\(\omega\)-pluripolar sets and subextension of
\(\omega\)-plurisubharmonic functions on compact Kähler manifolds

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Abstract. We establish some results on \(\omega\)-pluripolarity and complete \(\omega\)-pluripolarity for sets in a compact Kähler manifold \(X\) with fundamental form \(\omega\). Moreover, we study subextension of \(\omega\)-psh functions on a hyperconvex domain in \(X\) and prove a comparison principle for the class \(\mathcal{E}(X, \omega)\) recently introduced and investigated by Guedj-Zeriahi.

1. Introduction. Plurisubharmonic (psh) and holomorphic functions are very important objects of complex analysis. In order to study singularities of psh functions, Demaill, Lempert and Shiffman in [DLS] introduced the notion of quasi-psh functions, which are locally a sum of a psh function and a smooth function. Regarding this notion recently Kołodziej [Ko] and Guedj-Zeriahi [GZ1], [GZ2] introduced and investigated \(\omega\)-psh functions on a compact Kähler manifold with fundamental form \(\omega\). They studied some problems of pluripotential theory in a local setting (for bounded hyperconvex domains in \(\mathbb{C}^n\)) for \(\omega\)-psh functions, in particular, the Dirichlet problem.

The aim of this paper is to study some other problems of pluripotential theory of \(\omega\)-psh functions. Namely in Section 3 we study \(\omega\)-pluripolar and complete \(\omega\)-pluripolar sets in a compact Kähler manifold. In particular, we prove that a subset \(S\) of a compact Kähler manifold \(X\) with fundamental form \(\omega\) is locally pluripolar if and only if there exists a \(\varphi \in \mathcal{E}^\infty(X, \omega)\) (see Definition 2.3) such that \(\varphi = -\infty\) on \(S\). This result in a weaker form was proved by Guedj-Zeriahi in [GZ2]. Section 4 is devoted to investigating complete \(\omega\)-pluripolar sets in the projective space \(\mathbb{CP}^n\). We prove that a subset \(S \subset \mathbb{CP}^n\) is complete \(\omega\)-pluripolar in \(\mathbb{CP}^n\) if and only if \(S \cap U_j\) is complete pluripolar in the coordinate neighbourhood \(U_j = \{[z_0 : \ldots : z_n] \in \mathbb{CP}^n : z_j \neq 0\}\) for \(0 \leq j \leq n\). It is shown that a subset \(S \subset \mathbb{CP}^n\) is complete \(\omega\)-pluripolar in \(\mathbb{CP}^n\) iff \(\tilde{S} = \pi^{-1}(S) \cup \{0\}\) is complete pluripolar in \(\mathbb{C}^{n+1}\) where \(\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n\) denotes the canonical projection. Next in Sec-

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tion 5 we study the problem of subextension for $\omega$-psh functions. We show that every psh function in the class $\mathcal{F}$ (see Definition 2.2) on a hyperconvex domain in a compact Kähler manifold $X$ can be subextended to an $\omega$-psh function on $X$. Finally, in Section 6 we establish a comparison principle for the class $\mathcal{E}(X, \omega)$ introduced and investigated recently by Guedj–Zeriahi (see [GZ2]).

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2. Preliminaries. In this section we recall some elements of pluripotential theory in the local setting that can be found in Bedford–Taylor [BT], Klimek [KI], and Cegrell [Ce1], [Ce2].

2.1. Let $\Omega$ be a bounded domain in $\mathbb{C}^n$. The $C_n$-capacity in the sense of Bedford and Taylor on $\Omega$ is the set function given by

$$C_n(E) = C_n(E, \Omega) = \sup \left\{ (dd^c u)^n : u \in \text{PSH}(\Omega), -1 \leq u \leq 0 \right\}$$

for every Borel set $E$ in $\Omega$. It is known [BT] that

$$C_n(E) = \int_{\Omega} (dd^c h_{E,\Omega}^*)^n,$$

where $h_{E,\Omega}^*$ is the relative extremal psh function for $E$ (relative to $\Omega$) defined as the smallest upper semicontinuous majorant of

$$h_{E,\Omega}(z) = \sup\{u(z) : u \in \text{PSH}(\Omega), u \leq 0, u \leq -1 \text{ on } E\}.$$

2.2. Let $p \geq 1$. In [Ce1] and [Ce2] Cegrell introduced the following classes of psh functions on a bounded hyperconvex domain $\Omega$ in $\mathbb{C}^n$:

$$\mathcal{E}_0 = \mathcal{E}_0(\Omega) = \left\{ \varphi \in \text{PSH}(\Omega) \cap L^\infty(\Omega) : \lim_{z \to \partial \Omega} \varphi(z) = 0, \left( (dd^c \varphi)^n < \infty \right)_{\Omega} \right\},$$

$$\mathcal{F} = \mathcal{F}(\Omega) = \left\{ \varphi \in \text{PSH}(\Omega) : \exists \mathcal{E}_0 \ni \varphi_j \searrow \varphi, \sup_{j \geq 1} (dd^c \varphi_j)^n < \infty \right\},$$

$$\mathcal{E} = \mathcal{E}(\Omega) = \left\{ \varphi \in \text{PSH}(\Omega) : \forall z_0 \in \Omega, \text{ there exists a neighbourhood } U \ni z_0 \text{ and } \mathcal{E}_0 \ni \varphi_j \searrow \varphi \text{ on } U \text{ with } \sup_{j \geq 1} (dd^c \varphi_j)^n < \infty \right\}.$$

Recently Błocki [Bl] has proved that (belonging to) $\mathcal{E}$ is a local property. This result motivates the introduction of the following space:

$$\mathcal{D} = \mathcal{D}(\Omega) = \{ \varphi \in \text{PSH}(\Omega) : \forall z_0 \in \Omega, \text{ there is a neighbourhood } U \ni z_0 \text{ such that } \varphi|_U \in \mathcal{E}(U) + \mathbb{R} \}.$$
2.3. Let \( X \) be a compact Kähler manifold with fundamental form \( \omega \). For example, \( X \) is a projective space with the Fubini–Study Kähler form \( \omega = \omega_{FS} \). An upper semicontinuous function \( \varphi : X \to [-\infty, \infty) \) is said to be \( \omega \)-psh if \( \varphi \in L^1(X) \) and
\[
\omega + dd^c \varphi \geq 0.
\]
We denote by \( \text{PSH}(X, \omega) \) the set of all \( \omega \)-psh functions on \( X \). Along the lines of [Ce1], the following classes of \( \omega \)-psh functions were considered by Guedj and Zeriahi in [GZ2]:
\[
\mathcal{E}(X, \omega) = \{ \varphi \in \text{PSH}(X, \omega) : \forall z_0 \in X, \text{there is a neighbourhood } U \text{ of } z_0 \text{ and a potential } \theta \text{ of } \omega \text{ on } U \text{ such that } \varphi + \theta|_U \in \mathcal{D}(U) \},
\]
\[
\mathcal{E}^p(X, \omega) = \left\{ \varphi \in \text{PSH}(X, \omega) : \exists \text{ PSH}(X, \omega) \cap L^\infty(X) \ni \varphi_j \searrow \varphi, \quad \sup_{j \geq 1} \left\{ |\varphi_j|^p \omega_{\varphi_j}^n < \infty \right\} \right\}
\]
and
\[
\mathcal{E}^\infty(X, \omega) = \bigcap_{p \geq 1} \mathcal{E}^p(X, \omega).
\]

