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Integral transforms of functions with restricted derivatives

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Abstract. We show that functions whose derivatives lie in a half-plane are preserved under the Pommerenke, Chandra–Singh, Libera, Alexander and Bernardi integral transforms. We determine precisely how these transforms act on such functions. We prove that if the derivative of a function lies in a convex region then the derivative of its Pommerenke, Chandra–Singh, Libera, Alexander and Bernardi transforms lie in a strictly smaller convex region which can be determined. We also consider iterates of these transforms. Best possible results are obtained.

1. Introduction. Let $A(\mathbb{D})$ denote the class of functions f which are analytic in the unit disk $\mathbb{D}$ and normalized by $f(0) = 0$ and $f'(0) = 1$. The classical family of univalent functions in $A(\mathbb{D})$ is denoted by S. The following are well-known integral transforms on $A(\mathbb{D})$:

$$\mathbf{A}f(z) = \int_0^z \frac{f(\zeta)}{\zeta}\, d\zeta \qquad \text{(Alexander transform [1])},$$

$$\mathbf{L}f(z) = \frac{2}{z}\int_0^z f(\zeta)\, d\zeta \qquad \text{(Libera transform [9])},$$

$$\mathbf{B}_c f(z) = (c+1)\int_0^1 t^{c-1} f(tz)\, dt, \quad c > -1 \qquad \text{(Bernardi transform [2])}.$$

The Alexander and Libera transforms are special cases of the Bernardi transform with $c = 0$ and $c = 1$, respectively.

Biernacki [3] claimed that the Alexander transform preserved the class S, however a counterexample to this was constructed by Krzyż and Lewandowski [8]. Campbell and Singh [4] later showed that S is not preserved under the Libera transform either. Hence it was of interest to determine which subclasses of S and, more generally, of $A(\mathbb{D})$ are preserved under these and other transforms. It is known that the subclasses of S consisting of convex, starlike and close-to-convex functions (denoted by K, S^* and C,

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respectively) are each preserved under the Alexander and Libera transforms and also under the Bernardi transform for $c = 0, 1, \dots$ (see [2] for example). Ruscheweyh and Sheil-Small [13] also proved these same results using the theory of convolutions.

Another interesting integral transform was first introduced by Pommerenke [11]:

$$\mathbf{P}f(z) = \int_0^z \frac{f(z_1\zeta) - f(z_2\zeta)}{z_1\zeta - z_2\zeta}\, d\zeta, \tag{1.1}$$

for fixed $|z_1| \le 1$ and $|z_2| \le 1$. He proved that if $f \in C(\alpha)$ for $0 \le \alpha \le 1$, the class of *strongly close-to-convex functions of order* α (i.e., $|\arg\{f'(z)/h'(z)\}| \le \pi\alpha/2$ for some convex function h), then $\mathbf{P}f \in C(\alpha)$. Note that $\alpha = 0$ and $\alpha = 1$ correspond to the class of convex and close-to-convex functions, respectively. Recall that a function f is *close-to-convex of order* α if $\mathrm{Re}\{f'(z)/h'(z)\} > \alpha$.

Later, and apparently unaware of this result, Chandra and Singh [5] introduced a special case of the transform (1.1) defined by

$$\mathbf{P}_{\nu_1,\nu_2} f(z) = \frac{1}{e^{i\nu_1} - e^{i\nu_2}} \int_0^z \frac{f(te^{i\nu_1}) - f(te^{i\nu_2})}{t}\, dt, \tag{1.2}$$

where $0 \le \nu_1 < \nu_2 < 2\pi$, and proved that convex, starlike and close-to-convex functions of order α as well as strongly close-to-convex functions of order α are all preserved under the transform $\mathbf{P}_{\nu_1,\nu_2}$. Since integral transforms tend to smooth functions these results are not too surprising. In this paper we shall study these transforms on classes of functions in $A(\mathbb{D})$ with restricted derivatives.

A function $f \in A(\mathbb{D})$ is said to be of *bounded turning* of order β, where $0 \le \beta < 1$, if $\mathrm{Re}\{f'(z)\} > \beta$ for all $z \in \mathbb{D}$. We denote this class by R_β. By the Noshiro–Warschawski theorem we know that R_β is a subclass of S and is in fact a subclass of close-to-convex functions (see Duren [6]). It is easy to see that the Bernardi transform maps R_β into R_β:

$$\mathrm{Re}\{(\mathbf{B}_c f)'(z)\} = (c+1)\int_0^1 t^c \,\mathrm{Re}\{f'(tz)\}\, dt > (c+1)\int_0^1 t^c \beta\, dt = \beta.$$

It is also known for example that if $f \in R_0$ then $\mathbf{A}f \in S^*$ (see [14]).

