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Existence and nonexistence of solutions for a quasilinear elliptic system

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Abstract. By a sub-super solution argument, we study the existence of positive solutions for the system

$$\begin{cases} -\Delta_p u = a_1(x)F_1(x,u,v) & \text{in } \Omega, \\ -\Delta_q v = a_2(x)F_2(x,u,v) & \text{in } \Omega, \\ u,v>0 & \text{in } \Omega, \\ u=v=0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary or $\Omega = \mathbb{R}^N$. A nonexistence result is obtained for radially symmetric solutions.

1. Introduction. In this paper, we consider the existence and nonexistence of positive solutions for the system

$$\begin{cases} -\Delta_p u = a_1(x)F_1(x,u,v) & \text{in } \Omega, \\ -\Delta_q v = a_2(x)F_2(x,u,v) & \text{in } \Omega, \\ u,v>0 & \text{in } \Omega, \\ u=v=0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary or $\Omega = \mathbb{R}^N$ (when $\Omega = \mathbb{R}^N$, the condition $u = v = 0$ on $\partial\Omega$ should be understood as $u(x) \to 0$, $v(x) \to 0$ as $|x| \to \infty$), $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $p > 1$, $\Delta_q v = \operatorname{div}(|\nabla v|^{q-2}\nabla v)$, $q > 1$. Each $a_i(x)$ $(i = 1, 2)$ is a positive $C^{0,\alpha}(\overline{\Omega})$ $(\alpha \in (0, 1))$ function, and each function $F_i : \Omega \times (0,\infty) \times (0,\infty) \to (0,\infty)$ is continuously differentiable on its domain.

Systems of the above form are mathematical models occurring in studies of the p-Laplacian system, generalized reaction-diffusion theory, non-Newtonian fluid theory [AM], non-Newtonian filtration [K] and the turbulent flow of a gas in porous medium. Media with $p > 2$ are called dilatant

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fluids and those with $p < 2$ are called pseudoplastics. If $p = 2$, they are Newtonian fluids. When $p \neq 2$, the problem becomes more complicated since certain nice properties inherent to the case $p = 2$ seem to be lost or at least difficult to verify. The main differences between $p = 2$ and $p \neq 2$ can be found in [G2] and [GW].

There are many works dealing with the Lane–Emden system

$$
\begin{cases} -\Delta u = a_1(x)F_1(x,u,v) & \text{in } \Omega, \\ -\Delta v = a_2(x)F_2(x,u,v) & \text{in } \Omega, \\ u,v > 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}
$$

For example, [G1] and [Z] studied (1.2) with $a_1(x)F_1(x,u,v) = u^{-p}v^{-q}$, $a_2(x)F_2(x,u,v) = u^{-r}v^{-s}$, that is, F_i $(i = 1,2)$ are singular in all variables. We say that $F_i(x,u,v)$ is *singular in* u (or v) if $\lim_{u\to 0} F_i(x,u,v) = \infty$ (resp. $\lim_{v\to 0} F_i(x,u,v) = \infty$). By using the sub-super solution method, [G1] studied the existence, nonexistence, uniqueness, and C^1-regularity of solutions for (1.2). Furthermore, [Z] studied the existence, uniqueness and boundary behavior of solutions for (1.2) under different assumptions.

In [CMT], the authors considered the following system with nonsingular nonlinearities in all variables:

$$
\begin{cases} -\Delta U(x) = \nabla H(x,U(x)) & \text{in } \Omega, \\ U(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}
$$

where $U(x) = (u_1,u_2) : \Omega \to \mathbb{R}^2$, $H(x,u_1,u_2) = |u_1|^{\alpha_1}|u_2|^{\alpha_2}$ with $\alpha_i > 1$. By using variational methods, the authors provided the existence of nine nontrivial solutions characterized by sign properties of each component.