2.4. Following Bedford and Taylor [BT], Kołodziej [Ko] considered the \( \text{Cap}_\omega \)-capacity on \( X \) given by
\[
\text{Cap}_\omega(E) = \sup \left\{ \int_X \omega^n : \varphi \in \text{PSH}(X, \omega), \ 0 \leq \varphi \leq 1 \right\}
\]
for all Borel sets \( E \subset X \). In [Ko] (see also [GZ2]), it is proved that if \( \{U_\alpha\} \) is a finite cover of \( X \) by strictly pseudoconvex open subsets \( U_\alpha = \{ z \in X : \varphi_\alpha(z) < 0 \} \) where \( \varphi_\alpha \) is a strictly psh smooth function on a neighbourhood of \( U_\alpha \) then for every \( \delta > 0 \) there exists \( C > 0 \) such that
\[
\frac{1}{C} \text{Cap}_\omega(\cdot) \leq \text{Cap}_{\text{BT}}(\cdot) \leq C \text{Cap}_\omega(\cdot),
\]
where
\[
\text{Cap}_{\text{BT}}(E) = \sum_\alpha C_n(E \cap U_\alpha, U^\delta_\alpha), \quad U^\delta_\alpha = \{ z \in U_\alpha : \varphi_\alpha(z) < -\delta \}.
\]
The following equality was proved by Guedj and Zeriahi in [GZ1]:
\[
\text{Cap}_\omega(E) = \int_X (-h^{*}_{E, \omega}) \omega^n h^*_{E, \omega}
\]
for all Borel sets \( E \subset X \), where
\[
h_{E, \omega}(z) = \sup \{ \varphi(z) : \varphi \in \text{PSH}(X, \omega), \varphi \leq 0 \text{ and } \varphi \leq -1 \text{ on } E \}.
\]

2.5. Let \( S \subset X \). We say that \( S \) is \( \omega \)-pluripolar if there exists \( \varphi \in \text{PSH}(X, \omega) \) such that \( S \subset \varphi^{-1}(-\infty) \) and \( \varphi \not\equiv -\infty \). If \( \varphi \) can be chosen such that \( S = \varphi^{-1}(-\infty) \) then \( S \) is said to be a complete \( \omega \)-pluripolar set.
In [GZ1] the authors have shown that $S$ is $\omega$-pluripolar if and only if $S$ is locally pluripolar.

2.6. Given a domain $\Omega$ in $X$ and an $\omega$-psh function $\varphi$ on $\Omega$, an $\omega$-psh function $\tilde{\varphi}$ on $X$ is said to be a subextension of $\varphi$ if $\tilde{\varphi} \leq \varphi$ on $\Omega$.

2.7. In this paper we use Proposition 6.5 and Theorem 5.1 of [GZ2]. The latter is claimed to hold for $n > 2$ (see Theorem 7.5 in [GZ2]). However, it is not mentioned that Proposition 6.5 also holds for $n > 2$. We now prove that, using the notation of [GZ2]. Namely we establish the following. Let $\mu$ be a probability measure on a compact connected Kähler manifold, $\dim_{\mathbb{C}} X = n$, equipped with the Kähler form $\omega$. Assume that there exist $\alpha > p/(p + 1)$ and $A > 0$ such that

$$\mu(E) \leq A \text{Cap}_\omega(E)^\alpha$$

for all Borel sets $E \subset X$. Then $\mathcal{E}^p(X, \omega) \subset L^p(X)$.

First we recall that integration by parts on a compact manifold always holds since there is no boundary. Now the above claim follows from the following three results.

1) Let $\varphi \in \text{PSH}(X, \omega) \cap L^\infty(X)$. Then

$$\int_X (-\varphi)^p \omega^n \leq \int_X (-\varphi)^p \omega \wedge \omega^{n-1} \leq \cdots \leq \int_X (-\varphi)^p \omega_n.$$  

Indeed, let $T$ be a closed positive current. Then

$$\int_X (-\varphi)^p \omega \wedge T = \int_X (-\varphi)^p \omega \wedge T + \int_X (-\varphi)^p dd^c \varphi \wedge T$$

$$= \int_X (-\varphi)^p \omega \wedge T + p \int_X (-\varphi)^{p-1} d\varphi \wedge d^c \varphi \wedge T$$

$$\geq \int_X (-\varphi)^p \omega \wedge T,$$

and 1) follows.

2) Let $\varphi \in \mathcal{E}^p(X, \omega)$. Then

$$\text{Cap}_\omega(\varphi < -2t) \leq C(\varphi)/t^{p+1}.$$  

Indeed,

$$\text{Cap}_\omega(\varphi < -2t) = \sup \left\{ \int_X \omega^n_u : u \in \text{PSH}(X, \omega), -1 \leq u \leq 0 \right\}$$

$$\leq \sup \left\{ \int_X \omega^n_{u/t} : u \in \text{PSH}(X, \omega), -1 \leq u \leq 0 \right\}$$

$$\leq \sup \left\{ \int_{\varphi/t < u - 1} \omega^n_{\varphi/t} : u \in \text{PSH}(X, \omega), -1 \leq u \leq 0 \right\}$$

$$\leq \int_X (-\varphi)^p \omega \wedge T,$$
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\[
\leq \int_{\{\varphi < -t\}} (\omega + \omega \varphi/t)^n \leq \int_{\{\varphi < -t\}} \sum_{j=0}^{n} C_n^j \omega \varphi^{j} / t^j \land \omega^{n-j}
\]

\[
= \sum_{j=0}^{n} C_n^j \int_{\{\varphi < -t\}} \omega \varphi^{j} \land \omega^{n-j} = \int_{\{\varphi < -t\}} \omega^n + \sum_{j=1}^{n} C_n^j \int_{\{\varphi < -t\}} \omega \varphi^{j} \land \omega^{n-j}
\]

\[
\leq \int_{\{\varphi < -t\}} \omega^n + \sum_{j=1}^{n} C_n^j \int_{t^j+p}^X (-\varphi)^p \omega \varphi^{j} \land \omega^{n-j}
\]

\[
\leq \int_{\{\varphi < -t\}} \omega^n + \sum_{j=1}^{n} C_n^j \int_{t^j+p}^X (-\varphi)^p \omega \varphi^{j} \land \omega^{n-j}
\]

\[
\leq \frac{1}{t^{1+p}} \left[ \int_{X} (-\varphi)^p \omega^n + \sum_{j=1}^{n} C_n^j \int_{t^j+p}^X (-\varphi)^p \omega \varphi^{j} \land \omega^{n-j} \right] \leq \frac{C(\varphi)}{t^{p+1}}.
\]

**Proof of Proposition 6.5 of [GZ2] for \( n > 2 \).** Let \( \varphi \in \mathcal{E}^p(X, \omega) \) with \( \sup_X \varphi = -1 \). By the Fubini theorem we have

\[
\int_{X} (-\varphi)^p \omega^n = p \int_{1}^{\infty} t^{p-1} \mu(\varphi < -t) \, dt + \mu(\varphi < -t) \]

\[
\leq pA \int_{1}^{\infty} t^{p-1} (\text{Cap}_\omega(\varphi < -t))^\alpha \, dt + \mu(\varphi < -t).
\]

From 2) it follows that

\[
\int_{X} (-\varphi)^p \omega^n \leq 1 + C(\varphi) \int_{1}^{\infty} \frac{1}{t^{\alpha(p+1)+1-p}} \, dt < \infty
\]

because from the hypothesis we have \( \alpha(p+1) + 1 - p > 1 \).

