ANNALES
POLONICI MATHEMATICI
113.2 (2015)

Hölder regularity for solutions to complex Monge–Ampère equations

by Mohamad Charabati (Toulouse)

Abstract. We consider the Dirichlet problem for the complex Monge–Ampère equation in a bounded strongly hyperconvex Lipschitz domain in $\mathbb{C}^n$. We first give a sharp estimate on the modulus of continuity of the solution when the boundary data is continuous and the right hand side has a continuous density. Then we consider the case when the boundary value function is $\mathcal{C}^{1,1}$ and the right hand side has a density in $L^p(\Omega)$ for some $p > 1$, and prove the Hölder continuity of the solution.

1. Introduction. Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$. Given $\varphi \in \mathcal{C}(\partial\Omega)$ and $0 \le f \in L^1(\Omega)$. We consider the Dirichlet problem

$$\mathrm{Dir}(\Omega,\varphi,f):\begin{cases} u \in \mathrm{PSH}(\Omega)\cap\mathcal{C}(\bar{\Omega}), & \\ (dd^c u)^n = f\beta^n & \text{in } \Omega,\\ u=\varphi & \text{on } \partial\Omega,\end{cases}$$

where $\mathrm{PSH}(\Omega)$ is the set of plurisubharmonic (psh) functions in Ω. Here we write $d = \partial+\bar\partial$ and $d^c = (i/4)(\bar\partial-\partial)$; then $dd^c = (i/2)\partial\bar\partial$ and $(dd^c\cdot)^n$ stands for the complex Monge–Ampère operator.

If $u \in \mathcal{C}^2(\Omega)$ is a plurisubharmonic function, then

$$(dd^c u)^n = \det\left(\frac{\partial^2 u}{\partial z_j \partial \bar z_k}\right)\beta^n,$$

where $\beta = (i/2)\sum_{j=1}^n dz_j \wedge d\bar z_j$ is the standard Kähler form in $\mathbb{C}^n$.

In their seminal work, Bedford and Taylor proved that the complex Monge–Ampère operator can be extended to the set of bounded plurisubharmonic functions (see [BT76], [BT82]). Moreover, it is invariant under holomorphic changes of coordinates. We refer the reader to [BT76], [De89], [Kl91], [Ko05] for more details on its properties.

2010 *Mathematics Subject Classification*: Primary 32W20, 32U15; Secondary 35J96.
Key words and phrases: complex Monge–Ampère equation, plurisubharmonic function, Dirichlet problem, Hölder continuity, strongly hyperconvex Lipschitz domain.

DOI: 10.4064/ap113-2-1

The Dirichlet problem was studied extensively in the last decades by many authors. When Ω is a bounded strongly pseudoconvex domain with smooth boundary and $f \in \mathcal{C}(\bar{\Omega})$, Bedford and Taylor had showed that $\mathrm{Dir}(\Omega, \varphi, f)$ has a unique continuous solution $\mathtt{U} := \mathtt{U}(\Omega, \varphi, f)$. Furthermore, it was proved in [BT76] that $\mathtt{U} \in \mathrm{Lip}_\alpha(\bar{\Omega})$ when $\varphi \in \mathrm{Lip}_{2\alpha}(\partial\Omega)$ and $f^{1/n} \in \mathrm{Lip}_\alpha(\bar{\Omega})$ $(0 < \alpha \le 1)$. In the nondegenerate case, i.e. $0 < f \in \mathcal{C}^\infty(\bar{\Omega})$ and $\varphi \in \mathcal{C}^\infty(\partial\Omega)$, Caffarelli, Kohn, Nirenberg and Spruck [CK$^+$85] proved that $\mathtt{U} \in \mathcal{C}^\infty(\bar{\Omega})$. However a simple example of Gamelin and Sibony shows that the solution is not, in general, better than $\mathcal{C}^{1,1}$-smooth when $f \ge 0$ and f is smooth (see [GS80]). Krylov proved that if $\varphi \in \mathcal{C}^{3,1}(\partial\Omega)$ and $f^{1/n} \in \mathcal{C}^{1,1}(\bar{\Omega})$, $f \ge 0$, then $\mathtt{U} \in \mathcal{C}^{1,1}(\bar{\Omega})$ (see [Kr89]).

For B-regular domains, Błocki [Bł96] proved the existence of a continuous solution to the Dirichlet problem $\mathrm{Dir}(\Omega, \varphi, f)$ when $0 \le f \in \mathcal{C}(\bar{\Omega})$.

For a strongly pseudoconvex domain with smooth boundary, Kołodziej [Ko98] demonstrated that $\mathrm{Dir}(\Omega, \varphi, f)$ still admits a unique continuous solution under the milder assumption $f \in L^p(\Omega)$, for $p > 1$. Recently Guedj, Kołodziej and Zeriahi studied the Hölder continuity of the solution when $0 \le f \in L^p(\Omega)$, for some $p > 1$, is bounded near the boundary (see [GKZ08]).

For the complex Monge–Ampère equation on a compact Kähler manifold, the Hölder continuity of the solution was proved earlier by Kołodziej [Ko08] (see also [DD$^+$14]).

A viscosity approach to the complex Monge–Ampère equation has been developed in [EGZ11] and [Wan12].

In this paper, we consider the more general case where Ω is a bounded strongly hyperconvex Lipschitz domain (the boundary does not need to be smooth).

Our first result gives a sharp estimate for the modulus of continuity of the solution in terms of the modulus of continuity of the data φ, f.

Theorem A. *Let $\Omega \subset \mathbb{C}^n$ be a bounded strongly hyperconvex Lipschitz domain, $\varphi \in \mathcal{C}(\partial\Omega)$ and $0 \le f \in \mathcal{C}(\bar{\Omega})$. Assume that ω_φ is the modulus of continuity of φ, and $\omega_{f^{1/n}}$ is the modulus of continuity of $f^{1/n}$. Then the modulus of continuity of the unique solution $\mathtt{U}$ to $\mathrm{Dir}(\Omega, \varphi, f)$ satisfies the estimate*

$$\omega_{\mathtt{U}}(t) \le \eta(1 + \|f\|^{1/n}_{L^\infty(\bar{\Omega})}) \max\{\omega_\varphi(t^{1/2}), \omega_{f^{1/n}}(t), t^{1/2}\},$$

where η is a positive constant depending on Ω.

Here we will use an alternative description of the solution given by Proposition 3.2 to get optimal control for the modulus of continuity of this solution in a strongly hyperconvex Lipschitz domain. This result was suggested by E. Bedford [Be88] and proved in the case of strictly convex domains with $f = 0$ [Be82].

Our second result concerns the Hölder continuity of the solution when $f \in L^p(\Omega)$, $p > 1$.

THEOREM B. *Let $\Omega \Subset \mathbb{C}^n$ be a bounded strongly hyperconvex Lipschitz domain. Assume that $\varphi \in \mathcal{C}^{1,1}(\partial\Omega)$ and $f \in L^p(\Omega)$ for some $p > 1$. Then the unique solution* $\mathtt{U}$ *to* $\mathrm{Dir}(\Omega, \varphi, f)$ *is α-Hölder continuous on $\bar{\Omega}$ for any $0 < \alpha < 1/(nq+1)$ where $1/p + 1/q = 1$. Moreover, if $p \geq 2$, then the solution* $\mathtt{U}$ *is α-Hölder continuous on $\bar{\Omega}$ for any $0 < \alpha < \min\{1/2, 2/(nq+1)\}$.*

In [GKZ08] the Hölder continuity of the solution is obtained when $\varphi \in \mathcal{C}^{1,1}(\partial\Omega)$ and $f \in L^p(\Omega)$, for $p > 1$, is bounded near the boundary. Recently, N. C. Nguyen [N14] proved that the solution is Hölder continuous when the density f satisfies a growth condition near the boundary of Ω.

2. Preliminaries. We recall that a *hyperconvex domain* is a domain in $\mathbb{C}^n$ admitting a bounded plurisubharmonic exhaustion function. Let us define the class of hyperconvex domains which will be considered in this paper.

DEFINITION 2.1. A bounded domain $\Omega \subset \mathbb{C}^n$ is called a *strongly hyperconvex Lipschitz* (briefly SHL) domain if there exists a neighborhood Ω' of $\bar{\Omega}$ and a Lipschitz plurisubharmonic defining function $\rho : \bar{\Omega}' \to \mathbb{R}$ such that

(1) $\rho < 0$ in Ω and $\partial\Omega = \{\rho = 0\}$,
(2) there exists a constant $c > 0$ such that $dd^c\rho \geq c\beta$ in Ω in the weak sense of currents.

EXAMPLE 2.2.