Ponnusamy and Rønning [12] generalized R_β and studied the Bernardi transform of functions in $A(\mathbb{D})$ whose derivatives lie in an arbitrary half-plane. They defined this class of functions as

$$\mathcal{P}_\beta = \{f \in A(\mathbb{D}) : \exists \alpha \in \mathbb{R},\ \mathrm{Re}[e^{i\alpha}(f'(z) - \beta)] > 0,\ \forall z \in \mathbb{D}\},$$

where $\beta \in \mathbb{R}$, and proved a number of sharp results including finding the largest $\beta = \beta(c, \gamma)$ such that if $f \in \mathcal{P}_\beta$, then its Bernardi transform $\mathbf{B}_c f(z)$

is starlike of order γ, generalizing the result in [14]. We should point out that unlike R_β, the class $\mathcal{P}_\beta$ may contain non-univalent functions, as can be shown by the function $f(z) = z + z^2$ which belongs to every $\mathcal{P}_\beta$ for $\beta < -1$, but does not belong to S.

We define the class of functions R_β^α as follows:

$$R_\beta^\alpha = \{f \in A(\mathbb{D}) : \operatorname{Re}[e^{i\alpha}(f'(z) - \beta)] > 0,\ \forall z \in \mathbb{D}\}. \tag{1.3}$$

It is clear that if $f \in R_\beta^\alpha$ then $f'(0) = 1$ and so necessarily we must have

$$(1 - \beta)\cos\alpha > 0. \tag{1.4}$$

Note that for a fixed β, we have $R_\beta^\alpha \subset \mathcal{P}_\beta$. As above, it is easy to see that the Bernardi transform also maps R_β^α into R_β^α. It is natural to ask if the class R_β^α is preserved under the Chandra–Singh transform (1.2) and more generally the Pommerenke transform (1.1). We prove that this is indeed the case and also show that all these transforms actually map R_β^α into strictly smaller subclasses which can be determined.

We can now state our main results.

THEOREM 1. *Let* $\alpha, \beta \in \mathbb{R}$ *satisfy* (1.4). *If* $f \in R_\beta^\alpha$, *then*:

(a) $\mathbf{P}f \in R_{\beta_{\mathbf{P}}}^\alpha$, *where*

$$\mathbf{P}f(z) = \int_0^z \frac{f(z_1\zeta) - f(z_2\zeta)}{z_1\zeta - z_2\zeta}\, d\zeta \qquad (z_1, z_2 \in \overline{\mathbb{D}}),$$

$$\beta_{\mathbf{P}} = 2\beta - 1 + (1 - \beta)\,\frac{3 + \delta}{2 + 2\delta} \tag{1.5}$$

and $\delta = \max\{\min\{|z_1|, |z_2|\}, |z_1 + z_2|/2\}$.

(b) $\mathbf{P}_{\nu_1,\nu_2} f \in R_{\beta_*}^\alpha$, *where*

$$\mathbf{P}_{\nu_1,\nu_2} f(z) = \frac{1}{e^{i\nu_1} - e^{i\nu_2}} \int_0^z \frac{f(te^{i\nu_1}) - f(te^{i\nu_2})}{t}\, dt \qquad (0 \le \nu_1 < \nu_2 < 2\pi),$$

$$\beta_* = 2\beta - 1 + (1 - \beta)\nu/\sin\nu \tag{1.6}$$

and $\nu = \frac{1}{2}\min\{(\nu_2 - \nu_1),\ 2\pi - (\nu_2 - \nu_1)\} \in (0, \pi/2]$. *This result is best possible.*

(c) $\mathbf{B}_c f \in R_{\beta_c}^\alpha$ *for* $c = 0, 1, \ldots,$ *where*

$$\mathbf{B}_c f(z) = (c + 1)\int_0^1 t^{c-1} f(tz)\, dt,$$

$$\beta_c = 2\beta - 1 + (1 - \beta)\gamma_c \tag{1.7}$$

with $\gamma_0 = \log 4$ *and*

$$\gamma_c = 2(c+1)(-1)^c\left[\log 2 - \sum_{k=1}^{c}\frac{(-1)^{k+1}}{k}\right], \qquad c = 1, 2, \ldots, \tag{1.8}$$

and $1 < \gamma_c < 2$. *This result is best possible.*

REMARK 1. If both z_1 and z_2 lie on $|z| = 1$, then the Pommerenke transform (1.1) reduces to the Chandra–Singh transform (1.2). Consequently, without loss of generality, we shall henceforth assume when referring to the Pommerenke transform that at most one of z_1 and z_2 lies on $|z| = 1$. Thus we then have $0 \le \delta < 1$.

The proof of Theorem 1 is given in the next section. We first state and prove some applications.

COROLLARY 1. *If* $f \in R^\alpha_\beta$, *then*:

(i) $\mathbf{P}f \in R^\alpha_{\beta_\mathbf{P}} \subset R^\alpha_\beta$, *where* $\beta_\mathbf{P}$ *is given by* (1.5).
(ii) $\mathbf{P}_{\nu_1,\nu_2}f \in R^\alpha_{\beta_*} \subset R^\alpha_\beta$, *where* β_* *is given by* (1.6).
(iii) $\mathbf{B}_c f \in R^\alpha_{\beta_c} \subset R^\alpha_\beta$ *for* $c = 0, 1, \ldots,$ *where* β_c *is given by* (1.7).