3. **\( \omega \)-pluripolar and complete \( \omega \)-pluripolar sets.** In this section we investigate \( \omega \)-pluripolar and complete \( \omega \)-pluripolar sets on a compact Kähler manifold with fundamental form \( \omega \). Before stating the first result we would like to explain its origin. In Theorem 6.2 of [GZ1] the authors proved that every locally pluripolar set is an \( \omega \)-pluripolar set. Here we give another proof of this fact by applying a recent result on solution of the Monge–Ampère equation presented in [GZ2].

**3.1. Theorem.** Let \( S \) be a locally \( \omega \)-pluripolar set in \( X \). Then there exists \( \varphi \in \mathcal{E}^\infty(X, \omega) \) such that \( \varphi \equiv -\infty \) on \( S \) and \( \varphi \not\equiv -\infty \).

In order to prove the theorem we need the following lemma.
3.2. Lemma. Let $\Omega$ be a domain in $X$ which is biholomorphic to a ball in $\mathbb{C}^n$ and $D \subseteq \Omega$. Let $\varphi \in \mathcal{F}_\infty(\Omega)$. Then there exists $u \in \mathcal{E}_\infty(X, \omega)$ for some $a > 0$ such that $u \leq \varphi$ on $D$. Here

$$\mathcal{F}_\infty = \mathcal{F}_\infty(\Omega) = \bigcap_{p \geq 1} \mathcal{F}_p(\Omega)$$

with

$$\mathcal{F}_p = \mathcal{F}_p(\Omega) = \{ \varphi \in \text{PSH}(\Omega) : \exists \mathcal{E}_0 \ni \varphi \downarrow \varphi, \sup_{j \geq 1} \varphi_j^{p}(dd^c \varphi_j)^n < \infty \}.$$

Proof. By hypothesis, $\varphi \leq 0$ on $\Omega$. We can assume that $\varphi \neq 0$. Put

$$h_{D, \varphi}(z) = \sup\{ u(z) : u \in \text{PSH}(\Omega), u \leq 0 \text{ and } |D| \leq \varphi \}.$$

Since $\varphi \leq h_{D, \varphi}^*$ and $\varphi \in \mathcal{F}_\infty(\Omega)$ it follows that $h_{D, \varphi}^* \in \mathcal{F}_\infty(\Omega)$ ([Ce1]). Moreover, $h_{D, \varphi} = \varphi$ on $D$ and $\text{supp}(dd^c h_{D, \varphi}^*) \subset D$. It is easy to see that $(dd^c h_{D, \varphi}^*)^n \neq 0$. Indeed, otherwise Lemma 3.3 in [Ah] implies that $h_{D, \varphi}^* \equiv 0$ on $\Omega$, hence $\varphi \equiv 0$, which is a contradiction. Consider the probability measure

$$\mu = \alpha^{-1}(dd^c h_{D, \varphi}^*)^n$$

with $\alpha = \int_\Omega (dd^c h_{D, \varphi}^*)^n \neq 0$. It follows from the Hölder inequality (see [Ko]) that for each $p \geq 1$ there exist $A_p, B_p > 0$ such that

$$\mu(E) = \mu(E \cap D) \leq \frac{1}{\alpha} \int_\Omega (-h_{E \cap D, \Omega}^*)^{p}(dd^c h_{D, \varphi}^*)^n$$

$$\leq \frac{A_p}{\alpha} \left( \int_\Omega (-h_{E \cap D, \Omega}^*)^{p}(dd^c h_{D, \varphi}^*)^n \right)^{n/(p+n)}$$

$$\times \left( \int_\Omega (-h_{E \cap D, \Omega}^*)^{p}(dd^c h_{E \cap D, \Omega}^*)^n \right)^{p/(p+n)}$$

$$\leq \frac{A_p}{\alpha} \left( \int_\Omega (-h_{E \cap D, \varphi}^*)^{p}(dd^c h_{D, \varphi}^*)^n \right)^{n/(p+n)} C_n(E \cap D, \Omega)^{p/(p+n)}$$

$$\leq B_p \text{Cap}_\omega(E \cap D, X)^{p/(p+n)}$$

for all Borel sets $E \subseteq X$. Proposition 6.5 and Theorem 5.1 in [GZ2] imply that there exists $v \in \mathcal{E}_\infty(X, \omega)$ with $\omega^v = \mu$. Let $\theta$ be a negative potential of $\omega$ on $\Omega$, $\omega = dd^c \theta$. Since

$$(dd^c (v + \theta))^n = (dd^c v + \omega)^n = \frac{1}{\alpha} (dd^c h_{D, \varphi}^*)^n$$

on $\Omega$, by the comparison principle we have

$$v + \theta \leq \frac{1}{\alpha^{1/n}} h_{D, \varphi}^*$$

on $\Omega$. Notice that $\mathcal{E}_\infty(X, A\omega) = A\mathcal{E}_\infty(X, \omega)$ for all $A > 0$, and hence for $u = \alpha^{1/n}(v+c)$ where $c = \inf_D \theta$ it follows that $u \in \mathcal{E}_\infty(X, \alpha^{1/n} \omega)$. Moreover, $u \leq h_{D, \varphi}^*$ on $D$, and therefore $u \leq \varphi$ almost everywhere on $D$. Thus $u \leq \varphi$ on $D$. 

Proof of Theorem 3.1. Let $S$ be a locally $\omega$-pluripolar set in $X$. Then by [H] we can find hyperconvex subsets $V_s \subseteq U_s$ and $\varphi_s \in \mathcal{F}_\omega(U_s)$ such that $\varphi_s = -\infty$ on $S \cap U_s$ and $X = \bigcup_{s=1}^k V_s$. We may assume that every $U_s$ is biholomorphic to a ball in $\mathbb{C}^n$. For each $s = 1, \ldots, k$ applying Lemma 3.2 we can find $u_s \in \mathcal{E}^\infty(X, a_s \omega)$ with $a_s > 0$ such that $u_s \leq \varphi_s$ on $V_s$. Hence \( \{ \varphi_s = -\infty \} \cap V_s = \{ u_s = -\infty \} \cap V_s \) for $s = 1, \ldots, k$. Put
\[
u = \frac{1}{k} \sum_{s=1}^k \frac{u_s}{a_s}.
\]
From the convexity of $\mathcal{E}^\infty(X, \omega)$, we infer that $u \in \mathcal{E}^\infty(X, \omega)$ and $u = -\infty$ on $S$. This completes the proof of Theorem 3.1.

Remark. Theorem 3.1 also follows from [GZ2]. Indeed, by Example 6.3 in [GZ2], we can find $\varphi \in \mathcal{E}^1(X, \omega)$ such that $\varphi \equiv -\infty$ on $S$. It is enough to consider the function $u := -\log(-\varphi)$.

Next we investigate the completeness of $\omega$-pluripolar sets. Given a pluripolar set $S \subset X$, as in the local setting put
\[
S^* = \{ z \in X : \varphi(z) = -\infty, \forall \varphi \in \text{PSH}(X, \omega), \varphi|_S = -\infty \}.
\]
In the local setting (for pseudoconvex domains in $\mathbb{C}^n$) Zeriahi [Ze] proved that if $S$ is an $F_\sigma$ and $G_\delta$ pluripolar set such that $S = S^*$ then $S$ is complete pluripolar. By a similar argument using the approximation theorem of Demailly [De] for $\omega$-psh functions we also obtain

**3.3. Proposition.** Let $S$ be an $F_\sigma$ and $G_\delta$ $\omega$-pluripolar set such that $S = S^*$. Then $S$ is complete $\omega$-pluripolar.