(1) Let Ω be a *strictly convex* domain, that is, there exists a Lipschitz defining function ρ such that $\rho - c|z|^2$ is convex for some $c > 0$. It is clear that Ω is a strongly hyperconvex Lipschitz domain.
(2) A smooth strictly pseudoconvex bounded domain is a SHL domain (see [HL84]).
(3) The nonempty finite intersection of strictly pseudoconvex bounded domains with smooth boundary in $\mathbb{C}^n$ is a bounded SHL domain. In fact, it is sufficient to set $\rho = \max\{\rho_i\}$. More generally a finite intersection of SHL domains is a SHL domain.
(4) The domain

$$\Omega = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n : |z_1| + \cdots + |z_n| < 1\} \quad (n \geq 2)$$

is a bounded strongly hyperconvex Lipschitz domain in $\mathbb{C}^n$ with non-smooth boundary.
(5) The unit polydisc in $\mathbb{C}^n$ $(n \geq 2)$ is hyperconvex with Lipschitz boundary but it is not strongly hyperconvex Lipschitz.

Remark 2.3. Any bounded SHL domain is B-regular in the sense of Sibony ([Sib87], [Bł96]).

Let $\Omega \subset \mathbb{C}^n$ be a bounded domain. If $u \in \mathrm{PSH}(\Omega)$ then $dd^c u \geq 0$ in the sense of currents. We define

$$\Delta_H u := \sum_{j,k=1}^{n} h_{j\bar{k}} \frac{\partial^2 u}{\partial z_k \partial \bar{z}_j} \tag{2.1}$$

for every positive definite Hermitian matrix $H = (h_{j\bar{k}})$. We can view $\Delta_H u$ as a positive Radon measure in Ω.

The following lemma is elementary and important for what follows (see [Gav77]).

Lemma 2.4 ([Gav77]). *Let Q be a $n \times n$ nonnegative Hermitian matrix. Then*

$$(\det Q)^{1/n} = \inf\{\mathrm{tr}(HQ) : H \in H_n^+ \text{ and } \det H = n^{-n}\},$$

where H_n^+ denotes the set of all positive Hermitian $n \times n$ matrices.

Example 2.5. We calculate $\Delta_H(|z|^2)$ for every matrix $H \in H_n^+$ with $\det H = n^{-n}$:

$$\Delta_H(|z|^2) = \sum_{j,k=1}^{n} h_{j\bar{k}} \delta_{k\bar{j}} = \mathrm{tr}\, H.$$

Using the inequality of arithmetic and geometric means, we have

$$1 = (\det I)^{1/n} \leq \mathrm{tr}\, H,$$

hence $\Delta_H(|z|^2) \geq 1$ for every matrix $H \in H_n^+$ with $\det H = n^{-n}$.

The following result is well known (see [Bł96]), but we will give here an alternative proof using ideas from the theory of viscosity due to Eyssidieux, Guedj and Zeriahi [EGZ11].

Proposition 2.6. *Let $u \in \mathrm{PSH}(\Omega) \cap L^\infty(\Omega)$ and $0 \leq f \in \mathcal{C}(\Omega)$. Then the following conditions are equivalent:*

1. *$\Delta_H u \geq f^{1/n}$ in the weak sense of distributions, for any $H \in H_n^+$ with $\det H = n^{-n}$.*
2. *$(dd^c u)^n \geq f\beta^n$ in the weak sense of currents in Ω.*

Proof. First, suppose that $u \in \mathcal{C}^2(\Omega)$. Then by Lemma 2.4 the inequality

$$\Delta_H u = \sum_{j,k=1}^{n} h^{j\bar{k}} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \geq f^{1/n}, \qquad \forall H \in H_n^+, \det H = n^{-n},$$

is equivalent to

$$\left(\det\left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\right)\right)^{1/n} \geq f^{1/n}.$$

The latter means that

$$(dd^c u)^n \geq f\beta^n.$$

(1)⇒(2). Let (ρ_ϵ) be the standard family of regularizing kernels with $\operatorname{supp}\rho_\epsilon \subset B(0,\epsilon)$ and $\int_{B(0,\epsilon)} \rho_\epsilon = 1$. Then the sequence $u_\epsilon = u * \rho_\epsilon$ decreases to u, and we see that (1) implies $\Delta_H u_\epsilon \geq (f^{1/n})_\epsilon$. Since u_ϵ is smooth, we use the first case and get $(dd^c u_\epsilon)^n \geq ((f^{1/n})_\epsilon)^n \beta^n$, hence by applying the convergence theorem of Bedford and Taylor [BT82, Theorem 7.4] we obtain $(dd^c u)^n \geq f\beta^n$.

(2)⇒(1). Fix $x_0 \in \Omega$, and let q be a $\mathcal{C}^2$-function in a neighborhood B of x_0 such that $u \leq q$ in this neighborhood and $u(x_0) = q(x_0)$.

First step: We will show that $dd^c q_{x_0} \geq 0$. Indeed, for every small enough ball $B' \subset B$ centered at x_0, we have

$$u(x_0) - q(x_0) \geq \frac{1}{V(B')} \int_{B'} (u - q)\, dV,$$

therefore

$$\frac{1}{V(B')} \int_{B'} q\, dV - q(x_0) \geq \frac{1}{V(B')} \int_{B'} u\, dV - u(x_0) \geq 0.$$

Since q is $\mathcal{C}^2$-smooth and the radius of B' tends to 0, it follows from [H94, Proposition 3.2.10] that $\Delta q_{x_0} \geq 0$. For every positive definite Hermitian matrix H with $\det H = n^{-n}$, we make a linear change of complex coordinates T such that $\operatorname{tr}(HQ) = \operatorname{tr}(\tilde{Q})$ where $\tilde{Q} = (\partial^2 \tilde{q}/\partial w_j \partial \bar{w}_k)$ and $\tilde{q} = q \circ T^{-1}$. Then

$$\Delta_H q(x_0) = \operatorname{tr}(HQ) = \operatorname{tr}(\tilde{Q}) = \Delta \tilde{q}(y_0).$$

Hence $\Delta_H q(x_0) \geq 0$ for every $H \in H_n^+$ with $\det H = n^{-n}$, so $dd^c q_{x_0} \geq 0$.

Second step: We claim that $(dd^c q)^n_{x_0} \geq f(x_0)\beta^n$. Suppose that there exists a point $x_0 \in \Omega$ and a $\mathcal{C}^2$-function q which satisfies $u \leq q$ in a neighborhood of x_0 and $u(x_0) = q(x_0)$ such that $(dd^c q)^n_{x_0} < f(x_0)\beta^n$. We put

$$q^\epsilon(x) = q(x) + \epsilon(\|x - x_0\|^2 - r^2/2)$$

for $0 < \epsilon \ll 1$ small enough; we see that

$$0 < (dd^c q^\epsilon)^n_{x_0} < f(x_0)\beta^n.$$

Since f is lower semicontinuous on Ω, there exists $r > 0$ such that

$$(dd^c q^\epsilon)^n_x \leq f(x)\beta^n, \qquad x \in B(x_0, r).$$

Then $(dd^c q^\epsilon)^n \leq f\beta^n \leq (dd^c u)^n$ in $B(x_0, r)$ and $q^\epsilon = q + \epsilon r^2/2 \geq q \geq u$ on $\partial B(x_0, r)$, hence $q^\epsilon \geq u$ on $B(x_0, r)$ by the comparison principle. But $q^\epsilon(x_0) = q(x_0) - \epsilon r^2/2 = u(x_0) - \epsilon r^2/2 < u(x_0)$, a contradiction.

Hence, from the first part of the proof, we get $\Delta_H q(x_0) \geq f^{1/n}(x_0)$ for every point $x_0 \in \Omega$ and every $\mathcal{C}^2$-function q in a neighborhood of x_0 such that $u \leq q$ in this neighborhood and $u(x_0) = q(x_0)$.

Assume that $f > 0$ and $f \in \mathcal{C}^\infty(\Omega)$. Then there exists $g \in \mathcal{C}^\infty(\Omega)$ such that $\Delta_H g = f^{1/n}$. Hence $\varphi = u - g$ is Δ_H-subharmonic (by [H94, Proposition 3.2.10′]), from which it follows that $\Delta_H \varphi \geq 0$ and $\Delta_H u \geq f^{1/n}$.

In case $f > 0$ is merely continuous, we observe that

$$f = \sup\{w : w \in \mathcal{C}^\infty,\ f \geq w > 0\},$$

so $(dd^c u)^n \geq f\beta^n \geq w\beta^n$. Since $w > 0$ is smooth, we have $\Delta_H u \geq w^{1/n}$. Therefore, we get $\Delta_H u \geq f^{1/n}$.