Proof. Let α and β be fixed and let

$$\beta^{**} = 2\beta - 1 + M(1-\beta),$$

where $M > 1$ is fixed. We assert that $R^\alpha_{\beta^{**}} \subset R^\alpha_\beta$. The corollary then follows because if $f \in R^\alpha_\beta$ then from the theorem in each of the cases (i)–(iii) we simply let $M = (3+\delta)/(2+2\delta), \nu/\sin\nu$, γ_c, respectively, to conclude that the corresponding transform F belongs to $R^\alpha_{\beta^{**}}$.

To prove our assertion that $R^\alpha_{\beta^{**}} \subset R^\alpha_\beta$ we consider several cases. Suppose $F \in R^\alpha_{\beta^{**}}$ and recall that $(1-\beta)\cos\alpha > 0$.

CASE 1: $-\infty < \beta < 1$. In this case we have $\cos\alpha > 0$ and we obtain

$$\beta^{**} = 2\beta - 1 + M(1-\beta) > \beta.$$

Since $F \in R^\alpha_{\beta^{**}}$, i.e., $\operatorname{Re}\{e^{i\alpha}[F'(z) - \beta^{**}]\} > 0$, we obtain

$$\operatorname{Re}\{e^{i\alpha}F'(z)\} > \beta^{**}\cos\alpha > \beta\cos\alpha,$$

which implies that $F \in R^\alpha_\beta$.

CASE 2: $1 < \beta < \infty$. Here $\cos\alpha < 0$ and observe that $\beta^{**} < \beta$. Thus we have $\operatorname{Re}\{e^{i\alpha}F'(z)\} > \beta^{**}\cos\alpha > \beta\cos\alpha$ and hence $F \in R^\alpha_\beta$. ∎

In the above result, these transforms map R^α_β into strictly smaller subclasses, and since the values given by (1.6) and (1.7) are best possible, the Chandra–Singh and Bernardi transforms do not map R^α_β into any class smaller than the corresponding $R^\alpha_{\beta^{**}}$.

If the derivative of an arbitrary function in $A(\mathbb{D})$ lies in a region, then one might expect that the region in which the derivative of its integral transform lies should be related. We obtain the following result:

THEOREM 2. *Let* $f \in A(\mathbb{D})$ *and let* F *be its Pommerenke, Chandra–Singh or Bernardi transform with* $c = 0, 1, \ldots$. *If* $\Delta(f) = \{f'(z) : z \in \mathbb{D}\}$ *lies in a convex region* Ω, *then* $\Delta(F) = \{F'(z) : z \in \mathbb{D}\}$ *also lies in* Ω.

Proof. Note that $f \in R_\beta^\alpha$ if and only if $f(rz)/r \in R_\beta^\alpha$ for any $0 < r < 1$. Hence, without loss of generality, we may assume that $\Omega \subset \mathbb{C}$ is bounded. Furthermore, we may assume that Ω is a convex polygonal region. Consequently, it is sufficient to prove the theorem when Ω is a bounded convex polygonal region with m sides. Necessarily we have $1 \in \Omega$. Let $f \in A(\mathbb{D})$ and suppose that $\Delta(f) = \{f'(z) : z \in \mathbb{D}\} \subset \Omega$.

Assume first that $\partial\Omega$ contains no horizontal segments. Because $\overline{\Omega}$ may be obtained as the intersection of m closed half-planes, each containing 1, it follows that

$$f \in \bigcap_{j=1}^{m} R_{\beta_j}^{\alpha_j}$$

for suitable choices of α_j and β_j, each satisfying $(1-\beta_j)\cos\alpha_j > 0$. To see this, we let L_j be the line bounding a side of Ω, β_j its intersection with the real axis and μ_j $(0 < \mu_j < \pi)$ the angle L_j makes with the positive real axis. If $\beta_j > 1$, choose $\alpha_j = 3\pi/2 - \mu_j$, while if $\beta_j < 1$, set $\alpha_j = \pi/2 - \mu_j$. Hence $f \in R_{\beta_j}^{\alpha_j}$ for each j and by Corollary 1 the same holds for F. Thus $F \in \bigcap_{j=1}^{m} R_{\beta_j}^{\alpha_j}$ and so we conclude that $\Delta(F) \subset \Omega$.