*Proof.* Since $S$ is $F_\sigma$ and $G_\delta$, we can write $S$ and $X \setminus S$ as increasing unions of compact subsets
\[
S = \bigcup_{j=1}^\infty K_j, \quad X \setminus S = \bigcup_{j=1}^\infty L_j.
\]
Let $a \in L_j$. Then $a \notin S^*$. Hence, there exists $u_a^{(j)} \in \text{PSH}(X, \omega)$ such that $u_a^{(j)}|_S \equiv -\infty$ and $u_a^{(j)}(a) > -\infty$. Since $\varepsilon u_a^{(j)} \in \text{PSH}(X, \omega)$ for all $0 < \varepsilon < 1$, we can assume that
\[
u_a^{(j)}|_S \equiv -\infty, \quad u_a^{(j)}(a) > -1, \quad u_a^{(j)} \leq 0.
\]
By [De] there exists a sequence $\{ u_k^{(j)} \} \subset \text{PSH} \cap C^\infty(X, \omega)$ that decreases pointwise to $u_a^{(j)}$ on $X$. Applying Dini’s theorem we find $k_a$ such that
\[
u_a^{(j)}|_K \leq -2^j, \quad u_a^{(j)}(a) > -1, \quad u_a^{(j)} \leq 0.
\]
Let $U_a$ be a neighbourhood of $a$ such that $u_k^{(j)} > -1$ on $U_a$. Now a standard argument using the compactness of $L_j$ implies that there exists a continuous
function $v_j \in \text{PSH}(X, \omega)$ such that

(i) $v_j\models_K \leq -2^j$.
(ii) $v_j\models_L > -1$.
(iii) $v_j \leq 0$.

Set

$$v = \sum_{j=1}^{\infty} 2^{-j} v_j.$$ 

Then $v \in \text{PSH}(X, \omega)$ and $S = \{v = -\infty\}$, and the proposition follows.

3.4. Proposition. Let $S$ be a closed complete locally $\omega$-pluripolar set in $X$. Then $S$ is complete $\omega$-pluripolar.

Proof. From the proof of Theorem 1 in [Co] it follows that there exist finite open covers $D''_i \subseteq D'_i \subseteq D_i$, $1 \leq i \leq m$, of $X$ and negative psh functions $\varphi_i$ on $D_i$ such that

(i) $S \cap D_i = \{\varphi_i = -\infty\}$, $X = \bigcup_{i=1}^m D''_i$.
(ii) $\varphi_i = \varphi_j$ is bounded on $D_i \cap D_j \setminus S$.
(iii) $\omega = dd^c \theta_i$ on $D_i$ where $\theta_i$ is a strictly psh function on $D_i$ and $\theta_i < 0$.

As in the proof of Theorem 1 in [Co] we can choose $p_i \in C^\infty_0(X)$ with $p_i \geq 0$ and $\text{supp} p_i \subseteq D'_i$ such that

(1) $\varphi_i + p_i < \varphi_j + p_j$ on $(\partial D'_i \cap D''_j) \setminus S$.

Set

$$\varphi(z) = \frac{1}{M} \sup_{1 \leq i \leq m} \{\varphi_i(z) + p_i(z) : z \in D'_i\}$$

where $M > 0$ is chosen such that $p_i/M + \theta_i$ is psh on $D'_i$ for $1 \leq i \leq m$. From (1) we see that $\varphi$ is upper semicontinuous on $X$. Moreover, (iii) implies that $\varphi \in \text{PSH}(X, \omega)$. It is easy to check that $S = \varphi^{-1}(-\infty)$.

Remark. Proposition 3.4 was in fact proved in [DLS] by Demaylly–Lempert–Shiffman.

Now we investigate complete pluripolarity in the case $\dim X = 1$. We have the following result.

3.5. Proposition. Let $\dim X = 1$ and $S$ an $\omega$-pluripolar set in $X$. Then

(i) $S = S^*$.
(ii) $S$ is complete $\omega$-pluripolar if and only if $S$ is a $G_\delta$.

Proof. (i) Take an $\omega$-psh function $u$ on $X$ such that $u \not\equiv -\infty$ and $S \subseteq \{u = -\infty\}$. Let $z \not\in S$. Since $\dim X = 1$, by [Lan] there exists a decreasing neighbourhood basis $U_j$ of $z$ such that $\inf_{\partial U_j} u > -\infty$. Take $\varepsilon_j > 0$ such
that $\inf_{\partial U_j} \varepsilon_j u > -1$. Define

$$v_j(z) = \begin{cases} \max\{\varepsilon_j u(z), -1\} & \text{on } U_j, \\ \varepsilon_j u(z) & \text{on } X \setminus U_j. \end{cases}$$

It follows that $v_j$ is $\omega$-psh on $X$ with $v_j \geq -1$ on $U_j$ and $v_j = -\infty$ on $S \setminus U_j$. Let

$$v = \sum_{j=1}^{\infty} \frac{v_j}{2^j}.$$ 

From the convexity of $\text{PSH}(X, \omega)$ it follows that $v$ is $\omega$-psh on $X$ with $v(z) > -1$ and $v = -\infty$ on $S$. Hence $z \notin S^*$ and the desired conclusion follows.

(ii) The necessity is obvious. It remains to prove the sufficiency. Assume that $S$ is a $G_\delta$ $\omega$-pluripolar set. Fix $z \in X$. Take a coordinate neighbourhood $U_z$ of $z$ in $X$ and a smooth subharmonic function $\theta_z$ on a neighbourhood of $\overline{U_z}$ such that $\omega = dd^c \theta_z$. Since $S \cap U_z$ is a $G_\delta$ polar set, Deny's theorem (see [Lan]) implies that there exists a subharmonic function $u_z$ on $U_z$ such that $U_z \cap S = \{u_z = -\infty\}$. Let $\phi$ be an $\omega$-psh function on $X$ such that $S \subset \{\phi = -\infty\}$ and $\phi \neq -\infty$. As in the proof of (i) we can find an $\omega$-psh function $\varphi_z$ on $X$ such that $\varphi_z \geq -1$ on $\overline{U_z'}$ and $U_z' \cap S \subset \{\varphi_z = -\infty\}$ where $U_z'$ is some neighbourhood of $z$ with $U_z' \Subset U_z$. Define

$$\psi_z = \begin{cases} \max\{u_z - \sup_{\overline{U_z}} u_z - 1 - \theta_z + \inf_{\overline{U_z}} \theta_z, \varphi_z\} & \text{on } U_z', \\ \varphi_z & \text{on } X \setminus U_z'. \end{cases}$$

It follows that $\psi_z$ is $\omega$-psh on $X$ with $U_z' \cap S = \{\psi_z = -\infty\}$. By the compactness of $X$ we can find a finite open cover $U_{z_j'}, j = 1, \ldots, m$, of $X$. Put

$$\psi = \frac{1}{m} \sum_{j=1}^{m} \psi_{z_j}.$$ 

Then $\psi$ is $\omega$-psh with $S = \{\psi = -\infty\}$.