In the general case $0 \leq f \in \mathcal{C}(\Omega)$, we observe that $u^\epsilon(z) = u(z) + \epsilon|z|^2$ satisfies

$$(dd^c u^\epsilon)^n \geq (f + \epsilon^n)\beta^n,$$

and so

$$\Delta_H u^\epsilon \geq (f + \epsilon^n)^{1/n}.$$

Letting ϵ converge to 0, we get $\Delta_H u \geq f^{1/n}$ for all $H \in H_n^+$ with $\det H = n^{-n}$. ■

As a consequence of Proposition 2.6, we give an alternative description of the classical Perron–Bremermann family of subsolutions to the Dirichlet problem $\mathrm{Dir}(\Omega, \varphi, f)$.

Definition 2.7. We denote by $\mathcal{V}(\Omega, \varphi, f)$ the family of subsolutions of $\mathrm{Dir}(\Omega, \varphi, f)$, that is,

$$\mathcal{V}(\Omega, \varphi, f) = \{v \in \mathrm{PSH}(\Omega) \cap \mathcal{C}(\bar{\Omega}) : v|_{\partial\Omega} \leq \varphi \text{ and } \Delta_H v \geq f^{1/n} \text{ for all } H \in H_n^+ \text{ with } \det H = n^{-n}\}.$$

Remark 2.8. We observe that $\mathcal{V}(\Omega, \varphi, f) \neq \emptyset$. Indeed, let ρ be as in Definition 2.1 and $A, B > 0$ large enough; then $A\rho - B \in \mathcal{V}(\Omega, \varphi, f)$.

Furthermore, the family $\mathcal{V}(\Omega, \varphi, f)$ is stable under finite maximum, that is, if $u, v \in \mathcal{V}(\Omega, \varphi, f)$ then $\max(u, v) \in \mathcal{V}(\Omega, \varphi, f)$.

3. The Perron–Bremermann envelope. Bedford and Taylor [BT76] proved that the unique solution to $\mathrm{Dir}(\Omega, \varphi, f)$ in a bounded strongly pseudoconvex domain with smooth boundary is given as the *Perron–Bremermann envelope*

$$u = \sup\{v : v \in \mathcal{B}(\Omega, \varphi, f)\},$$

where $\mathcal{B}(\Omega, \varphi, f) = \{v \in \mathrm{PSH}(\Omega) \cap \mathcal{C}(\bar{\Omega}) : v|_{\partial\Omega} \leq \varphi \text{ and } (dd^c v)^n \geq f\beta^n\}$.

Thanks to Proposition 2.6, we get the following corollary:

Corollary 3.1. $\mathcal{V}(\Omega, \varphi, f) = \mathcal{B}(\Omega, \varphi, f)$.

Hence we get an alternative description of the Perron–Bremermann envelope in a bounded SHL domain. More precisely, we consider the upper envelope

$$\mathtt{U}(z) = \sup\{v(z) : v \in \mathcal{V}(\Omega, \varphi, f)\}.$$

PROPOSITION 3.2. *Let $\Omega \subset \mathbb{C}^n$ be a bounded strongly hyperconvex Lipschitz domain, $0 \le f \in \mathcal{C}(\bar{\Omega})$ and $\varphi \in \mathcal{C}(\partial\Omega)$. Then the Dirichlet problem* $\mathrm{Dir}(\Omega, \varphi, f)$ *has a unique solution* $\mathtt{U}$. *Moreover the solution is given by*

$$\mathtt{U} = \sup\{v : v \in \mathcal{V}(\Omega, \varphi, f)\},$$

where $\mathcal{V}$ is defined in Definition 2.7 *and Δ_H is the Laplacian associated to a positive definite Hermitian matrix H as in* (2.1).

Proof. The uniqueness follows from the comparison principle [BT76]. Our domain Ω is B-regular in the sense of Sibony, therefore the existence of the solution follows from [Bł96, Theorem 4.1]. The description of the solution given in the proposition follows from Corollary 3.1 and [Bł96, Theorem 4.1]. ∎

REMARK 3.3. Let $\varphi_1, \varphi_2 \in \mathcal{C}(\partial\Omega)$ and $f_1, f_2 \in \mathcal{C}(\bar{\Omega})$. Then the solutions $\mathtt{U}_1 = \mathtt{U}(\Omega, \varphi_1, f_1)$, $\mathtt{U}_2 = \mathtt{U}(\Omega, \varphi_2, f_2)$ satisfy the stability estimate

$$\|\mathtt{U}_1 - \mathtt{U}_2\|_{L^\infty(\bar{\Omega})} \le d^2 \|f_1 - f_2\|^{1/n}_{L^\infty(\bar{\Omega})} + \|\varphi_1 - \varphi_2\|_{L^\infty(\partial\Omega)}, \tag{3.1}$$

where $d := \mathrm{diam}(\Omega)$. Indeed, fix $z_0 \in \Omega$ and define

$$v_1(z) = \|f_1 - f_2\|^{1/n}_{L^\infty(\bar{\Omega})}(|z - z_0|^2 - d^2) + \mathtt{U}_2(z),$$
$$v_2(z) = \mathtt{U}_1(z) + \|\varphi_1 - \varphi_2\|_{L^\infty(\partial\Omega)}.$$

It is clear that $v_1, v_2 \in \mathrm{PSH}(\Omega) \cap \mathcal{C}(\bar{\Omega})$. Hence, by the comparison principle, we get $v_1 \le v_2$ on $\bar{\Omega}$. Then we conclude that

$$\mathtt{U}_2 - \mathtt{U}_1 \le d^2 \|f_1 - f_2\|^{1/n}_{L^\infty(\bar{\Omega})} + \|\varphi_1 - \varphi_2\|_{L^\infty(\partial\Omega)}.$$

Reversing the roles of $\mathtt{U}_1$ and $\mathtt{U}_2$, we get the inequality (3.1).

We will need in Section 5 an estimate, proved by Błocki [Bł93], for the L^n-L^1 stability of solutions to the Dirichlet problem $\mathrm{Dir}(\Omega, \varphi, f)$:

$$\|\mathtt{U}_1 - \mathtt{U}_2\|_{L^n(\Omega)} \le \lambda(\Omega)\|\varphi_1 - \varphi_2\|_{L^\infty(\partial\Omega)} + \frac{r^2}{4}\|f_1 - f_2\|^{1/n}_{L^1(\Omega)}, \tag{3.2}$$

where $r = \min\{r' > 0 : \Omega \subset B(z_0, r') \text{ for some } z_0 \in \mathbb{C}^n\}$.

4. The modulus of continuity of the Perron–Bremermann envelope. Recall that a real function ω on $[0, l]$, $0 < l < \infty$, is called a *modulus of continuity* if ω is continuous, subadditive, nondecreasing and satisfies $\omega(0) = 0$. In general, ω fails to be concave; we denote by $\bar{\omega}$ the minimal concave majorant of ω. The following property of $\bar{\omega}$ is well known (see [Kor82] and [Ch14]).

Lemma 4.1. *Let ω be a modulus of continuity on $[0,l]$ and $\bar{\omega}$ be the minimal concave majorant of ω. Then $\omega(\eta t) < \bar{\omega}(\eta t) < (1+\eta)\omega(t)$ for any $t > 0$ and $\eta > 0$.*

4.1. Modulus of continuity of the solution. Now, we will start the first step to establish an estimate for the modulus of continuity of the solution to $\mathrm{Dir}(\Omega, \varphi, f)$. For this purpose, it is natural to investigate the relation between the modulus of continuity of $\mathtt{U}$ and the modulus of continuity of a subbarrier and a superbarrier. We prove the following:

Proposition 4.2. *Let $\Omega \subset \mathbb{C}^n$ be a bounded SHL domain, $\varphi \in \mathcal{C}(\partial\Omega)$ and $0 \le f \in \mathcal{C}(\bar{\Omega})$. Suppose that there exist $v \in \mathcal{V}(\Omega, \varphi, f)$ and $w \in \mathrm{SH}(\Omega) \cap \mathcal{C}(\bar{\Omega})$ such that $v = \varphi = -w$ on $\partial\Omega$. Then there is a constant $C > 0$ depending on* $\mathrm{diam}(\Omega)$ *such that the modulus of continuity of $\mathtt{U}$ satisfies*

$$\omega_{\mathtt{U}}(t) \le C \max\{\omega_v(t), \omega_w(t), \omega_{f^{1/n}}(t)\}.$$