If a side of $\overline{\Omega}$ is a horizontal segment then we construct a larger convex polygonal region containing all non-horizontal sides of Ω but replace each horizontal side by two non-horizontal sides as follows. Let $0 < \varepsilon < 1$ and define the convex set $\Omega(\varepsilon)$ to be bounded by all the lines bounding $\overline{\Omega}$ except the horizontal lines. Each horizontal line is to be replaced by two intersecting lines. In particular, if say Ω is bounded by the horizontal line L_h through the vertices $\omega_1 = a + i\lambda$ and $\omega_2 = b + i\lambda$ with $a < b$ and $\lambda > 0$, then instead of bounding $\Omega(\varepsilon)$ by L_h, we bound it by the two lines $L_h^{(1)}$ and $L_h^{(2)}$ which pass through the pair ω_1 and $\omega_\varepsilon = (b+a)/2 + i[\lambda + \varepsilon(b-a)]$ and the pair ω_2 and ω_ε, respectively. With this construction, it is clear that $\Omega \subset \Omega(\varepsilon)$ for all $0 < \varepsilon < 1$ and that $\Omega(\varepsilon)$ has no horizontal lines bounding it. A similar construction holds for $\lambda < 0$. Apply the above argument to $\Omega(\varepsilon)$ and let $\varepsilon \to 0$ to complete the proof of the theorem. ∎

REMARK 2. It should be pointed out that by Corollary 1, since the transforms map R_β^α strictly into itself, we actually have $\Delta(F) \subset \Omega' \subset \Omega$, where Ω' is a convex region strictly inside Ω. The convex region Ω' can be determined, once Ω is known.

Finally, we consider iterates of integral transforms. Because these integral transforms map R^α_β into strictly smaller subclasses the following result obtains:

THEOREM 3. *If f is an arbitrary function in R^α_β and $\mathbf{T}f$ is its Pommerenke, Chandra–Singh or Bernardi transform with $c = 0, 1, \ldots,$ then*

$$\lim_{n\to\infty} \mathbf{T}^{(n)} f(z) = z,$$

where $\mathbf{T}^{(n)} = \mathbf{T} \circ \cdots \circ \mathbf{T}$ is the n^{th} iterate of $\mathbf{T}$ and the convergence is uniform on compact subsets in $\mathbb{D}$.

We shall also prove this theorem in the next section.

2. Proof of the main results. We begin with a few preliminaries about the class R^α_β. Assume throughout that α and β are fixed and satisfy (1.4).

It is clear that the function K defined by

$$K(z) = e^{-i\alpha}[Az + B\log(1-z)], \tag{2.1}$$

where

$$A = -\lambda\cos\alpha + i\sin\alpha, \qquad B = -(1+\lambda)\cos\alpha, \qquad \lambda = 1 - 2\beta \tag{2.2}$$

belongs to the class R^α_β and so it is non-empty. The class R^α_β is convex: if $f, g \in R^\alpha_\beta$ then $tf + (1-t)g \in R^\alpha_\beta$ for all $0 \le t \le 1$. It is also rotationally invariant: $f \in R^\alpha_\beta$ if and only if $e^{-i\mu} f(e^{i\mu} z) \in R^\alpha_\beta$ for $\mu \in \mathbb{R}$.

The Carathéodory class $\mathfrak{P}$ consists of all functions p which are analytic in $\mathbb{D}$ with $\operatorname{Re} p(z) > 0$ and normalized by $p(0) = 1$. Observe that $g \in R^\alpha_\beta$ if and only if

$$p(z) = \frac{e^{i\alpha}(g'(z) - \beta) - i(1-\beta)\sin\alpha}{(1-\beta)\cos\alpha} \tag{2.3}$$

belongs to $\mathfrak{P}$. From this and the distortion theorems for $p \in \mathfrak{P}$ (see [6] or [7] for example), we see that if $g \in R^\alpha_\beta$, then $|g'(z)|$ and hence $|g(z)|$ are bounded on all compact sets in $\mathbb{D}$, and so the normalization for functions in R^α_β makes it a compact family.

The extreme points of the Carathéodory class $\mathfrak{P}$ are well-known [7]:

$$\mathcal{E}(\mathfrak{P}) = \left\{ \frac{1+xz}{1-xz} : |x| = 1 \right\}. \tag{2.4}$$

From (2.3) and (2.4) it follows that the extreme points for the class R^α_β are precisely

$$\mathcal{E}(R^\alpha_\beta) = \{\overline{x} K(xz) : |x| = 1\} \tag{2.5}$$

where K is defined by (2.1) and (2.2).

We will make use of the following result, essentially due to Marx [10].