4. Complete $\omega$-pluripolar sets in $\mathbb{CP}^n$. This section is devoted to studying the complete $\omega$-pluripolarity of subsets in $\mathbb{CP}^n$ equipped with the Fubini–Study Kähler form $\omega = \omega_{FS}$. First we prove the following

4.1. Proposition. Let $S \subset \mathbb{CP}^n$. Then $S$ is complete $\omega$-pluripolar if and only if $S \cap U_j$ is complete pluripolar in $U_j$ for $0 \leq j \leq n$ where

$$U_j = \{z = [z_0 : \ldots : z_n] \in \mathbb{CP}^n : z_j \neq 0\}$$

Proof. Necessity. Let $S$ be a complete $\omega$-pluripolar subset in $\mathbb{CP}^n$. Then there exists an $\omega$-psh function $\varphi$ on $\mathbb{CP}^n$ such that $\varphi \neq -\infty$ and $S = \{\varphi = -\infty\}$. Let $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ be the canonical projection. Then $\pi|_{V_j} : V_j \to U_j$ is biholomorphic where

$$V_j = \{(z_0, \ldots, z_{j-1}, 1, z_{j+1}, \ldots, z_n) \} \subset \mathbb{C}^{n+1} \setminus \{0\}. $$
The function \( \psi(z) = \varphi(\pi(z)) + \frac{1}{2} \log(\sum_{k=0}^{n} |z_k|^2) \), \( z \in V_j \), is plurisubharmonic on \( V_j \) and \( (\pi|_{V_j})^{-1}(S) \cap V_j = \{ \psi = -\infty \} \). Hence \( (\pi|_{V_j})^{-1}(S) \cap V_j \) is complete pluripolar in \( V_j \). From \( S \cap U_j = \pi|_{V_j}((\pi|_{V_j})^{-1}(S) \cap V_j) \) it follows that \( S \cap U_j \) is complete pluripolar in \( U_j \) for \( 0 \leq j \leq n \).

**Sufficiency.** Assume that \( S \cap U_j \) is complete pluripolar for \( 0 \leq j \leq n \). Since \( \mathbb{C}P^n \setminus U_j \) is complete \( \omega \)-pluripolar, we can find an \( \omega \)-psh function \( u_j \) on \( \mathbb{C}P^n \) such that

\[
\{ u_j = -\infty \} = \mathbb{C}P^n \setminus U_j.
\]

By [Si] there exists \( v_j \in \mathcal{L}(U_j) \) such that \( \{ v_j = -\infty \} = S \cap U_j \). The example 1.2 in [GZ1] shows that the function

\[
\tilde{v}_j(z) = \begin{cases} 
v_j(z) - \frac{1}{2} \log(1 + \|z\|^2) & \text{for } z \in U_j, \\
\lim_{U_j \ni w \to z} (v_j(w) - \frac{1}{2} \log(1 + \|w\|^2)) & \text{for } z \in \mathbb{C}P^n \setminus U_j,
\end{cases}
\]

belongs to \( \text{PSH}(\mathbb{C}P^n, \omega) \). Moreover \( \{ \tilde{v}_j = -\infty \} \cap U_j = S \cap U_j \). Let

\[
\varphi_j = \frac{u_j + \tilde{v}_j}{2}.
\]

Then

\[
\varphi_j \in \text{PSH}(\mathbb{C}P^n, \omega), \quad \{ \varphi_j = -\infty \} \cap U_j = S \cap U_j,
\]

\[
\varphi_j = -\infty \quad \text{on } \mathbb{C}P^n \setminus U_j.
\]

By (2) if \( \varphi = \max \{ \varphi_j : 0 \leq j \leq n \} \) then \( \varphi \) is \( \omega \)-psh on \( \mathbb{C}P^n \) and \( \{ \varphi = -\infty \} = S \). The proof of Proposition 4.1 is complete.

Next we establish a result on complete \( \omega \)-pluripolarity of a subset in \( \mathbb{C}P^n \).

**4.2. PROPOSITION.** Let \( \pi : \mathbb{C}^{n+1} \setminus \{ 0 \} \to \mathbb{C}P^n \) be the canonical projection and \( S \subset \mathbb{C}P^n \). Then \( S \) is complete \( \omega \)-pluripolar if and only if \( \tilde{S} = \pi^{-1}(S) \cup \{ 0 \} \) is complete pluripolar in \( \mathbb{C}^{n+1} \).

**Proof.** Assume that \( S \) is complete \( \omega \)-pluripolar. Take an \( \omega \)-psh function \( \varphi \) on \( \mathbb{C}P^n \) with \( \varphi \not\equiv -\infty \) and \( S = \{ \varphi = -\infty \} \). Consider \( \tilde{\varphi}(z) = \varphi(\pi(z)) + \log \|z\| \) for \( z \in \mathbb{C}^{n+1} \setminus \{ 0 \} \). Since \( \varphi \) is an \( \omega \)-psh function on \( \mathbb{C}P^n \) it follows that \( \tilde{\varphi} \) is plurisubharmonic on \( \mathbb{C}^{n+1} \setminus \{ 0 \} \), and hence on \( \mathbb{C}^{n+1} \). Because \( \tilde{\varphi}(0) = \lim_{z \to 0} (\varphi(\pi(z)) + \log \|z\|) = -\infty \) we infer that \( \tilde{S} = \{ \tilde{\varphi} = -\infty \} \). Hence \( \tilde{S} \) is complete pluripolar in \( \mathbb{C}^{n+1} \).

Conversely, assume that \( \tilde{S} \) is complete pluripolar in \( \mathbb{C}^{n+1} \). For each \( 0 \leq j \leq n \), let \( V_j = \{ (z_0, \ldots, z_{j-1}, 1, z_{j+1}, \ldots, z_n) \} \subset \mathbb{C}^{n+1} \setminus \{ 0 \} \) and \( U_j = \{ z = [z_0 : \ldots : z_n] \in \mathbb{C}P^n : z_j \neq 0 \} \). Then \( \pi|_{V_j} : V_j \to U_j \) is biholomorphic and \( \tilde{S} \cap V_j = \pi^{-1}(S) \cap V_j \) is complete pluripolar in \( V_j \). This implies that \( S \cap U_j \) is complete pluripolar in \( U_j \). Proposition 4.1 implies that \( S \) is complete \( \omega \)-pluripolar in \( \mathbb{C}P^n \).
**4.3. Proposition.** Let $S$ be an $\omega$-pluripolar set in $\mathbb{CP}^n$. Then

$$[S]^*_{\mathbb{CP}^n} \cap U_j = [S \cap U_j]^*_{U_j} \quad \text{for } j = 0, \ldots, n,$$

and hence

$$[S]^*_{\mathbb{CP}^n} = \bigcup_{j=0}^n [S \cap U_j]^*_{U_j}.$$

**Proof.** It is easy to see that $[S \cap U_j]^*_{U_j} \subset [S]^*_{\mathbb{CP}^n} \cap U_j$. Hence, it remains to show that $[S]^*_{\mathbb{CP}^n} \cap U_j \subset [S \cap U_j]^*_{U_j}$ for $0 \leq j \leq n$. Let $z_0 \in [S]^*_{\mathbb{CP}^n} \cap U_j$ and $u \in \text{PSH}(U_j)$ with $u = -\infty$ on $S \cap U_j$. By [Si] we may assume that $u \in \mathcal{L}(U_j)$. As in the proof of Proposition 4.1 the function

$$\tilde{u}(z) = \left\{ \begin{array}{ll}
    u(z) - \frac{1}{2} \log(1 + \|z\|^2) & \text{for } z \in U_j, \\
    \lim_{U_j \ni w \to z} (u(w) - \frac{1}{2} \log(1 + \|w\|^2)) & \text{for } z \in \mathbb{CP}^n \setminus U_j,
\end{array} \right.$$

is $\omega$-psh on $\mathbb{CP}^n$ and $\tilde{u} = -\infty$ on $S \cap U_j$. Let $v$ be an $\omega$-psh function on $\mathbb{CP}^n$ such that $\{v = -\infty\} = \mathbb{CP}^n \setminus U_j$, and set $\varphi = (\tilde{u} + v)/2$. Then $\varphi$ is psh on $\mathbb{CP}^n$ and $\varphi = -\infty$ on $S$. Hence $\varphi(z_0) = -\infty$. Thus $\tilde{u}$, and therefore $u$, is equal to $-\infty$ at $z_0$. This shows that $z_0 \in [S \cap U_j]^*_{U_j}$.