Proof. Set $g(t) := \max\{\omega_v(t), \omega_w(t), \omega_{f^{1/n}}(t)\}$ and $d := \mathrm{diam}(\Omega)$. As $v = \varphi = -w$ on $\partial\Omega$, for all $z \in \bar{\Omega}$ and $\xi \in \partial\Omega$ we have

$$-g(|z-\xi|) \le v(z) - \varphi(\xi) \le \mathtt{U}(z) - \varphi(\xi) \le -w(z) - \varphi(\xi) \le g(|z-\xi|).$$

Hence

$$|\mathtt{U}(z) - \mathtt{U}(\xi)| \le g(|z-\xi|), \quad \forall z \in \bar{\Omega}, \forall \xi \in \partial\Omega. \tag{4.1}$$

Fix a point $z_0 \in \Omega$. For any vector $\tau \in \mathbb{C}^n$ with small enough norm, we set $\Omega_{-\tau} := \{z - \tau : z \in \Omega\}$ and define in $\Omega \cap \Omega_{-\tau}$ the function

$$v_1(z) = \mathtt{U}(z+\tau) + g(|\tau|)|z - z_0|^2 - d^2 g(|\tau|) - g(|\tau|),$$

which is a well defined psh function in $\Omega \cap \Omega_{-\tau}$ and continuous on $\bar{\Omega} \cap \bar{\Omega}_{-\tau}$. By (4.1), if $z \in \bar{\Omega} \cap \partial\Omega_{-\tau}$ we can see that

$$v_1(z) - \mathtt{U}(z) \le g(|\tau|) + g(|\tau|)|z - z_0|^2 - d^2 g(|\tau|) - g(|\tau|) \le 0. \tag{4.2}$$

Moreover, we assert that $\Delta_H v_1 \ge f^{1/n}$ in $\Omega \cap \Omega_{-\tau}$ for all $H \in H_n^+$ with $\det H = n^{-n}$. Indeed, we have

$$\begin{aligned}\Delta_H v_1(z) &\ge f^{1/n}(z+\tau) + g(|\tau|)\Delta_H(|z-z_0|^2) \ge f^{1/n}(z+\tau) + g(|\tau|) \\ &\ge f^{1/n}(z+\tau) + |f^{1/n}(z+\tau) - f^{1/n}(z)| \ge f^{1/n}(z)\end{aligned}$$

for all $H \in H_n^+$ with $\det H = n^{-n}$. Hence, by the above properties of v_1, we find that

$$V_\tau(z) = \begin{cases} \mathtt{U}(z), & z \in \bar{\Omega} \setminus \Omega_{-\tau}, \\ \max(\mathtt{U}(z), v_1(z)), & z \in \bar{\Omega} \cap \Omega_{-\tau}, \end{cases}$$

is a well defined function and belongs to $\mathrm{PSH}(\Omega) \cap \mathcal{C}(\bar{\Omega})$. It is clear that $\Delta_H V_\tau \ge f^{1/n}$ for all $H \in H_n^+$ with $\det H = n^{-n}$. We claim that $V_\tau = \varphi$ on $\partial\Omega$. If $z \in \partial\Omega \setminus \Omega_{-\tau}$ then $V_\tau(z) = \mathtt{U}(z) = \varphi(z)$. On the other hand

$z \in \partial\Omega \cap \Omega_{-\tau}$, and by (4.2) we get $V_\tau(z) = \max(\mathtt{U}(z), v_1(z)) = \mathtt{U}(z) = \varphi(z)$. Consequently, $V_\tau \in \mathcal{V}(\Omega, \varphi, f)$ and this implies that

$$V_\tau(z) \le \mathtt{U}(z), \quad \forall z \in \bar{\Omega}.$$

Then for all $z \in \bar{\Omega} \cap \Omega_{-\tau}$ we have

$$\mathtt{U}(z+\tau) + g(|\tau|)|z - z_0|^2 - d^2 g(|\tau|) - g(|\tau|) \le \mathtt{U}(z).$$

Hence,

$$\mathtt{U}(z+\tau) - \mathtt{U}(z) \le (d^2+1)g(|\tau|) - g(|\tau|)|z-z_0|^2 \le Cg(|\tau|).$$

Reversing the roles of $z+\tau$ and z, we get

$$|\mathtt{U}(z+\tau) - \mathtt{U}(z)| \le Cg(|\tau|), \quad \forall z, z+\tau \in \bar{\Omega}.$$

Thus, finally,

$$\omega_{\mathtt{U}}(|\tau|) \le C \max\{\omega_v(|\tau|), \omega_w(|\tau|), \omega_{f^{1/n}}(|\tau|)\}. \ \blacksquare$$

Remark 4.3. Let H_φ be the harmonic extension of φ in a bounded SHL domain Ω. We can replace w in the last proposition by H_φ. It is known in the classical harmonic analysis (see [Ai10]) that the harmonic extension H_φ does not have, in general, the same modulus of continuity of φ.

Let us define, for small positive t, the modulus of continuity

$$\psi_{\alpha,\beta}(t) = (-\log(t))^{-\alpha} t^\beta$$

with $\alpha \ge 0$ and $0 \le \beta < 1$. It is clear that $\psi_{\alpha,0}$ is weaker than Hölder continuity and $\psi_{0,\beta}$ is Hölder continuity. It was shown in [Ai02] that $\omega_{H_\varphi}(t) \le c\psi_{0,\beta}(t)$ for some $c > 0$ if $\omega_\varphi(t) \le c_1\psi_{0,\beta}(t)$ for $\beta < \beta_0$, where $\beta_0 < 1$ depends only on n and the Lipschitz constant of the defining function ρ. Moreover, a similar result was proved in [Ai10] for the modulus of continuity $\psi_{\alpha,0}(t)$. However, the same argument of Aikawa gives $\omega_{H_\varphi}(t) \le c\psi_{\alpha,\beta}(t)$ for some $c > 0$ if $\omega_\varphi(t) \le c_1\psi_{\alpha,\beta}(t)$ for $\alpha \ge 0$ and $0 \le \beta < \beta_0 < 1$.

This leads us to the conclusion that if there exists a barrier v to the Dirichlet problem such that $v = \varphi$ on $\partial\Omega$ and $\omega_v(t) \le \lambda\psi_{\alpha,\beta}(t)$ with α, β as above, then the last proposition gives

$$\omega_{\mathtt{U}} \le \lambda_1 \max\{\psi_{\alpha,\beta}(t), \omega_{f^{1/n}}(t)\},$$

where $\lambda_1 > 0$ depends on λ and $\operatorname{diam}(\Omega)$.

4.2. Construction of barriers. In this subsection, we will construct a subsolution to the Dirichlet problem with boundary value φ and estimate its modulus of continuity.

Proposition 4.4. *Let $\Omega \subset \mathbb{C}^n$ be a bounded SHL domain, assume that $\varphi \in \mathcal{C}(\partial\Omega)$ and $0 \le f \in \mathcal{C}(\bar{\Omega})$. Then there exists a subsolution $v \in \mathcal{V}(\Omega, \varphi, f)$ such that $v = \varphi$ on $\partial\Omega$ and the modulus of continuity of v satisfies*

$$\omega_v(t) \le \lambda(1 + \|f\|_{L^\infty(\bar{\Omega})}^{1/n}) \max\{\omega_\varphi(t^{1/2}), t^{1/2}\},$$

where $\lambda > 0$ depends on Ω.

Observe that we do not assume any smoothness on $\partial\Omega$.

Proof. First of all, fix $\xi \in \partial\Omega$. We claim that there exists $v_\xi \in \mathcal{V}(\Omega, \varphi, f)$ such that $v_\xi(\xi) = \varphi(\xi)$. It is sufficient to prove that there exists a constant $C > 0$ depending on Ω such that for every point $\xi \in \partial\Omega$ and $\varphi \in \mathcal{C}(\partial\Omega)$, there is a function $h_\xi \in \mathrm{PSH}(\Omega) \cap \mathcal{C}(\bar{\Omega})$ satisfying

(1) $h_\xi(z) \le \varphi(z),\ \forall z \in \partial\Omega$,
(2) $h_\xi(\xi) = \varphi(\xi)$,
(3) $\omega_{h_\xi}(t) \le C\omega_\varphi(t^{1/2})$.

Assume this is true. We fix $z_0 \in \Omega$ and write $K_1 := \sup_{\bar{\Omega}} f^{1/n} \ge 0$. Hence

$$\Delta_H(K_1|z - z_0|^2) = K_1\Delta_H|z - z_0|^2 \ge f^{1/n}, \qquad \forall H \in H_n^+,\ \det H = n^{-n}.$$

We also set $K_2 := K_1|\xi - z_0|^2$. Then for the continuous function

$$\tilde{\varphi}(z) := \varphi(z) - K_1|z - z_0|^2 + K_2,$$

we have h_ξ such that (1)–(3) hold.