LEMMA 1. *If*

$$H(\theta,\mu)=\operatorname{Im}\left\{-e^{-i\theta}\log\frac{1-e^{i(\theta+\mu)}}{1-e^{i(\theta-\mu)}}\right\},\qquad 0\le\theta,\mu\le\pi,$$

then

$$\min_{0\le\theta\le\pi} H(\theta,\mu)=\begin{cases}\mu, & 0\le\mu\le\pi/2,\\ \pi-\mu, & \pi/2<\mu\le\pi.\end{cases}$$

Proof. Observe that if $\theta\neq\mu$ then

$$H(\theta,\mu)=\frac{\sin\theta}{2}\log\frac{1-\cos(\theta+\mu)}{1-\cos(\theta-\mu)}-\gamma\cos\theta,$$

where

$$\gamma=\begin{cases}\mu, & 0\le\mu<\theta\le\pi,\\ \mu-\pi, & 0\le\theta<\mu\le\pi.\end{cases}$$

After a calculation we obtain

$$\frac{\partial H}{\partial\theta}=\frac{\cos\theta}{2}\log\frac{1-\cos(\theta+\mu)}{1-\cos(\theta-\mu)}+\frac{\sin\theta\sin\mu}{\cos\theta-\cos\mu}+\gamma\sin\theta.$$

A further calculation leads to

$$\text{(2.6)}\qquad \frac{\partial}{\partial\mu}\left(\frac{\partial H}{\partial\theta}\right)=\frac{\sin\theta}{(\cos\theta-\cos\mu)^2}\,(2\cos\theta\cos\mu-\cos^2\theta-1)$$
$$\le-\frac{(\sin\theta)(1-|\cos\theta|)^2}{(\cos\theta-\cos\mu)^2}.$$

Consequently, for fixed $0\le\theta_0\le\pi$, the function $\partial H/\partial\theta$ is non-increasing with μ.

Suppose first that $0\le\theta_0<\mu\le\pi$. Then

$$\frac{\partial H}{\partial\theta}(\theta_0,\mu)\ge\frac{\partial H}{\partial\theta}(\theta_0,\pi)=0,$$

and so for $0\le\theta<\mu\le\pi$, we see that H is a non-decreasing function of θ and thus

$$H(\theta,\mu)\ge H(0,\mu)=\pi-\mu.$$

Next, if $0\le\mu<\theta_0\le\pi$ then

$$\frac{\partial H}{\partial\theta}(\theta_0,\mu)\le\frac{\partial H}{\partial\theta}(\theta_0,0)=0.$$

In this case, H is a non-increasing function of θ and hence for $0\le\mu<\theta\le\pi$ we get

$$H(\theta,\mu)\ge H(\pi,\mu)=\mu.$$

Thus if $\theta\neq\mu$ then $H(\theta,\mu)\ge\min\{\mu,\,(\pi-\mu)\}$ and the function is unbounded as $\theta\to\mu$. This proves the lemma. ∎

It should be pointed out that there is a typo in formula (65) in Marx [10]. It should read:

$$\frac{\partial}{\partial\phi}\left(4\sin\phi\,\frac{\partial p(\phi,\theta)}{\partial\theta}\right)=\frac{(2\sin\theta)(2\cos\theta\cos\phi-\cos^2\theta-1)}{(\cos\theta-\cos\phi)^2}.$$

Fortunately, his conclusion that the function on the left is non-positive still holds as our (2.6) shows.

Lemma 2. *If*

$$\Phi(\zeta_1,\zeta_2)=\frac{1}{\zeta_2-\zeta_1}\log\frac{1-\zeta_1}{1-\zeta_2}$$

and $\zeta_1,\zeta_2\in\overline{\mathbb{D}}\,(\zeta_1\neq\zeta_2)$, *then*

$$\operatorname{Re}\Phi(\zeta_1,\zeta_2)\geq\frac{3+\delta}{4+4\delta}$$

where $\delta=\max\{\min\{|\zeta_1|,|\zeta_2|\},\ |\zeta_1+\zeta_2|/2\}$.

Proof. Let $\omega(t)=\zeta_1+(\zeta_2-\zeta_1)t,\ 0\leq t\leq 1$, be the line segment from ζ_1 to ζ_2 in $\overline{\mathbb{D}}$. It follows that $|\omega(t)|\leq\delta$ for $0\leq t\leq 1/2$ or $1/2\leq t\leq 1$. To see this, suppose say $\delta=|\zeta_1|$; then

$$|\omega(1/2)|=|\zeta_1+\zeta_2|/2\leq|\zeta_1|=|\omega(0)|=\delta$$

and hence $|\omega(t)|\leq\delta$ for $0\leq t\leq 1/2$. The proof of the other cases follows a similar pattern. Using this we conclude that

$$\operatorname{Re}\Phi(\zeta_1,\zeta_2)=\operatorname{Re}\left\{\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\frac{1}{1-z}\,dz\right\}=\operatorname{Re}\int_0^1\frac{1}{1-\omega(t)}\,dt$$

$$\geq\int_0^1\frac{1}{1+|\omega(t)|}\,dt\geq\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{1+\delta}=\frac{3+\delta}{4+4\delta}.\ \blacksquare$$

We can now prove the main results.

Proof of Theorem 1. We consider each transform separately.