For the case $p \neq 2$, $q \neq 2$, Lee et al. [LSY1], [LSY2] studied the existence of solutions for the singular system

$$
\begin{cases} -\Delta_p u = \lambda(f_1(u,v) - u^{-\gamma_1}) & \text{in } \Omega, \\ -\Delta_q v = \lambda(f_2(u,v) - v^{-\gamma_2}) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}
$$

where $\gamma_i \in (0,1)$, $f_i \in C([0,\infty) \times [0,\infty))$, f_i are nondecreasing in both u and v, $i = 1,2$, $\lambda > 0$, $p,q > 1$.

In [YY2], Yin and Yang studied the existence and nonexistence of entire positive solutions for the nonlinear elliptic system

$$
\begin{cases} -\Delta_p u = a(x)u^m + \lambda c(x)v^n, & x \in \mathbb{R}^N, \\ -\Delta_q v = b(x)v^l + \theta c(x)u^n, & x \in \mathbb{R}^N, \\ u,v > 0, & x \in \mathbb{R}^N, \\ u \to 0,\ v \to 0 & \text{as } |x| \to \infty, \end{cases} \tag{1.5}
$$

where $1 < p,q < N$, λ, $\theta \geq 0$ are nonnegative parameters, $a, b, c : \mathbb{R}^N \to [0,\infty)$

are locally Hölder continuous functions not identically zero, $-\infty < m < p-1$, $-\infty < l < q-1$, $\max\{p-1, q-1\} < n$.

Moreover, when the nonlinearities are nonsingular in all variables, a lot of articles deal with blow-up solutions: see, for example, [WY1], [MY1] and [WY2].

Motivated by the above results, we establish results when the nonlinearities are singular in one of the variables and nonsingular in the others. Thus we assume F_i $(i = 1, 2)$ satisfy the following conditions:

(F_1) F_i $(i = 1, 2)$ are locally Hölder continuous.

(F_2) For each $i \in \{1, 2\}$, there exists a continuous function g_i : $(0, \infty) \to (0, \infty)$ satisfying $F_i(x, t_1, t_2) \le g_i(t_i)$ for all (x, t_1, t_2) in $\Omega \times (0, \infty) \times (0, \infty)$ with $g_1(s)/s^{p-1}$ and $g_2(s)/s^{q-1}$ decreasing on $(0, \infty)$, and
$$\lim_{s\to\infty} \frac{g_1(s)}{s^{p-1}} = 0, \qquad \lim_{s\to\infty} \frac{g_2(s)}{s^{q-1}} = 0.$$

(F_3) For each $i \in \{1, 2\}$, there exists $\delta_i \in (0, 1)$ and a continuous nonincreasing function $h_i : (0, \delta_1) \times (0, \delta_2) \to (0, \infty)$ satisfying $F_i(x, t_1, t_2) \ge h_i(t_1, t_2)$ for all $(x, t_1, t_2) \in \Omega \times (0, \delta_1) \times (0, \delta_2)$, and $\lim_{s\to 0} h_i(s, s) \in (0, \infty]$.

Now, we give an example of nonlinearities F_1, F_2 satisfying the assumptions (F_1)–(F_3). Let
$$F_1(x, t_1, t_2) = t_1^{(p-1)\alpha_1}(t_2 + \epsilon_2)^{\alpha_2} \quad \text{with } \alpha_1,\ \alpha_2 < 0,\ \epsilon_2 > 1,$$
$$F_2(x, t_1, t_2) = (t_1 + \epsilon_1)^{\alpha_1} t_2^{(q-1)\alpha_2} \quad \text{with } \alpha_1, \alpha_2 < 0,\ \epsilon_1 > 1.$$
Then, we choose
$$h_1(t_1, t_2) = t_1^{(p-1)\alpha_1}(t_2 + \epsilon_2)^{\alpha_2}, \qquad h_2(t_1, t_2) = (t_1 + \epsilon_1)^{\alpha_1} t_2^{(q-1)\alpha_2},$$
$$g_1(t_1) = \epsilon_2^{-\alpha_2} t_1^{(p-1)\alpha_1}, \qquad g_2(t_2) = \epsilon_1^{-\alpha_1} t_2^{(q-1)\alpha_2}.$$
By a direct computation, we can easily show that the functions F_i, h_i, g_i satisfy the assumptions (F_1)–(F_3).