Then the desired function $v_\xi \in \mathcal{V}(\Omega, \varphi, f)$ is given by

$$v_\xi(z) = h_\xi(z) + K_1|z - z_0|^2 - K_2.$$

Thus $h_\xi(z) \le \tilde{\varphi}(z) = \varphi(z) - K_1|z - z_0|^2 + K_2$ on $\partial\Omega$, so $v_\xi(z) \le \varphi$ on $\partial\Omega$ and $v_\xi(\xi) = \varphi(\xi)$.

Moreover, it is clear that

$$\Delta_H v_\xi = \Delta_H h_\xi + K_1\Delta_H(|z - z_0|^2) \ge f^{1/n}, \qquad \forall H \in H_n^+,\ \det H = n^{-n}.$$

Furthermore, using the hypothesis on h_ξ, we can control the modulus of continuity of v_ξ:

$$\begin{aligned}\omega_{v_\xi}(t) = \sup_{|z-y|\le t} |v_\xi(z) - v_\xi(y)| &\le \omega_{h_\xi}(t) + K_1\omega_{|z-z_0|^2}(t)\\ &\le C\omega_{\tilde{\varphi}}(t^{1/2}) + 4d^{3/2}K_1t^{1/2}\\ &\le C\omega_\varphi(t^{1/2}) + 2dK_1(C + 2d^{1/2})t^{1/2}\\ &\le (C + 2d^{1/2})(1 + 2dK_1)\max\{\omega_\varphi(t^{1/2}), t^{1/2}\},\end{aligned}$$

where $d := \mathrm{diam}(\Omega)$. Hence, we conclude that

$$\omega_{v_\xi}(t) \le \lambda(1 + K_1)\max\{\omega_\varphi(t^{1/2}), t^{1/2}\},$$

where $\lambda := (C + 2d^{1/2})(1 + 2d)$ is a positive constant depending on Ω.

Now we will construct $h_\xi \in \mathrm{PSH}(\Omega) \cap \mathcal{C}(\bar{\Omega})$ which satisfies the three conditions above. Let $B > 0$ be large enough such that the function

$$g(z) = B\rho(z) - |z - \xi|^2$$

is psh in Ω. Let $\bar{\omega}_\varphi$ be the minimal concave majorant of ω_φ and define

$$\chi(x) = -\bar{\omega}_\varphi((-x)^{1/2}),$$

which is a convex nondecreasing function on $[-d^2, 0]$. Now fix $r > 0$ so small that $|g(z)| \le d^2$ in $B(\xi, r) \cap \Omega$ and define for $z \in B(\xi, r) \cap \bar{\Omega}$ the function

$$h(z) = \chi \circ g(z) + \varphi(\xi).$$

It is clear that h is a continuous psh function on $B(\xi, r) \cap \Omega$ and we see that $h(z) \le \varphi(z)$ if $z \in B(\xi, r) \cap \partial\Omega$ and $h(\xi) = \varphi(\xi)$. Moreover by the subadditivity of $\bar{\omega}_\varphi$ and Lemma 4.1 we have

$$\begin{aligned}\omega_h(t) &= \sup_{|z-y|\le t} |h(z) - h(y)| \\ &\le \sup_{|z-y|\le t} \bar{\omega}_\varphi\big[\big||z-\xi|^2 - |y-\xi|^2 - B(\rho(z) - \rho(y))\big|^{1/2}\big] \\ &\le \sup_{|z-y|\le t} \bar{\omega}_\varphi[(|z-y|(2d + B_1))^{1/2}] \le C.\omega_\varphi(t^{1/2}),\end{aligned}$$

where $C := 1 + (2d + B_1)^{1/2}$ depends on Ω.

Recall that $\xi \in \partial\Omega$ and fix $0 < r_1 < r$ and $\gamma_1 \ge d/r_1$ such that

$$-\gamma_1 \bar{\omega}_\varphi[(|z-\xi|^2 - B\rho(z))^{1/2}] \le \inf_{\partial\Omega} \varphi - \sup_{\partial\Omega} \varphi$$

for $z \in \partial\Omega \cap \partial B(\xi, r_1)$. Set $\gamma_2 = \inf_{\partial\Omega} \varphi$. Then

$$\gamma_1(h(z) - \varphi(\xi)) + \varphi(\xi) \le \gamma_2 \quad \text{for } z \in \partial B(\xi, r_1) \cap \bar{\Omega}.$$

Now set

$$h_\xi(z) = \begin{cases} \max[\gamma_1(h(z) - \varphi(\xi)) + \varphi(\xi), \gamma_2], & z \in \bar{\Omega} \cap B(\xi, r_1), \\ \gamma_2, & z \in \bar{\Omega} \setminus B(\xi, r_1), \end{cases}$$

which is a well defined psh function on Ω, continuous on $\bar{\Omega}$ and such that $h_\xi(z) \le \varphi(z)$ for all $z \in \partial\Omega$. Indeed, on $\partial\Omega \cap B(\xi, r_1)$ we have

$$\gamma_1(h(z) - \varphi(\xi)) + \varphi(\xi) = -\gamma_1 \bar{\omega}_\varphi(|z-\xi|) + \varphi(\xi) \le -\bar{\omega}_\varphi(|z-\xi|) + \varphi(\xi) \le \varphi(z).$$

Hence it is clear that h_ξ satisfies the three conditions above.

We have just proved that for each $\xi \in \partial\Omega$, there is a $v_\xi \in \mathcal{V}(\Omega, \varphi, f)$ with $v_\xi(\xi) = \varphi(\xi)$ and

$$\omega_{v_\xi}(t) \le \lambda(1 + K_1) \max\{\omega_\varphi(t^{1/2}), t^{1/2}\}.$$

Set

$$v(z) = \sup\{v_\xi(z) : \xi \in \partial\Omega\}.$$

Since $0 \le \omega_v(t) \le \lambda(1 + K_1) \max\{\omega_\varphi(t^{1/2}), t^{1/2}\}$, we see that $\omega_v(t)$ converges to zero when t converges to zero. Consequently, $v \in \mathcal{C}(\bar{\Omega})$ and $v = v^* \in \mathrm{PSH}(\Omega)$. Thanks to Choquet's lemma, we can choose a nondecreasing sequence (v_j), where $v_j \in \mathcal{V}(\Omega, \varphi, f)$, converging to v almost everywhere. This implies that

$$\Delta_H v = \lim_{j\to\infty} \Delta_H v_j \ge f^{1/n}, \quad \forall H \in H_n^+, \det H = n^{-n}.$$

It is clear that $v(\xi) = \varphi(\xi)$ for any $\xi \in \partial\Omega$. Finally, $v \in \mathcal{V}(\Omega, \varphi, f)$, $v = \varphi$ on $\partial\Omega$ and $\omega_v(t) \le \lambda(1 + K_1)\max\{\omega_\varphi(t^{1/2}), t^{1/2}\}$. ∎

REMARK 4.5. If we assume that Ω has a smooth boundary and φ is $\mathcal{C}^{1,1}$-smooth, then it is possible to construct a Lipschitz barrier v to the Dirichlet problem $\mathrm{Dir}(\Omega, \varphi, f)$ (see [BT76, Theorem 6.2]).

COROLLARY 4.6. *Under the same assumption of Proposition* 4.4, *there exists a plurisuperharmonic function* $\tilde{v} \in \mathcal{C}(\bar{\Omega})$ *such that* $\tilde{v} = \varphi$ *on* $\partial\Omega$ *and*

$$\omega_{\tilde{v}}(t) \le \lambda(1 + \|f\|^{1/n}_{L^\infty(\bar{\Omega})})\max\{\omega_\varphi(t^{1/2}), t^{1/2}\},$$

where $\lambda > 0$ *depends on* Ω.