(a) Let $F=\mathbf{P}f$. Now for fixed $z_0\in\mathbb{D}$ we have

$$\operatorname{Re}\{e^{i\alpha}F'(z_0)\}=\operatorname{Re}\left\{e^{i\alpha}\,\frac{f(z_1z_0)-f(z_2z_0)}{z_1z_0-z_2z_0}\right\}.$$

The linear functional

$$L(f)=e^{i\alpha}\,\frac{f(z_1z_0)-f(z_2z_0)}{z_1z_0-z_2z_0}$$

attains its minimum real part over the set of extreme points of R^α_β. (This follows e.g. from Thm. 4.5, p. 44, in [7] by observing that $-\min\operatorname{Re}\{L(f)\}=\max\operatorname{Re}\{J(f)\}$, where $J(f)=-L(f)$.) Consequently,

$$\operatorname{Re}\{e^{i\alpha}F'(z_0)\}\geq\min_{|x|=1}\operatorname{Re}\left\{e^{i\alpha}\,\frac{K(xz_1z_0)-K(xz_2z_0)}{xz_1z_0-xz_2z_0}\right\},$$

where K is given by (2.1). Using (2.1) and (2.2) we obtain

$$\operatorname{Re}\{e^{i\alpha}F'(z_0)\} \geq \min_{|x|=1} \operatorname{Re}\left\{ A + B\, \frac{1}{(xz_1z_0 - xz_2z_0)} \log \frac{1-xz_1z_0}{1-xz_2z_0} \right\}.$$

The above function is analytic in the variable $z = xz_0$ and hence by the minimum principle and Lemma 2 we may conclude that

$$\begin{aligned}\operatorname{Re}\{e^{i\alpha}F'(z_0)\} &> (2\beta-1)\cos\alpha + 2(1-\beta)(\cos\alpha)\min_{|z|=1}\operatorname{Re}\{\Phi(\{zz_1\},\{zz_2\})\} \\ &\geq (2\beta-1)\cos\alpha + 2(1-\beta)(\cos\alpha)\frac{3+\delta}{4+4\delta} = \beta_{\mathbf{P}}\cos\alpha.\end{aligned}$$

Thus for any $z_0 \in \mathbb{D}$, we get $\operatorname{Re}\{e^{i\alpha}[F'(z_0)-\beta_{\mathbf{P}}]\} > 0$ and so $F = \mathbf{P}f \in R^{\alpha}_{\beta_{\mathbf{P}}}$.

(b) Let $F = \mathbf{P}_{\nu_1,\nu_2}f$. Note that the function $F \in R^{\alpha}_{\beta_*}$ if and only if $G(z) = e^{-i\mu}F(e^{i\mu}z) \in R^{\alpha}_{\beta_*}$ for any $\mu \in \mathbb{R}$. Hence we see that

$$G(z) = \frac{1}{e^{i(\nu_1+\mu)} - e^{i(\nu_2+\mu)}} \int_0^z \frac{f(se^{i(\nu_1+\mu)}) - f(se^{i(\nu_2+\mu)})}{s}\, ds. \tag{2.7}$$

If $\nu = (\nu_2-\nu_1)/2$ then setting $\mu = -(\nu_1+\nu_2)/2$ in (2.7) gives

$$G(z) = \frac{1}{e^{i\nu} - e^{-i\nu}} \int_0^z \frac{f(se^{i\nu}) - f(se^{-i\nu})}{s}\, ds.$$

On the other hand, if $\nu = \pi - (\nu_2-\nu_1)/2$, set $\mu = \pi - (\nu_1+\nu_2)/2$ to obtain the same form of $G(z)$. Thus it is sufficient to show that if $f \in R^{\alpha}_{\beta}$, then $G \in R^{\alpha}_{\beta_*}$ where

$$G(z) = \frac{1}{e^{i\nu} - e^{-i\nu}} \int_0^z \frac{f(se^{i\nu}) - f(se^{-i\nu})}{s}\, ds \tag{2.8}$$

with $0 < \nu \leq \pi/2$ and

$$\beta_* = 2\beta - 1 + (1-\beta)\,\frac{\nu}{\sin\nu}.$$

For fixed $0 < \nu \leq \pi/2$ we see from (2.8) that

$$\operatorname{Re}\{e^{i\alpha}G'(z)\} = \operatorname{Re}\left\{ \frac{e^{i\alpha}}{2i\sin\nu}\, \frac{f(ze^{i\nu}) - f(ze^{-i\nu})}{z} \right\}.$$

Now fix $z_0 \in \mathbb{D}$ and consider the linear functional on $A(\mathbb{D})$ defined by

$$L(f) = \frac{e^{i\alpha}}{2i\sin\nu}\, \frac{f(z_0e^{i\nu}) - f(z_0e^{-i\nu})}{z_0}.$$

The minimum real part of L is achieved at an extreme point of R^{α}_{β}. Hence

$$\operatorname{Re}\{e^{i\alpha}G'(z_0)\} \geq \min_{|x|=1} \operatorname{Re}\{L(\overline{x}K(xz))\},$$

where K is given by (2.1). A calculation shows that

$$\{L(\overline{x}K(xz))\} = A + \frac{B}{2i\sin\nu}\,\frac{1}{xz_0}\log\frac{1-e^{i\nu}xz_0}{1-e^{-i\nu}xz_0}.$$

