.

Also by the hypotheses

$$f'(z) = a(z) + B(z - z_1)^{q_1} \cdots (z - z_n)^{q_n} Q(z)$$

and

$$f''(z) = a(z) + C(z - z_1)^{r_1} \cdots (z - z_n)^{r_n} Q(z) R(z),$$

where Q, R are polynomials such that $q_1 + \cdots + q_n + \deg Q = d - 1$, $r_1 + \cdots + r_n + \deg Q + \deg R = d - 2$ and B ($\neq 0$), C are constants.

First we suppose that $C = 0$. Then $f''(z) \equiv a(z)$ and so

$$f(z) = \frac{\alpha^2}{6} z^3 + \frac{\beta}{2} z^2 + \gamma z + \delta \quad \text{and} \quad f'(z) = \frac{\alpha}{2} z^2 + \beta z + \gamma,$$

where γ, δ are constants. Since $E(a; f) \subset E(a; f')$, we see that $f(z) - a(z) = 0$ must have one multiple root, say z_0. If its multiplicity is three, then by the hypotheses we have $a(z_0) = a'(z_0) = a''(z_0)$, which is impossible because $\alpha \neq 0$. So $f(z) - a(z) = 0$ has one double root and it is a root of $a(z) - a'(z) = 0$. Hence $z = (\alpha - \beta)/\alpha$ is a double root of $f(z) - a(z) = 0$. Also it is a root of $f'(z) - a(z) = 0$ and so $\gamma = (\alpha^2 + \beta^2)/2\alpha$. Hence

$$f'(z) - a(z) = \frac{\alpha}{2} \left(z - \frac{\alpha - \beta}{\alpha} \right)^2.$$

Since $E(a; f) \subset E(a; f')$ and $f(z) - a(z) = 0$ has two distinct roots, we arrive at a contradiction. Therefore $C \neq 0$.

Since $E(a; f) \subset E(a; f')$, we see that the roots of $f(z) - a(z) = 0$ cannot all be simple. By the hypotheses we see that a multiple root of $f(z) - a(z) = 0$ must be a root of $a(z) - a'(z) = 0$ and so it is $(\alpha - \beta)/\alpha$. If its multiplicity is greater than two, then it is a root of $a(z) - a''(z) = 0$ and so $\alpha = 0$, which is impossible. So $z = (\alpha - \beta)/\alpha$ is a double root of $f(z) - a(z) = 0$. Without loss of generality we put $z_1 = (\alpha - \beta)/\alpha$ and $p_1 = 2$. Then $z_2, \ldots, z_n$ are all simple roots of $f(z) - a(z) = 0$. Therefore $d = n + 1$ and so $q_1 = \cdots = q_n = 1$ and $\deg Q = 0$. Since $E(a; f') \subset E(a; f'')$, we get $r_j \geq 1$ for $j = 1, \ldots, n$. Hence $n + \deg R \leq r_1 + \cdots + r_n + \deg R = n - 1$, which is a contradiction.

Therefore f is a transcendental entire function. Let

$$\psi = \frac{(a - a')f'' - a(f' - a')}{f - a}.$$

If $\psi \equiv 0$, then

$$\frac{f''}{f' - \alpha} \equiv 1 + \frac{\alpha}{\alpha z + \beta - \alpha}.$$

This gives on integration $f' = \alpha + A(\alpha z + \beta - \alpha)\exp\{z\}$ and $f = \alpha z + A(\alpha z + \beta - 2\alpha)\exp\{z\} + B$, where A ($\neq 0$) and B are constants. Also $f'' = A(\alpha z + \beta)\exp\{z\}$. Since $E(a; f) \subset E(a; f')$ and $E(a; f') \subset E(a; f'')$, we see that $f(z) - a(z) = 0$ has the unique solution $z_0 = (2\alpha - B)/\alpha$. Also $f(z) - (\alpha z + B) = 0$ has only one solution $z_1 = (2\alpha - \beta)/\alpha$. Hence by Nevanlinna's three small functions theorem we get $B = \beta$. So

$$f = \alpha z + \beta + A(\alpha z + \beta - 2\alpha)\exp\{z\}.$$

Also since $E(a; f) \subset E(a; f')$, it follows that $\alpha + A(\alpha z_0 + \beta - \alpha)\exp\{z_0\} = \alpha z_0 + \beta$ and so $A = \exp\{(\beta - 2\alpha)/\alpha\}$. Therefore

$$f = \alpha z + \beta + (\alpha z + \beta - 2\alpha)\exp\left\{\frac{\alpha z + \beta - 2\alpha}{\alpha}\right\}.$$