Proof. We can perform the same construction as in the proof of Proposition 4.4 for the function $\varphi_1 = -\varphi \in \mathcal{C}(\partial\Omega)$; then we get $v_1 \in \mathcal{V}(\Omega, \varphi_1, f)$ such that $v_1 = \varphi_1$ on $\partial\Omega$ and $\omega_{v_1}(t) \le \lambda(1 + \|f\|^{1/n}_{L^\infty(\bar{\Omega})})\max\{\omega_\varphi(t^{1/2}), t^{1/2}\}$. Hence, we set $\tilde{v} = -v_1$ which is a plurisuperharmonic function on Ω, continuous on $\bar{\Omega}$ and satisfying $\tilde{v} = \varphi$ on $\partial\Omega$ and

$$\omega_{\tilde{v}}(t) \le \lambda(1 + \|f\|^{1/n}_{L^\infty(\bar{\Omega})})\max\{\omega_\varphi(t^{1/2}), t^{1/2}\}. \ ∎$$

4.3. Proof of Theorem A. Thanks to Proposition 4.4, we have a subsolution $v \in \mathcal{V}(\Omega, \varphi, f)$ with $v = \varphi$ on $\partial\Omega$ and

$$\omega_v(t) \le \lambda(1 + \|f\|^{1/n}_{L^\infty(\bar{\Omega})})\max\{\omega_\varphi(t^{1/2}), t^{1/2}\}.$$

From Corollary 4.6, we get $w \in \mathrm{PSH}(\Omega) \cap \mathcal{C}(\bar{\Omega})$ such that $w = -\varphi$ on $\partial\Omega$ and

$$\omega_w(t) \le \lambda(1 + \|f\|^{1/n}_{L^\infty(\bar{\Omega})})\max\{\omega_\varphi(t^{1/2}), t^{1/2}\},$$

where $\lambda > 0$ is a constant. Applying Proposition 4.2 we obtain the required result, that is,

$$\omega_{\mathrm{U}}(t) \le \eta(1 + \|f\|^{1/n}_{L^\infty(\bar{\Omega})})\max\{\omega_\varphi(t^{1/2}), \omega_{f^{1/n}}(t), t^{1/2}\},$$

where $\eta > 0$ depends on Ω. ∎

COROLLARY 4.7. *Let* Ω *be a bounded SHL domain in* $\mathbb{C}^n$. *Let* $\varphi \in \mathcal{C}^{0,\alpha}(\partial\Omega)$ *and* $0 \le f^{1/n} \in \mathcal{C}^{0,\beta}(\bar{\Omega})$, $0 < \alpha, \beta \le 1$. *Then the solution* U *to the Dirichlet problem* $\mathrm{Dir}(\Omega, \varphi, f)$ *belongs to* $\mathcal{C}^{0,\gamma}(\bar{\Omega})$ *for* $\gamma = \min(\beta, \alpha/2)$.

The following example illustrates that the estimate of ω_{U} in Theorem A is optimal.

EXAMPLE 4.8. Let ψ be a concave modulus of continuity on $[0, 1]$ and

$$\varphi(z) = -\psi[\sqrt{(1 + \Re z_1)/2}] \quad \text{for } z = (z_1, \dots, z_n) \in \partial\mathbb{B} \subset \mathbb{C}^n.$$

It is easy to show that $\varphi \in \mathcal{C}(\partial\mathbb{B})$ with modulus of continuity

$$\omega_\varphi(t) \le C\psi(t)$$

for some $C > 0$.

Let $v(z) = -(1+\Re z_1)/2 \in \mathrm{PSH}(\mathbb{B}) \cap \mathcal{C}(\bar{\mathbb{B}})$ and $\chi(\lambda) = -\psi(\sqrt{-\lambda})$ be a convex increasing function on $[-1, 0]$. Hence we see that

$$u(z) = \chi \circ v(z) \in \mathrm{PSH}(\mathbb{B}) \cap \mathcal{C}(\bar{\mathbb{B}})$$

and satisfies $(dd^c u)^n = 0$ in $\mathbb{B}$ and $u = \varphi$ on $\partial\mathbb{B}$. The modulus of continuity of $\mathtt{U}$ has the estimate

$$C_1\psi(t^{1/2}) \le \omega_{\mathtt{U}}(t) \le C_2\psi(t^{1/2})$$

for $C_1, C_2 > 0$. Indeed, let $z_0 = (-1, 0, \dots, 0)$ and $z = (z_1, 0, \dots, 0) \in \mathbb{B}$ where $z_1 = -1 + 2t$ and $0 \le t \le 1$. Hence, by Lemma 4.1, we conclude that

$$\psi(t^{1/2}) = \psi[\sqrt{|z - z_0|/2}] = \psi[\sqrt{(1+\Re z_1)/2}] = |\mathtt{U}(z) - \mathtt{U}(z_0)| \le 3\omega_{\mathtt{U}}(t).$$

Definition 4.9. Let ψ be a modulus of continuity, $E \subset \mathbb{C}^n$ be a bounded set and $g \in \mathcal{C} \cap L^\infty(E)$. We define the *norm of g with respect to ψ* (briefly, the *ψ-norm*) as follows:

$$\|g\|_\psi := \sup_{z\in E} |g(z)| + \sup_{z\neq y\in E} \frac{|g(z) - g(y)|}{\psi(|z-y|)}.$$

Proposition 4.10. *Let $\Omega \subset \mathbb{C}^n$ be a bounded SHL domain, $\varphi \in \mathcal{C}(\partial\Omega)$ with modulus of continuity ψ_1 and $f^{1/n} \in \mathcal{C}(\bar{\Omega})$ with modulus of continuity ψ_2. Then there exists a constant $C > 0$ depending on Ω such that*

$$\|\mathtt{U}\|_\psi \le C(1 + \|f\|^{1/n}_{L^\infty(\bar{\Omega})}) \max\{\|\varphi\|_{\psi_1}, \|f^{1/n}\|_{\psi_2}\},$$

where $\psi(t) = \max\{\psi_1(t^{1/2}), \psi_2(t)\}$.

Proof. By hypothesis, we see that $\|\varphi\|_{\psi_1} < \infty$ and $\|f^{1/n}\|_{\psi_2} < \infty$. Let $z \neq y \in \bar{\Omega}$. By Theorem A, we get

$$\begin{aligned}|\mathtt{U}(z) - \mathtt{U}(y)| &\le \eta(1 + \|f\|^{1/n}_{L^\infty(\bar{\Omega})}) \max\{\omega_\varphi(|z-y|^{1/2}), \omega_{f^{1/n}}(|z-y|)\}\\ &\le \eta(1 + \|f\|^{1/n}_{L^\infty(\bar{\Omega})}) \max\{\|\varphi\|_{\psi_1}, \|f^{1/n}\|_{\psi_2}\}\psi(|z-y|),\end{aligned}$$

where $\psi(|z-y|) = \max\{\psi_1(|z-y|^{1/2}), \psi_2(|z-y|)\}$. Hence

$$\sup_{z\neq y\in\bar{\Omega}} \frac{|\mathtt{U}(z) - \mathtt{U}(y)|}{\psi(|z-y|)} \le \eta(1 + \|f\|^{1/n}_{L^\infty(\bar{\Omega})}) \max\{\|\varphi\|_{\psi_1}, \|f^{1/n}\|_{\psi_2}\},$$

where $\eta \ge d^2 + 1$ and $d = \mathrm{diam}(\Omega)$ (see Proposition 4.2). From Remark 3.3, we note that

$$\|\mathtt{U}\|_{L^\infty(\bar{\Omega})} \le d^2\|f\|^{1/n}_{L^\infty(\bar{\Omega})} + \|\varphi\|_{L^\infty(\partial\Omega)} \le \eta \max\{\|\varphi\|_{\psi_1}, \|f^{1/n}\|_{\psi_2}\}.$$

Then we conclude that

$$\|\mathtt{U}\|_\psi \le 2\eta(1+\|f\|^{1/n}_{L^\infty(\bar\Omega)})\max\{\|\varphi\|_{\psi_1}, \|f^{1/n}\|_{\psi_2}\}. \blacksquare$$

Finally, it is natural to try to relate the modulus of continuity of $\mathtt{U} := \mathtt{U}(\Omega,\varphi,f)$ to the modulus of continuity of $\mathtt{U}_0 := \mathtt{U}(\Omega,\varphi,0)$, the solution to the Bremermann problem in a bounded SHL domain.

PROPOSITION 4.11. *Let Ω be a bounded SHL domain in $\mathbb{C}^n$, $0 \le f \in \mathcal{C}(\bar\Omega)$ and $\varphi \in \mathcal{C}(\partial\Omega)$. Then there exists a positive constant $C = C(\Omega)$ such that*

$$\omega_{\mathtt{U}}(t) \le C(1+\|f\|^{1/n}_{L^\infty(\bar\Omega)})\max\{\omega_{\mathtt{U}_0}(t), \omega_{f^{1/n}}(t)\}.$$

Proof. First, we search for a subsolution $v \in \mathcal{V}(\Omega,\varphi,f)$ such that $v|_{\partial\Omega} = \varphi$ and estimate its modulus of continuity. Since Ω is a bounded SHL domain, there exists a Lipschitz defining function ρ on $\bar\Omega$. Define

$$v(z) = \mathtt{U}_0(z) + A\rho(z),$$

where $A := \|f\|^{1/n}_{L^\infty}/c$ and $c > 0$ is as in Definition 2.1. It is clear that $v \in \mathcal{V}(\Omega,\varphi,f)$, $v=\varphi$ on $\partial\Omega$ and

$$\omega_v(t) \le \tilde{C}\omega_{\mathtt{U}_0}(t)$$

where $\tilde C := \gamma(1+\|f\|^{1/n}_{L^\infty(\bar\Omega)})$ and $\gamma \ge 1$ depends on Ω.