This is an analytic function of $\omega = xz_0$. By (2.2) it follows from the minimum principle and symmetry that

$$\begin{aligned}\operatorname{Re}\{e^{i\alpha}G'(z_0)\} &\geq \min_{|x|=1}\operatorname{Re}\{L(\overline{x}K(xz))\}\\ &> (2\beta-1)\cos\alpha + \frac{(1-\beta)\cos\alpha}{\sin\nu}\min_{0\leq\theta\leq\pi}H(\theta,\nu)\end{aligned}$$

where

$$H(\theta,\nu) = \operatorname{Im}\left\{-e^{-i\theta}\log\frac{1-e^{i(\theta+\nu)}}{1-e^{i(\theta-\nu)}}\right\}.$$

We may now apply Lemma 1 with $\mu = \nu$ and $0 < \nu \leq \pi/2$ to see that

$$\operatorname{Re}\{e^{i\alpha}G'(z_0)\} \geq (2\beta-1)\cos\alpha + (1-\beta)(\cos\alpha)\frac{\nu}{\sin\nu} = \beta_*\cos\alpha.$$

Hence $\operatorname{Re}\{e^{i\alpha}[G'(z_0)-\beta_*]\} > 0$ for any $z_0 \in \mathbb{D}$ and so $G \in R^{\alpha}_{\beta_*}$.

To show that β_* is best possible, consider the function $f = K$ defined in (2.1) and let $z = -r$. A calculation gives

$$\operatorname{Re}\{e^{i\alpha}G'(-r)\} = \left[2\beta - 1 + \frac{1-\beta}{\sin\nu}\operatorname{Im}\left\{\frac{1}{r}\log\frac{1+re^{i\nu}}{1+re^{-i\nu}}\right\}\right]\cos\alpha$$

and hence

$$\lim_{r\to1}\operatorname{Re}\{e^{i\alpha}[G'(-r)-\beta_*]\} = 0.$$

(c) Let $F = \mathbf{B}_c f$. For $z_0 \in \mathbb{D}$ arbitrary but fixed, the linear functional $L(f) = (c+1)\int_0^1 e^{i\alpha}t^c f'(tz_0)\,dt$ assumes its minimum real part over the set of extreme points of R^{α}_{β} and hence

$$\operatorname{Re}\{e^{i\alpha}F'(z_0)\} \geq \min_{|x|=1}\operatorname{Re}\left\{(c+1)\int_0^1 e^{i\alpha}t^c K'(xtz_0)\,dt\right\} \tag{2.9}$$

where K is given by (2.1) and (2.2). Next, by the minimum principle, we see that

$$\begin{aligned}\min_{|x|=1}\int_0^1\operatorname{Re}\left\{\frac{t^c}{1-txz_0}\right\}dt &> \min_{-\pi<\theta\leq\pi}\int_0^1\operatorname{Re}\left\{\frac{t^c}{1-te^{i\theta}}\right\}dt\\ &\geq \int_0^1\frac{t^c}{1+t}\,dt = (-1)^c\left[\log 2 - \sum_{k=1}^{c}\frac{(-1)^{k+1}}{k}\right].\end{aligned}$$

Using (2.9) and this estimate we obtain after a calculation

$$\begin{aligned}\operatorname{Re}\{e^{i\alpha}F'(z_0)\} \\ &> \left\{2\beta - 1 + 2(1-\beta)(c+1)(-1)^c\left[\log 2 - \sum_{k=1}^{c}\frac{(-1)^{k+1}}{k}\right]\right\}\cos\alpha \\ &= \{2\beta - 1 + (1-\beta)\gamma_c\}\cos\alpha = \beta_c\cos\alpha.\end{aligned}$$

(If $c = 0$, then $\int_0^1 \operatorname{Re}\{1/(1-te^{i\theta})\}\,dt \geq \log 2$ and from (2.9) we get the above result with $\gamma_0 = \log 4$.) Thus $\operatorname{Re}\{e^{i\alpha}[F'(z_0)-\beta_c]\} > 0$ and hence we conclude that $F \in R^{\alpha}_{\beta_c}$. Because

$$\frac{1}{2(c+1)} = \int_0^1 \frac{t^c}{2}\,dt < \int_0^1 \frac{t^c}{1+t}\,dt < \int_0^1 t^c\,dt = \frac{1}{c+1}$$

and

$$\int_0^1 \frac{t^c}{1+t}\,dt = (-1)^c\left[\log 2 - \sum_{k=1}^{c}\frac{(-1)^{k+1}}{k}\right] = \frac{\gamma_c}{2(c+1)}$$

we must have

$$1 < \gamma_c < 2.$$

To show that β_c is best possible we consider the function $f = K$ given by (2.1) and let $z = -r$:

$$\begin{aligned}\operatorname{Re}\{e^{i\alpha}F'(-r)\} &= (c+1)\int_0^1 \operatorname{Re}\{e^{i\alpha}K'(-tr)\}\,dt \\ &= \left[2\beta - 1 + 2(1-\beta)(c+1)\int_0^1 \frac{t^c}{1+rt}\,dt\right]\cos\alpha.\end{aligned}$$