Now we suppose that $\psi \not\equiv 0$. Then

$$f - a \equiv \frac{1}{\psi}\left[(a - a')f'' - a(f' - a')\right]$$

and so

$$\left[1 + a\left(\frac{1}{\psi}\right)' + \frac{a'}{\psi}\right](f' - a) \equiv (a' - a)\left[1 + \left(\frac{1}{\psi}\right)'(a - a') + \frac{2a'}{\psi}\right] + (a' - a)\left[\frac{1}{\psi} - \left(\frac{1}{\psi}\right)'\right](f'' - a') - (a' - a)\frac{f'''}{\psi}. \tag{3.1}$$

Let

$$\Delta = 1 + \left(\frac{1}{\psi}\right)'(a - a') + \frac{2a'}{\psi} \equiv 0.$$

Then

$$\psi^2 + 2\alpha\psi \equiv \psi'(\alpha z + \beta - \alpha). \tag{3.2}$$

If ψ is transcendental, then from (3.2) we get

$$T(r, \psi) = m(r, \psi) + N(r, \psi) \le m(r, \psi'/\psi) + O(\log r) = S(r, \psi),$$

a contradiction.

Hence ψ is a rational function. If ψ has a pole, then by the hypotheses we see that $z = (\alpha - \beta)/\alpha$ is the only pole of ψ. If p is its multiplicity, then from (3.2) we get $2p = p$. So ψ has no pole at all. If n is the degree of ψ, then from (3.2) we get $2n = n$ and so $n = 0$. Hence ψ is a constant and from (3.2) we get $\psi = -2\alpha$. Therefore

$$(\alpha z + \beta - \alpha)f'' - (\alpha z + \beta)(f' - \alpha) + 2\alpha(f - \alpha z - \beta) \equiv 0.$$

Differentiating twice we get

$$\frac{f^{(4)}}{f^{(3)}} = 1 - \frac{\alpha}{\alpha z + \beta - \alpha}.$$

On integration we obtain

$$f^{(3)} = \frac{A}{\alpha z + \beta - \alpha} \exp\{z\},$$

where A ($\neq 0$) is a constant. This is impossible because f is entire. Therefore $\Delta \not\equiv 0$ and so from (3.1) we get

$$\frac{1}{f' - a} \equiv \frac{1 + a\left(\frac{1}{\psi}\right)' + \frac{a'}{\psi}}{(a' - a)\Delta} - \frac{\frac{1}{\psi} - \left(\frac{1}{\psi}\right)'}{\Delta} \cdot \frac{f'' - a'}{f' - a} + \frac{1}{\psi\Delta} \cdot \frac{f'''}{f' - a}.$$

Since $T(r, \psi) = S(r, f)$ and f is transcendental, we get

$$m\left(r, \frac{1}{f' - a}\right) = S(r, f). \tag{3.3}$$

By the hypotheses we see that $z = (\alpha - \beta)/\alpha$ is the only possible multiple (actually double) zero of $f' - a$. So $N(r, a; f' \mid f \neq a) \leq N(r, 0; \psi) + O(\log r) = S(r, f)$. Therefore

$$\begin{aligned} N(r, a; f') &= N(r, a; f) + N(r, a; f' \mid f \neq a) + O(\log r) \\ &= N(r, a; f) + S(r, f). \end{aligned} \tag{3.4}$$

Again since f is entire and

$$f = a + \frac{f' - a'}{\psi}\left[(a - a')\,\frac{f''}{f' - a'} - a\right],$$

we get

$$\begin{aligned} T(r, f) = m(r, f) &\leq m(r, f' - a') + S(r, f) \\ &\leq m(r, f') + S(r, f) = T(r, f') + S(r, f). \end{aligned}$$

Also

$$T(r, f') = m(r, f') \leq m(r, f) + m(r, f'/f) = T(r, f) + S(r, f).$$

Therefore

$$T(r, f) = T(r, f') + S(r, f). \tag{3.5}$$

From (3.3)–(3.5) we get

$$\begin{aligned} m\left(r, \frac{1}{f - a}\right) &= T(r, f) - N\left(r, \frac{1}{f - a}\right) + S(r, f) \\ &= T(r, f') - N\left(r, \frac{1}{f - a}\right) + S(r, f) \\ &= N\left(r, \frac{1}{f' - a}\right) - N\left(r, \frac{1}{f - a}\right) + S(r, f) \\ &= S(r, f). \end{aligned}$$

Therefore by Lemma 2.1 we get $f = A\exp\{z\}$. This proves the theorem. ∎

Proof of Corollary 1.1. If

$$f = (\alpha z + \beta) + (\alpha z + \beta - 2\alpha) \exp\left\{\frac{\alpha z + \beta - 2\alpha}{\alpha}\right\},$$

then we see that $E(a; f)$ contains only one element but $E(a; f')$ contains infinitely many elements. This contradicts the hypothesis $E(a; f) = E(a; f')$. Therefore by Theorem 1.1 we get $f = A \exp\{z\}$. This proves the corollary. ■

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