On the other hand, by the comparison principle we get $\mathtt{U} \le \mathtt{U}_0$. So,

$$v \le \mathtt{U} \le \mathtt{U}_0 \quad \text{in } \Omega \quad \text{and} \quad v = \mathtt{U} = \mathtt{U}_0 = \varphi \quad \text{on } \partial\Omega.$$

Thanks to Proposition 4.2, there exists $\lambda > 0$ depending on Ω such that

$$\omega_{\mathtt{U}}(t) \le \lambda \max\{\omega_v(t), \omega_{\mathtt{U}_0}(t), \omega_{f^{1/n}}(t)\}.$$

Hence, for some $C > 0$ depending on Ω,

$$\omega_{\mathtt{U}}(t) \le C(1+\|f\|^{1/n}_{L^\infty(\bar\Omega)})\max\{\omega_{\mathtt{U}_0}(t), \omega_{f^{1/n}}(t)\}. \blacksquare$$

5. Hölder continuous solutions for the Dirichlet problem with L^p density. In this section we will prove the existence and the Hölder continuity of the solution to the Dirichlet problem $\mathrm{Dir}(\Omega,\varphi,f)$ when $f \in L^p(\Omega)$, $p > 1$, in a bounded SHL domain.

It is well known (see [Ko98]) that there exists a weak continuous solution to this problem when Ω is a bounded strongly pseudoconvex domain with smooth boundary.

The Hölder continuity of this solution was studied in [GKZ08] under some additional conditions on the density and on the boundary data, that is, when f is bounded near the boundary and $\varphi \in \mathcal{C}^{1,1}(\partial\Omega)$.

The following weak stability estimate plays an important role in the proof of the Hölder continuity of the solution.

THEOREM 5.1 ([GKZ08]). *Fix $0 \le f \in L^p(\Omega)$, $p > 1$. Let u, v be two bounded plurisubharmonic functions in Ω such that $(dd^c u)^n = f\beta^n$ in Ω and let $u \ge v$ on $\partial\Omega$. Fix $r \ge 1$ and $0 \le \gamma < r/(nq+r)$, $1/p + 1/q = 1$. Then there exists a uniform constant $C = C(\gamma, n, q) > 0$ such that*

$$\sup_{\Omega}(v-u) \le C(1+\|f\|^{\tau}_{L^p(\Omega)})\|(v-u)_+\|^{\gamma}_{L^r(\Omega)},$$

where $\tau := \frac{1}{n} + \frac{\gamma q}{r-\gamma(r+nq)}$ and $(v-u)_+ := \max(v-u, 0)$.

In [GKZ08], the authors constructed a Lipschitz continuous barrier to the Dirichlet problem when $\varphi \in \mathcal{C}^{1,1}(\partial\Omega)$ and f is bounded near the boundary. Moreover, it was shown in this case that the total mass of $\Delta\mathtt{U}$ is finite in Ω. Finally, they concluded that $\mathtt{U} \in \mathcal{C}^{0,\alpha}(\bar{\Omega})$ for any $\alpha < 2/(nq+1)$. The following theorem summarizes the work in [GKZ08].

THEOREM 5.2 ([GKZ08]). *Let $0 \le f \in L^p(\Omega)$ for some $p > 1$, and $\varphi \in \mathcal{C}(\partial\Omega)$. Suppose that there exist $v, w \in \mathrm{PSH}(\Omega) \cap \mathcal{C}^{0,\alpha}(\bar{\Omega})$ such that $v \le \mathtt{U} \le -w$ on $\bar{\Omega}$ and $v = \varphi = -w$ on $\partial\Omega$. If the total mass of $\Delta\mathtt{U}$ is finite in Ω, then $\mathtt{U} \in \mathcal{C}^{0,\alpha'}(\bar{\Omega})$ for $\alpha' < \min\{\alpha, 2/(nq+1)\}$.*

Let $\Omega \subset \mathbb{C}^n$ be a bounded SHL domain. Using the stability Theorem 5.1 we will ensure the existence of the solution to the Dirichlet problem $\mathrm{Dir}(\Omega, \varphi, f)$ when $f \in L^p(\Omega)$, $p > 1$.

PROPOSITION 5.3. *Let $\Omega \subset \mathbb{C}^n$ be a bounded SHL domain, $\varphi \in \mathcal{C}(\partial\Omega)$ and $0 \le f \in L^p(\Omega)$ for some $p > 1$. Then there exists a unique solution $\mathtt{U}$ to the Dirichlet problem $\mathrm{Dir}(\Omega, \varphi, f)$.*

Proof. Let (f_j) be a sequence of smooth functions on $\bar{\Omega}$ which converges to f in $L^p(\Omega)$. Thanks to Proposition 3.2, there exists a unique solution $\mathtt{U}_j$ to $\mathrm{Dir}(\Omega, \varphi, f_j)$, that is, $\mathtt{U}_j \in \mathrm{PSH}(\Omega) \cap \mathcal{C}(\bar{\Omega})$, $\mathtt{U}_j = \varphi$ on $\partial\Omega$ and $(dd^c\mathtt{U}_j)^n = f_j\beta^n$ in Ω. We claim that

$$\|\mathtt{U}_k - \mathtt{U}_j\|_{L^\infty(\bar{\Omega})} \le A(1+\|f_k\|^{\tau}_{L^p(\Omega)})(1+\|f_j\|^{\tau}_{L^p(\Omega)})\|f_k - f_j\|^{\gamma/n}_{L^1(\Omega)}, \tag{5.1}$$

where $0 \le \gamma < 1/(q+1)$ is fixed, $\tau := \frac{1}{n} + \frac{\gamma q}{n-\gamma n(1+q)}$, $1/p + 1/q = 1$ and $A = A(\gamma, n, q, \mathrm{diam}(\Omega))$.

Indeed, by the stability theorem 5.1 and for $r = n$, we get

$$\begin{aligned}\sup_{\Omega}(\mathtt{U}_k - \mathtt{U}_j) &\le C(1+\|f_j\|^{\tau}_{L^p(\Omega)})\|(\mathtt{U}_k - \mathtt{U}_j)_+\|^{\gamma}_{L^n(\Omega)} \\ &\le C(1+\|f_j\|^{\tau}_{L^p(\Omega)})\|\mathtt{U}_k - \mathtt{U}_j\|^{\gamma}_{L^n(\Omega)},\end{aligned}$$

where $0 \le \gamma < 1/(q+1)$ is fixed and $C = C(\gamma, n, q) > 0$. Hence by the L^n-L^1 stability theorem of [Bł93] (see our Remark 3.3),

$$\|\mathtt{U}_k - \mathtt{U}_j\|_{L^n(\Omega)} \le \tilde{C}\|f_k - f_j\|^{1/n}_{L^1(\Omega)},$$

where $\tilde{C}$ depends on $\mathrm{diam}(\Omega)$. Then, from the last two inequalities and reversing the role of $\mathtt{U}_j$ and $\mathtt{U}_k$, we deduce

$$\|\mathtt{U}_k - \mathtt{U}_j\|_{L^\infty(\Omega)} \le C\tilde{C}^\gamma(1 + \|f_k\|^\tau_{L^p(\Omega)})(1 + \|f_j\|^\tau_{L^p(\Omega)})\|f_k - f_j\|^{\gamma/n}_{L^1(\Omega)}.$$

Since $\mathtt{U}_k = \mathtt{U}_j = \varphi$ on $\partial\Omega$, the inequality (5.1) holds.

As f_j converges to f in $L^p(\Omega)$, there is a uniform constant $B > 0$ such that

$$\|\mathtt{U}_k - \mathtt{U}_j\|_{L^\infty(\bar{\Omega})} \le B\|f_k - f_j\|^{\gamma/n}_{L^1(\Omega)}.$$

This implies that the sequence $\mathtt{U}_j$ converges uniformly in $\bar{\Omega}$. Set $\mathtt{U} = \lim \mathtt{U}_j$. It is clear that $\mathtt{U} \in \mathrm{PSH}(\Omega) \cap \mathcal{C}(\bar{\Omega})$ and $\mathtt{U} = \varphi$ on $\partial\Omega$. Moreover, $(dd^c\mathtt{U}_j)^n$ converges to $(dd^c\mathtt{U})^n$ in the sense of currents, thus $(dd^c\mathtt{U})^n = f\beta^n$ in Ω. The uniqueness of the solution follows from the comparison principle (see [BT76]). ∎

Our next step is to construct Hölder continuous subbarriers and superbarriers to the Dirichlet problem when $f \in L^p(\Omega)$ for some $p > 1$ and $\varphi \in \mathcal{C}^{0,1}(\partial\Omega)$.