Using this and (1.7) we let $r \to 1$ to find that $\operatorname{Re}\{e^{i\alpha}[F'(-r)-\beta_c]\} \to 0$, and hence β_c is best possible. This completes the proof of Theorem 1. ■

Proof of Theorem 3. Fix α and β satisfying (1.4). By Corollary 1, we conclude that each of the transforms $\mathbf{P}$, $\mathbf{P}_{\nu_1,\nu_2}$ and $\mathbf{B}_c$ (for $c = 0, 1, \ldots$) maps R^{α}_{β} into $R^{\alpha}_{\beta^{**}}$, where

$$\beta^{**} = 2\beta - 1 + M(1-\beta) = \beta(2-M) + M - 1 \tag{2.10}$$

and

$$M = \frac{3+\delta}{2+2\delta}, \quad M = \frac{\nu}{\sin\nu} \quad \text{or} \quad M = \gamma_c,$$

respectively. Recall that $0 \leq \delta < 1$ (see Remark 1) and $0 < \nu \leq \pi/2$. Consequently, in each case we have $1 < M < 2$. Now let

$$F_0 = f, \quad F_1 = \mathbf{T}f, \quad \ldots, \quad F_n = (\mathbf{T} \circ \cdots \circ \mathbf{T})f,$$

where $\mathbf{T}$ is the Pommerenke transform $\mathbf{P}$, the Chandra–Singh transform $\mathbf{P}_{\nu_1,\nu_2}$ or the Bernardi transform $\mathbf{B}_c$. For convenience, set $x = 2 - M$ in (2.10). From Corollary 1, we may apply an induction argument to show that $F_n \in R^\alpha_{\beta(n)} \subset R^\alpha_\beta$ where

$$\beta(n) = \beta x^n + 1 - x^n.$$

Let $\varepsilon > 0$ be given. It suffices to show that $|F_n(z) - z| < \varepsilon$ for all $|z| \le r < 1$ and all $n > N(\varepsilon)$. Since $F_n \in R^\alpha_{\beta(n)}$, it follows from (2.3) that

$$F_n'(z) = e^{-i\alpha}\{p(z) - 1\}(1 - \beta(n))\cos\alpha + 1 \tag{2.11}$$

for some $p \in \mathfrak{P}$. Using (2.11) and the estimate $|p(re^{i\theta})| \le (1+r)/(1-r)$ for any $p \in \mathfrak{P}$, we obtain

$$\begin{aligned}|F_n(z) - z| &= \left|\int_0^z [e^{-i\alpha}\{p(\zeta) - 1\}(1 - \beta(n))\cos\alpha]\, d\zeta\right| \\ &= \left|ze^{-i\alpha}(1 - \beta(n))\cos\alpha \int_0^1 \{p(tz) - 1\}\, dt\right| \\ &\le r(1 - \beta(n))\cos\alpha \int_0^1 \left\{\frac{2}{1 - rt}\right\} dt \\ &= x^n\{-2(1 - \beta)(\cos\alpha)\log(1 - r)\}.\end{aligned}$$

Hence, since $0 < x < 1$, by choosing n sufficiently large we obtain the desired estimate, and this completes the proof of Theorem 3. ∎

3. Remarks. (1) Our results show that the Pommerenke, Chandra–Singh and Bernardi transforms map R^α_β into strictly smaller classes. It is not too difficult to see that these transforms map K, S^* and C into smaller classes but these subclasses are not given explicitly as we have for R^α_β. It is known however that the Alexander transform maps S^* one-to-one and onto K, i.e., $f \in S^*$ if and only if $\mathbf{A}f \in K$. This is in fact Alexander's original theorem in [1].

(2) The search for invariant subclasses under these transforms stemmed from the fact that S was not preserved under $\mathbf{L}$ or $\mathbf{A}$. The Chandra–Singh transform does not preserve S either. In fact, simply consider the spirallike function in S given in [8]:

$$f(z) = \frac{z}{(1 - iz)^{1-i}},$$

where the principal branch of $(1 - iz)^{1-i}$ is chosen. If we let $\nu_1 = 0$ and $\nu_2 = \pi$ and apply (1.2) to this f, then

$$F(z) = \mathbf{P}_{0,\pi} f(z) = \frac{1}{2}\{e^{i\,\mathrm{Log}(1-iz)} - e^{i\,\mathrm{Log}(1+iz)}\}.$$

A check shows that for all $k \in \mathbb{N}$, we get $F(z_k) = 0$ where

$$z_k = i\,\frac{1 - e^{-2\pi k}}{1 + e^{-2\pi k}}.$$

This shows that the Chandra–Singh transform of the univalent function f is of *infinite* valence.

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