**4.4. Proposition.** Let $S$ be a $G_\delta$ set which is a countable union of compact complete pluripolar sets in $\mathbb{C}^n$. Then $S$ is complete $\omega$-pluripolar in $\mathbb{CP}^n = \mathbb{C}^n \cup H_\infty$.

**Proof.** We write $S = \bigcup_{j=1}^\infty S_j$, where $S_j$ are compact complete pluripolar sets in $\mathbb{C}^n$. Proposition 3.4 implies that $S_j$ is complete $\omega$-pluripolar in $\mathbb{CP}^n$. On the other hand,

$$[S]^*_{\mathbb{CP}^n} = \bigcup_{j=1}^\infty [S_j]^*_{\mathbb{CP}^n} = \bigcup_{j=1}^\infty S_j = S.$$

Now the desired conclusion follows from Proposition 3.3.

**4.5. Examples.** (a) Let $f$ be an entire function on $\mathbb{C}$ and $E = \{(z, f(z)) : z \in \mathbb{C}\} = \{(1 : z : f(z)) : z \in \mathbb{C}\} \subset \mathbb{C}^2 \subset \mathbb{CP}^2$. We have

$$[E]^*_{\mathbb{CP}^2} \cap U_0 = [E \cap U_0]^*_{U_0} = E,$$

$$[E]^*_{\mathbb{CP}^2} \cap U_1 = [E \cap U_1]^*_{U_1} = \{(1/z, f(z)/z) : z \in \mathbb{C}^\ast\}^*_{\mathbb{C}^2}$$

$$= \{((z, zf(1/z)) : z \in \mathbb{C}^\ast\}^*_{\mathbb{C}^2} = E \cap U_1 \quad \text{(by [Wie]).}$$

Let now $f(z) = e^z$. We have

$$[E]^*_{\mathbb{CP}^2} \cap U_2 = [E \cap U_2]^*_{U_2} = \{((e^{-z}, z e^{-z}) : z \in \mathbb{C})\}^*_{\mathbb{C}^2}$$

$$= \{((e^z, -ze^z) : z \in \mathbb{C})\}^*_{\mathbb{C}^2} \quad \text{(by Corollary 2.6 in [Edi])}$$

$$= \{(e^z, -ze^z) : z \in \mathbb{C}\} = \{(e^{-z}, ze^{-z}) : z \in \mathbb{C}\} = E \cap U_2.$$

Thus $\{(z, e^z) : z \in \mathbb{C}\}$ is complete $\omega$-pluripolar in $\mathbb{CP}^2$. 
Now we give an example in which the pluripolar hull of a graph for the class of $\omega$-psh functions may not coincide with the graph. Let $P(t) = c_d t^d + \cdots + c_0$ be a polynomial of degree $d > 1$. Consider the graph

\[ E = \{(\lambda, P(\lambda)) : \lambda \in \mathbb{C}\} = \{[1 : \lambda : P(\lambda)] : \lambda \in \mathbb{C}\} \subset \mathbb{C}^2 \subset \mathbb{CP}^2 \]

We show that

\[ [E]_{\mathbb{CP}^2}^* = E \cup \{[0 : 0 : 1]\} \]

where $[E]_{\mathbb{CP}^2}^*$ denotes the pluripolar envelope of $E$ for the class $\text{PSH}(\mathbb{CP}^2, \omega)$. It is easy to see that $E \subset [E]_{\mathbb{CP}^2}^*$. We show that $\{[0 : 0 : 1]\} \in [E]_{\mathbb{CP}^2}^*$. Let $u \in \text{PSH}(\mathbb{CP}^2, \omega)$ be such that $u([1 : \lambda : P(\lambda)]) = -\infty$ for $\lambda \in \mathbb{C}$. Define

\[ \varphi(\xi, \eta) = u([\xi : \eta : 1]) + \frac{1}{2} \log(1 + |\xi|^2 + |\eta|^2) \]

for $(\xi, \eta) \in \mathbb{C}^2$. From the $\omega$-plurisubharmonicity of $u$ it follows that $\varphi$ is psh on $\mathbb{C}^2$ and

\[
\varphi\left(\frac{1}{P(\lambda)}, \frac{\lambda}{P(\lambda)}\right) = u\left(\frac{1}{P(\lambda)} : \frac{\lambda}{P(\lambda)} : 1\right) + \frac{1}{2} \log\left(1 + \frac{1 + |\lambda|^2}{|P(\lambda)|^2}\right) \\
= u([1 : \lambda : P(\lambda)]) + \frac{1}{2} \log\left(1 + \frac{1 + |\lambda|^2}{|P(\lambda)|^2}\right) = -\infty
\]

for $\lambda \in \mathbb{C} \setminus P^{-1}(0)$. Take $R > 0$ sufficiently large such that $P(\lambda) = 0$ for all $\lambda \in \mathbb{C}$ with $|\lambda| < R$. Thus

\[ \varphi\left(\frac{1}{P(1/\lambda)}, \frac{1}{\lambda P(1/\lambda)}\right) = -\infty \quad \text{on} \ \mathbb{C} \setminus \{|\lambda| < R\}, \]

and hence

\[ \varphi\left(\frac{1}{P(1/\lambda)}, \frac{1}{\lambda P(1/\lambda)}\right) = -\infty \quad \text{for} \ 0 < |\lambda| < 1/R. \]

Consider the function

\[ \psi(\lambda) = \varphi\left(\frac{\lambda^d}{c_d + \cdots + c_0 \lambda^d}, \frac{\lambda^{d-1}}{c_d + \cdots + c_0 \lambda^d}\right) \]

for $|\lambda| < 1/R$. Then $\psi$ is subharmonic on $\{|\lambda| < 1/R\}$ and $\psi(\lambda) = -\infty$ for $0 < |\lambda| < 1/R$. Therefore $\psi(0) = -\infty$, and consequently

\[ u([0 : 0 : 1]) = \varphi(0, 0) = \psi(0) = -\infty. \]

Thus $\{[0 : 0 : 1]\} \in [E]_{\mathbb{CP}^2}^*$.