The main purpose of this paper is to investigate the existence and nonexistence of positive solutions for (1.1). Our main results are:

THEOREM 1.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary, and assume* (F_1)–(F_3) *hold. Then problem* (1.1) *has a solution.*

THEOREM 1.2. *Let $\Omega = \mathbb{R}^N$, and assume* (F_1)–(F_3) *hold, and a_i satisfy*
$$\int_0^\infty rA(r)\,dr < \infty \quad \textit{where} \quad A(r) = \max_{|x|=r}(a_1(x) + a_2(x)).$$
Then problem (1.1) *has a solution.*

THEOREM 1.3. *Let $\Omega = \mathbb{R}^N$, and assume that $F_i : \mathbb{R}^N \times [0,\infty) \times [0,\infty) \to \mathbb{R}^N$ are continuous functions. If there exist $\epsilon > 0$, $r_0 \geq 0$, and a continuous function $B : [r_0,\infty) \to (0,\infty)$ satisfying*

$$\int_{r_0}^{\infty} rB(r)\,dr = \infty$$

such that for all $x \in \mathbb{R}^N$ with $|x| \geq r_0$, we have

$$\sum_{i=1}^{2} a_i(x)F_i(x,u,v) \geq B(r) \quad \text{for all } |(u,v)| \leq \epsilon,$$

then problem (1.1) *has no radial positive bounded solutions.*

2. Preliminaries

DEFINITION 2.1. Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary. We say $(\underline{u},\underline{v})$ is a *subsolution* of (1.1) provided

$$\begin{cases} -\Delta_p \underline{u} \leq a_1(x)F_1(x,\underline{u},\underline{v}) & \text{in } \Omega, \\ -\Delta_q \underline{v} \leq a_2(x)F_2(x,\underline{u},\underline{v}) & \text{in } \Omega, \\ \underline{u},\underline{v} > 0 & \text{in } \Omega, \\ \underline{u} = \underline{v} = 0 & \text{on } \partial\Omega. \end{cases}$$

A *supersolution* $(\overline{u},\overline{v})$ is defined by reversing the inequalities.

LEMMA 2.2. *Assume that $(\underline{u},\underline{v})$ is a subsolution and $(\overline{u},\overline{v})$ is a supersolution of problem* (1.1), *with $\underline{u} \leq \overline{u}$, $\underline{v} \leq \overline{v}$ in Ω, and $\underline{u} = \underline{v} = \overline{u} = \overline{v} = 0$ on $\partial\Omega$. Then problem* (1.1) *has a solution (u,v) with $\underline{u} \leq u \leq \overline{u}$, $\underline{v} \leq v \leq \overline{v}$. In particular, $u = v = 0$ on $\partial\Omega$.*

LEMMA 2.3 (Diaz-Saa Inequality, see also [MY2, Lemma 2.7]). *Let $\Omega \subset \mathbb{R}^N$ be an open set. For $i = 1,2$, let $\omega_i \in L^\infty(\Omega)$ be such that $\omega_i > 0$ a.e. in Ω, $\omega_i \in W^{1,p}(\Omega)$, $\Delta_p \omega_i^{1/p} \in L^\infty(\Omega)$ and $\omega_1 = \omega_2$ on $\partial\Omega$. Then*

$$\int_{\Omega} \left[\frac{-\Delta_p \omega_1^{1/p}}{\omega_1^{(p-1)/p}} - \frac{-\Delta_p \omega_2^{1/p}}{\omega_2^{(p-1)/p}} \right] (\omega_1 - \omega_2)\,dx \geq 0$$

if $\omega_i/\omega_j \in L^\infty(\Omega)$ for $i \neq j$, $i,j = 1,2$.