Proposition 5.4. *Let $\varphi \in \mathcal{C}^{0,1}(\partial\Omega)$ and $0 \le f \in L^p(\Omega)$ for some $p > 1$. Then there exist $v, w \in \mathrm{PSH}(\Omega) \cap \mathcal{C}^{0,\alpha}(\bar{\Omega})$ where $\alpha < 1/(nq + 1)$ such that $v = \varphi = -w$ on $\partial\Omega$ and $v \le \mathtt{U} \le -w$ on Ω.*

Proof. Fix a large ball $B \subset \mathbb{C}^n$ so that $\Omega \Subset B \subset \mathbb{C}^n$. Let $\tilde{f}$ be a trivial extension of f to B. Since $\tilde{f} \in L^p(\Omega)$ is bounded near ∂B, the solution h_1 to $\mathrm{Dir}(B, 0, \tilde{f})$ is Hölder continuous on $\bar{B}$ with exponent $\alpha_1 < 2/(nq + 1)$ (see [GKZ08]). Now let h_2 denote the solution to the Dirichlet problem in Ω with boundary value $\varphi - h_1$ and the zero density. Thanks to Theorem A, we see that $h_2 \in \mathcal{C}^{0,\alpha_2}(\bar{\Omega})$ where $\alpha_2 = \alpha_1/2$. Therefore, the required barrier will be $v = h_1 + h_2$. It is clear that $v \in \mathrm{PSH}(\Omega) \cap \mathcal{C}(\bar{\Omega})$, $v|_{\partial\Omega} = \varphi$ and $(dd^cv)^n \ge f\beta^n$ in the weak sense in Ω. Hence, by the comparison principle we get $v \le \mathtt{U}$ in Ω and $v = \mathtt{U} = \varphi$ on $\partial\Omega$. Moreover $v \in \mathcal{C}^{0,\alpha}(\bar{\Omega})$ for any $\alpha < 1/(nq + 1)$.

Finally, it is enough to set $w = \mathtt{U}(\Omega, -\varphi, 0)$ to obtain a superbarrier to the Dirichlet problem $\mathrm{Dir}(\Omega, \varphi, f)$. We note that $w \in \mathrm{PSH}(\Omega) \cap \mathcal{C}(\bar{\Omega})$, $-w = \varphi$ on $\partial\Omega$ and $\mathtt{U} \le -w$ on $\bar{\Omega}$. Furthermore, by Theorem A, $w \in \mathcal{C}^{0,1/2}(\bar{\Omega})$ and then $w \in \mathcal{C}^{0,\alpha}(\bar{\Omega})$ for any $\alpha < 1/(nq + 1)$. ∎

When $f \in L^p(\Omega)$ for $p \ge 2$, we are able to find a Hölder continuous barrier to the Dirichlet problem with better Hölder exponent. The following theorem was proved in [Ch14] for the complex Hessian equation, and it is enough here to put $m = n$ to get the complex Monge–Ampère equation.

Theorem 5.5 ([Ch14]). *Let $\varphi \in \mathcal{C}^{0,1}(\partial\Omega)$ and $0 \le f \in L^p(\Omega)$, $p \ge 2$. Then there exist $v, w \in \mathrm{PSH}(\Omega) \cap \mathcal{C}^{0,1/2}(\bar{\Omega})$ such that $v = \varphi = -w$ on $\partial\Omega$ and $v \le \mathtt{U} \le -w$ in Ω.*

We recall the comparison principle for the total mass of the Laplacian of plurisubharmonic functions.

LEMMA 5.6. *Let $u, v \in \mathrm{PSH}(\Omega) \cap \mathcal{C}(\bar{\Omega})$ be such that $v \le u$ on Ω and $u = v$ on $\partial\Omega$. Then*

$$\int_\Omega dd^c u \wedge \beta^{n-1} \le \int_\Omega dd^c v \wedge \beta^{n-1}.$$

5.1. Proof of Theorem B. Let $\mathtt{U}_0$ be the solution to the Dirichlet problem $\mathrm{Dir}(\Omega, 0, f)$. We first claim that the total mass of $\Delta \mathtt{U}_0$ is finite in Ω. Indeed, let ρ be the defining function of Ω; then by [Ce04, Corollary 5.6] we get

$$\begin{aligned}(5.2)\qquad \int_\Omega dd^c\mathtt{U}_0 \wedge (dd^c\rho)^{n-1} &\le \Big(\int_\Omega (dd^c\mathtt{U}_0)^n\Big)^{1/n} \Big(\int_\Omega (dd^c\rho)^n\Big)^{(n-1)/n}\\ &\le \Big(\int_\Omega f\beta^n\Big)^{1/n} \Big(\int_\Omega (dd^c\rho)^n\Big)^{(n-1)/n}.\end{aligned}$$

Since Ω is a bounded SHL domain, there exists a constant $c > 0$ such that $dd^c\rho \ge c\beta$ in Ω. Hence (5.2) yields

$$\begin{aligned}\int_\Omega dd^c\mathtt{U}_0 \wedge \beta^{n-1} &\le \frac{1}{c^{n-1}} \int_\Omega dd^c\mathtt{U}_0 \wedge (dd^c\rho)^{n-1}\\ &\le \frac{1}{c^{n-1}} \Big(\int_\Omega f\beta^n\Big)^{1/n} \Big(\int_\Omega (dd^c\rho)^n\Big)^{(n-1)/n}.\end{aligned}$$

Now we note that the total mass of the complex Monge–Ampère measure of ρ is finite in Ω by the Chern–Levine–Nirenberg inequality and since ρ is psh and bounded in a neighborhood of $\bar{\Omega}$ (see [BT76]). Therefore, the total mass of $\Delta\mathtt{U}_0$ is finite in Ω.

Let $\tilde{\varphi}$ be a $\mathcal{C}^{1,1}$-extension of φ to $\bar{\Omega}$ with $\|\tilde{\varphi}\|_{\mathcal{C}^{1,1}(\bar{\Omega})} \le C\|\varphi\|_{\mathcal{C}^{1,1}(\partial\Omega)}$ for some $C > 0$. Now, let $v = A\rho + \tilde{\varphi} + \mathtt{U}_0$ where $A \gg 1$ such that $A\rho + \tilde{\varphi} \in \mathrm{PSH}(\Omega)$. By the comparison principle, $v \le \mathtt{U}$ in Ω and $v = \mathtt{U} = \varphi$ on $\partial\Omega$. Since ρ is psh in a neighborhood of $\bar{\Omega}$ and $\|\Delta\mathtt{U}_0\|_\Omega < \infty$, we deduce that $\|\Delta v\|_\Omega < \infty$. Then $\|\Delta\mathtt{U}\|_\Omega < \infty$ by Lemma 5.6.

Proposition 5.4 gives the existence of Hölder continuous barriers to the Dirichlet problem. Then using Theorem 5.2 we obtain the final result, that is, if $f \in L^p(\Omega)$ for some $p > 1$, then $\mathtt{U} \in \mathrm{PSH}(\Omega) \cap \mathcal{C}^{0,\alpha}(\bar{\Omega})$ where $\alpha < 1/(nq+1)$.

Moreover, if $f \in L^p(\Omega)$ for some $p \ge 2$, we can get a better result: by Theorems 5.5 and 5.2, $\mathtt{U} \in \mathrm{PSH}(\Omega) \cap \mathcal{C}^{0,\alpha}(\bar{\Omega})$ where $\alpha < \min\{1/2, 2/(nq+1)\}$. ∎

REMARK 5.7. It is shown in [GKZ08] that we cannot expect a better Hölder exponent than $2/(nq)$ (see also [Pl05]).

Acknowledgements. I would like to express my deepest and sincere gratitude to my advisor, Professor Ahmed Zeriahi, for all his help and sacrificing his very valuable time for me. I would also like to thank Hoang Chinh Lu for valuable discussions. I wish to express my acknowledgement to Professor Vincent Guedj for useful discussions.

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Mohamad Charabati
Institut de Mathématiques de Toulouse
Université Paul Sabatier
118 Route de Narbonne
31062 Toulouse Cedex 09, France
E-mail: mohamad.charabati@math.univ-toulouse.fr

Received 25.6.2014
and in final form 15.9.2014 (3429)