Conversely, we show that $[E]_{\mathbb{CP}^2}^* \subset E \cup \{[0 : 0 : 1]\}$. Assume that $[x_0 : y_0 : z_0] \in \mathbb{CP}^2 \setminus E \cup \{[0 : 0 : 1]\}$. Consider the function
\[ u([x, y, z]) \]
\[ = \begin{cases} \frac{1}{d} \log \frac{z}{x} - P \left( \frac{y}{x} \right) - \frac{1}{2} \log \left( 1 + \frac{|y|^2 + |z|^2}{|x|^2} \right) \\ \lim_{[x', y', z'] \to [0, y, z]} \left\{ \frac{1}{d} \log \frac{z'}{x'} - P \left( \frac{y'}{x'} \right) - \frac{1}{2} \log \left( 1 + \frac{|y'|^2 + |z'|^2}{|x'|^2} \right) \right\} \end{cases} \]
for \( x \neq 0 \),
\[ \frac{1}{d} \log |c_d| + \frac{1}{2} \log \frac{|y_0|^2}{|y_0|^2 + |z_0|^2} > -\infty. \]

Finally, if \( y_0 = 0 \) then \( z_0 \neq 0 \) and \([0 : 0 : z_0] = [0 : 0 : 1] \), which is also impossible. Thus \([E]_{\mathbb{C}P^2} = E \cup \{0 : 0 : 1\}\).

(b) Let \( f(z) = e^{1/z} \), \( z \neq 0 \), and \( E = \{(z, e^{1/z}) : z \neq 0\} = \{[1 : z : e^{1/z}] : z \neq 0\} \subset \mathbb{C}^2 \subset \mathbb{C}P^2 \). We have
\[ E_{\mathbb{C}P^2} \cap U_0 = [E \cap U_0]_{U_0} = \{(z, e^{1/z}) : z \neq 0\} \mid \mathbb{C}^2 = E \cap U_0 \quad \text{(by [Wiel])}, \]

\[ E_{\mathbb{C}P^2} \cap U_1 = [E \cap U_1]_{U_1} = \{(1/z, e^{1/z}/z) : z \neq 0\} \mid \mathbb{C}^2 \]
\[ = \{(z, e^{z}) : z \neq 0\} \mid \mathbb{C}^2 = (E \cap U_1) \cup \{0 : 1 : 0\}, \]
\[ [E]_{\mathbb{C}P^2} \cap U_2 = [E \cap U_2]_{U_2} = \{(e^{-1/z}, ze^{-1/z}) : z \neq 0\} \mid \mathbb{C}^2 \]
\[ = \{(e^z, -e^z/z) : z \neq 0\} \mid \mathbb{C}^2 = [\pi(P)] \mid \mathbb{C}^2, \]
where \( \pi : \mathbb{C}^2 \to \mathbb{C}^2 \), \( \pi(z, w) = (e^z, w) \) and \( P = \{(z, -e^z/z) : z \neq 0\} \). Since \( P \) is locally closed in \( \mathbb{C}^2 \) and \( \pi \) is an \( A \)-covering map (see the precise definition in [Edi]) and by Theorem 2.5 in [Edi] we have
\[ [\pi(P)] \mid \mathbb{C}^2 = \pi(P^*) = \pi(P) = E \cap U_2. \]
Thus \([E]_{\mathbb{C}P^2} \cap U_2 = E \cap U_2 \). Therefore
\[ [E]_{\mathbb{C}P^2} = E \cup \{0 : 1 : 0\}. \]

5. Subextension of \( \omega \)-psh functions. Let \( X \) be a compact Kähler manifold with fundamental form \( \omega \), and \( \Omega \) a hyperconvex domain in \( X \). Assume that \( \varphi \in \text{PSH}(\Omega) \). In this section we investigate the existence of an \( \omega \)-psh function \( \tilde{\varphi} \) on \( X \) such that \( \tilde{\varphi} \leq \varphi \) on \( \Omega \). Such an \( \omega \)-psh function is said to be a subextension of \( \varphi \). Now we have
5.1. Theorem. Let $\Omega$ be a hyperconvex domain in $X$ such that $\omega$ has a negative potential $\theta$ on $\Omega$. Assume that $\varphi \in \mathcal{F}(\Omega)$. Then there exist $a > 0$ and $\tilde{\varphi} \in \text{PSH}(X, a\omega)$ such that $\tilde{\varphi} \not\equiv -\infty$ and $\tilde{\varphi} \leq \varphi$ on $\Omega$.

Proof. Let $\mathcal{E}_0(\Omega) \ni \varphi \searrow \varphi$ be such that $\alpha = \int_{\Omega} (dd^c \varphi)^n < \infty$. Take an increasing exhaustion sequence $\{\Omega_j\}$ of $\Omega$ by relatively compact subdomains $\Omega_j \Subset \Omega$. For each $j \geq 1$, put

$$h_j = h_{\Omega_j, \varphi_j} = \sup\{v \in \text{PSH}(\Omega) : v \leq 0 \text{ and } v|_{\Omega_j} \leq \varphi_j\}.$$ 

Then $\mathcal{E}_0(\Omega) \ni h_j \searrow \varphi$ and

$$\alpha_j = \int_{\Omega} (dd^c h_j)^n \leq \int_{\Omega} (dd^c h_{j+1})^n = \alpha_{j+1} \to \alpha$$

(see Proposition 5.1 in [Ce2]). Consider the probability measure $\mu_j = (1/\alpha_j)(dd^c h_j)^n$ on $X$. Notice that $\text{supp}(dd^c h_j)^n \subset \overline{\Omega}_j$. Theorem 5.1 in [Ko] and Proposition 2.10 in [GZ1] imply that for each $j, p \geq 1$ there exist $A_p, B_p^j > 0$ such that

$$\mu_j(E) = \mu_j(E \cap \overline{\Omega}_j) \leq \frac{1}{\alpha_j} \int_{\Omega} (-h_j^*)_{E \cap \overline{\Omega}_j}^p (dd^c h_j)^n$$

$$\leq \frac{A_p}{\alpha_j} \left( \int_{\Omega} (-h_j)^p (dd^c h_j)^n \right)^{n/(p+n)} \left( \int_{\Omega} (-h_j^*)_{E \cap \overline{\Omega}_j}^p (dd^c h_j^*)_{E \cap \overline{\Omega}_j}^n \right)^{p/(p+n)}$$

$$\leq \frac{A_p}{\alpha_j} \left( \int_{\Omega} (-h_j)^p (dd^c h_j)^n \right)^{n/(p+n)} C_n (E \cap \overline{\Omega}_j, \Omega)^{p/(p+n)}$$

$$\leq B_p^j \text{Cap}_\omega(E \cap \overline{\Omega}_j, X)^{p/(p+n)}$$

for all Borel sets $E \subset X$. Proposition 6.5 and Theorem 5.1 in [GZ2] imply that there exists $v_j \in \mathcal{E}^p(X, \omega)$ such that

$$\omega_{v_j}^n = \mu_j \quad \text{and} \quad \sup_X v_j = -1.$$ 

Since

$$(dd^c (v_j + \theta))^n = \omega_{v_j}^n = \mu_j = \left( dd^c \left( \frac{1}{\alpha_j} h_j \right) \right)^n$$

on $\Omega$, by the comparison principle in [BT] it follows that

$$v_j + \theta \leq \frac{1}{\alpha_j^{1/n}} h_j$$

on $\Omega$. Thus for $u_j = \alpha_j^{1/n}(v_j + c)$ with $c = \inf_{\Omega} \theta < 0$ we have $u_j \in \text{PSH}(X, \alpha_j^{1/n} \omega) \cap L^\infty(X) \subset \text{PSH}(X, \alpha_j^{1/n} \omega) \cap L^\infty(X)$ and $\sup_X u_j = \alpha_j^{1/n}(c-1)$, $u_j \leq h_j \leq \varphi_j$ on $\Omega_j$. Define $\tilde{\varphi} = (\lim_{j \to \infty} u_j)^*$. Since $\sup_X u_j = \alpha_j^{1/n}(c-1) \to \alpha^{1/n}(c-1)$ as $j \to \infty$, we have $\tilde{\varphi} \not\equiv -\infty$ and it is easy to
see that $\tilde{\varphi} \in \text{PSH}(X, a\omega)$ with $a = a^{1/n}$ and $\tilde{\varphi} \leq \varphi$ on $\Omega$. Theorem 5.1 is completely proved.