LEMMA 2.4. *Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary, $a_i(\cdot) \in C^{0,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$, and $a_i(x) > 0$ for all $x \in \overline{\Omega}$. Then the problem*

$$\begin{cases} -\Delta_p w = a_i(x) & \text{in } \Omega, \\ w(x) > 0 & \text{in } \Omega, \\ w(x) = 0 & \text{on } \partial\Omega \end{cases}$$

has a unique solution.

3. Proof of Theorem 1.1. Consider the eigenvalue problem

$$\begin{cases} -\Delta_p \phi = \lambda a_i(x)|\phi|^{p-2}\phi & \text{in } \Omega, \\ \phi(x) > 0 & \text{in } \Omega, \\ \phi(x) = 0 & \text{on } \partial\Omega. \end{cases}$$

Let ϕ_1^i $(i = 1, 2)$ be the eigenfunctions corresponding to the first eigenvalues λ_1^i $(i = 1, 2)$ respectively. Then $\phi_1^i > 0$ $(i = 1, 2)$ in Ω.

Using the assumptions on h_i $(i = 1, 2)$, we get

$$\lim_{s \to 0^+} \frac{h_1(s, s)}{s^{p-1}} = \infty, \qquad \lim_{s \to 0^+} \frac{h_2(s, s)}{s^{q-1}} = \infty.$$

Then there exist $\epsilon_1, \epsilon_2 > 0$ satisfying

$$\frac{h_1(s, s)}{s^{p-1}} \geq \lambda_1^1, \ \ \forall s \in (0, \epsilon_1), \qquad \frac{h_2(s, s)}{s^{q-1}} \geq \lambda_1^2, \ \ \forall s \in (0, \epsilon_2).$$

Let $(\underline{u}, \underline{v}) = (C_1\phi_1^1, C_2\phi_1^2)$, where C_i $(i = 1, 2)$ satisfy

$$0 < C_i < \min\left\{1, \frac{\epsilon}{2\max_{x \in \bar{\Omega}} \phi_1^i(x)}\right\}, \qquad \epsilon = \min\{\epsilon_1, \epsilon_2\}.$$

We get

$$\begin{aligned} -\Delta_p \underline{u} &= -C_1^{p-1} \operatorname{div}(|\nabla\phi_1^1|^{p-2}\nabla\phi_1^1) = C_1^{p-1}\lambda_1^1 a_1(x)|\phi_1^1|^{p-2}\phi_1^1 \\ &\leq \lambda_1^1 a_1(x)(C_1\phi_1^1 + C_2\phi_1^2)^{p-1} \leq a_1(x)h_1(C_1\phi_1^1 + C_2\phi_1^2, C_1\phi_1^1 + C_2\phi_1^2) \\ &\leq a_1(x)h_1(C_1\phi_1^1, C_2\phi_1^2) \leq a_1(x)F_1(x, \underline{u}, \underline{v}). \end{aligned}$$

Using a similar method, we can obtain

$$-\Delta_q \underline{v} \leq a_2(x)F_2(x, \underline{u}, \underline{v}).$$

Thus, $(\underline{u}, \underline{v}) = (C_1\phi_1^1, C_2\phi_1^2)$ is a subsolution of (1.1).

Next, we will construct a supersolution. By the assumptions on g_i $(i = 1, 2)$, we define

$$\overline{g}_i(t) = \frac{2}{t} \int_{t/2}^{t} \widehat{g}_i(s)\, ds, \qquad t > 0,$$

where

$$\widehat{g}_1(s) = \sup_{t \geq s > 0} \frac{g_1(t)}{t^{p-1}}, \qquad \widehat{g}_2(s) = \sup_{t \geq s > 0} \frac{g_2(t)}{t^{q-1}}.$$

Then $\overline{g}_i(\cdot) \in C^1((0, \infty), (0, \infty))$, $\overline{g}_1(t) > g_1(t)/t^{p-1}$ and $\overline{g}_2(t) > g_2(t)/t^{p-1}$ for all $t > 0$, and $\overline{g}_i(\cdot)$ is nonincreasing on $(0, \infty)$.