5.2. Corollary. Let $\Omega$ be a bounded hyperconvex domain in $\mathbb{C}^n$ and $\varphi \in \mathcal{F}(\Omega)$. Then there exists $\tilde{\varphi} \in \mathcal{L}_\varepsilon(\mathbb{C}^n)$ such that $\tilde{\varphi} \leq \varphi$ on $\Omega$. Here

$$\varepsilon = \left[ \frac{1}{\Omega} \right]^{1/n},$$

$$\mathcal{L}_\varepsilon = \{ u \in \text{PSH}(\mathbb{C}^n) : u(z) \leq \varepsilon \log^+ ||z|| + O(1) \}.$$

Proof. Consider $\Omega$ as a domain in $\mathbb{C}^n = \mathbb{C}^n \cup H_\infty$. By Theorem 5.1 there exists $\psi \in \text{PSH}(\mathbb{C}^n, \varepsilon \omega)$ such that $\psi \leq \varphi$ on $\Omega$ and $\psi \not\equiv -\infty$. Define

$$\tilde{\varphi}(z) = \psi(z) + \frac{\varepsilon}{2} \log(1 + ||z||^2) - c$$

with

$$c = \sup_{\Omega} \frac{\varepsilon}{2} \log(1 + ||z||^2).$$

It follows that $\tilde{\varphi} \in \mathcal{L}_\varepsilon(\mathbb{C}^n)$ and $\tilde{\varphi} \leq \varphi$ on $\Omega$.

Remark. Corollary 5.2 was proved as Theorem 5.1 of [CKZ].

6. Appendix: The comparison principle in the class $\mathcal{E}(X, \omega)$. In [Ko] Kolodziej proved the comparison principle for bounded $\omega$-psh functions by using the approximation theorem of Demailly [De]. The aim of this section is to establish this principle in the class $\mathcal{E}(X, \omega)$. Notice that here we give a direct proof without using Demailly’s theorem.

6.1. Theorem. Let $\varphi, \psi, \varphi_1, \ldots, \varphi_{n-1} \in \mathcal{E}(X, \omega)$ and $T = \omega_{\varphi_1} \wedge \cdots \wedge \omega_{\varphi_{n-1}}$. Then

$$\int_{\{ \varphi < \psi \}} \omega_\psi \wedge T \leq \int_{\{ \varphi < \psi \}} \omega_\varphi \wedge T + \int_{\{ \varphi = \psi = -\infty \}} \omega_\varphi \wedge T.$$

Proof. We split the proof into the following two steps.

Step 1. First we prove that

$$\int_{\{ \varphi < \psi \}} \omega_\psi \wedge T \leq \int_{\{ \varphi < \psi \}} \omega_\varphi \wedge T. \quad (3)$$

For this, we establish the equality

$$\int_x (dd^c \varphi + \omega) \wedge T = \int_x \omega \wedge T. \quad (4)$$

Assume for the moment that (4) is true. Put $\varphi_\varepsilon = \max(\varphi + \varepsilon, \psi)$, $\varepsilon > 0$. From (4) it follows that

$$\int_x (dd^c \varphi_\varepsilon + \omega) \wedge T = \int_x \omega^n = \int_x (dd^c \varphi + \omega) \wedge T.$$
This equality together with the equality
\[(dd^c \varphi_\varepsilon + \omega) \wedge T|_{\{\varphi + \varepsilon > \psi\}} = (dd^c \varphi + \omega) \wedge T|_{\{\varphi + \varepsilon > \psi\}} \quad \text{(see [KH])}\]
implies that
\[
\int_{\{\varphi + \varepsilon \leq \psi\}} (dd^c \varphi_\varepsilon + \omega) \wedge T \leq \int_{\{\varphi \leq \psi\}} (dd^c \varphi + \omega) \wedge T.
\]
On the other hand,
\[(dd^c \varphi_\varepsilon + \omega) \wedge T|_{\{\varphi + \varepsilon < \psi\}} = (dd^c \psi + \omega) \wedge T|_{\{\varphi + \varepsilon < \psi\}}\]
so we obtain
\[
\int_{\{\varphi + \varepsilon < \psi\}} (dd^c \psi + \omega) \wedge T = \int_{\{\varphi + \varepsilon < \psi\}} (dd^c \varphi_\varepsilon + \omega) \wedge T \leq \int_{\{\varphi + \varepsilon < \psi\}} (dd^c \varphi + \omega) \wedge T.
\]
Letting \(\varepsilon\) tend to 0 we obtain
\[
\int_{\{\varphi < \psi\}} \omega_\psi \wedge T \leq \int_{\{\varphi \leq \psi\}} \omega_\varphi \wedge T,
\]
because \(\{\varphi + \varepsilon < \psi\} \not\supset \{\varphi < \psi\}\) as \(\varepsilon \to 0\). Thus (3) follows.

To prove (4), we first observe that by Stokes’ formula, if \(\varphi\) is bounded then
\[
(5) \quad \int_X dd^c \varphi \wedge T = 0.
\]
Next consider the case \(\varphi \in \mathcal{E}(X, \omega)\). Set \(\varphi_j = \max(\varphi, -j)\). Notice that \(\varphi_j \in \mathcal{E}(X, \omega) \cap L^\infty(X)\) and \(\varphi_j \searrow \varphi\). Therefore \(dd^c \varphi_j \wedge T\) weakly converges to \(dd^c \varphi \wedge T\). Using the above result we have \(\int_X dd^c \varphi_j \wedge T = 0\) for all \(j\), and hence \(\int_X dd^c \varphi \wedge T = 0\).

**Step 2.** Applying Step 1 to \(\varphi + \varepsilon\) and \(\psi\) we get
\[
\int_{\{\varphi + \varepsilon < \psi\}} \omega_\psi \wedge T \leq \int_{\{\varphi + \varepsilon \leq \psi\}} \omega_\varphi \wedge T.
\]
Letting \(\varepsilon\) tend to 0 we have
\[
\int_{\{\varphi < \psi\}} \omega_\psi \wedge T \leq \int_{\{\varphi < \psi\}} \omega_\varphi \wedge T + \int_{\{\varphi = \psi = -\infty\}} \omega_\varphi \wedge T,
\]
because \(\{\varphi + \varepsilon \leq \psi\} \not\supset \{\varphi < \psi\} \cup \{\varphi = \psi = -\infty\}\) as \(\varepsilon \to 0\).

**6.2. COROLLARY.** Let \(\varphi, \psi \in \mathcal{E}(X, \omega)\). Then
\[
\int_{\{\varphi < \psi\}} \omega_\psi^n \leq \int_{\{\varphi < \psi\}} \omega_\varphi^n + \int_{\{\varphi = \psi = -\infty\}} \omega_\varphi^n.
\]
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