Let $w_{a_1+a_2}(x)$ be the solution to the problem

$$\begin{cases} -\Delta_p w = a_1(x) + a_2(x) & \text{in } \Omega, \\ w(x) > 0 & \text{in } \Omega, \\ w(x) = 0 & \text{on } \partial\Omega, \end{cases}$$

and $C_0 = \max_{x\in\overline{\Omega}} w_{a_1+a_2}(x)$. Then we define $\overline{u} : \overline{\Omega} \to (0,\infty)$ implicitly by

$$w_{a_1+a_2} = \frac{1}{C_1}\int_0^{\overline{u}} \left(\frac{s^{p-1}}{s^{p-1}\overline{g}_1(s)+1}\right)^{\frac{1}{p-1}} ds$$

where C_1 satisfies

$$C_0C_1 < \int_0^{C_1} \left(\frac{s^{p-1}}{s^{p-1}\overline{g}_1(s)+1}\right)^{\frac{1}{p-1}} ds.$$

Then we have $0 \le \overline{u} \le C_1$. Thus,

$$\begin{aligned} C_1^{p-1}(a_1(x)+a_2(x)) &= -C_1^{p-1}\operatorname{div}(|\nabla w|^{p-2}\nabla w) \\ &= -\operatorname{div}\left(\frac{1}{\overline{g}_1(\overline{u}) + (\overline{u})^{-(p-1)}}|\nabla\overline{u}|^{p-2}\nabla\overline{u}\right) \\ &= -\frac{1}{\overline{g}_1(\overline{u}) + (\overline{u})^{-(p-1)}}\operatorname{div}(|\nabla\overline{u}|^{p-2}\nabla\overline{u}) \\ &\quad -|\nabla\overline{u}|^p\frac{d}{d\overline{u}}\left(\frac{1}{\overline{g}_1(\overline{u}) + (\overline{u})^{-(p-1)}}\right) \\ &\le -\frac{1}{\overline{g}_1(\overline{u}) + (\overline{u})^{-(p-1)}}\operatorname{div}(|\nabla\overline{u}|^{p-2}\nabla\overline{u}). \end{aligned}$$

Then we have

$$\begin{aligned} -\Delta_p\overline{u} &\ge C_1^{p-1}(a_1(x)+a_2(x))[\overline{g}_1(\overline{u}) + (\overline{u})^{-(p-1)}] \\ &\ge (a_1(x)+a_2(x))(\overline{u})^{p-1}\left[\frac{g_1(\overline{u})}{(\overline{u})^{p-1}} + \frac{1}{(\overline{u})^{p-1}}\right] \\ &\ge a_1(x)g_1(\overline{u}) \ge a_1(x)F_1(x,\overline{u},\overline{v}). \end{aligned}$$

Using a similar method, we can find a function $\overline{v} : \overline{\Omega} \to (0,\infty)$ satisfying

$$-\Delta_q\overline{v} \ge a_2(x)g_2(\overline{v}) \ge a_2(x)F_2(x,\overline{u},\overline{v}).$$

Thus, we have constructed a supersolution $(\overline{u},\overline{v})$.

Now, we show that $\underline{u} \le \overline{u}$ for all $x \in \overline{\Omega}$. Let

$$\Omega_{\underline{u},\overline{u}} = \{x \in \Omega : \underline{u} > \overline{u}\}.$$

We have to show that $\Omega_{\underline{u},\overline{u}} = \emptyset$. Assume, on the contrary, that $\Omega_{\underline{u},\overline{u}} \neq \emptyset$.

By exploiting Lemma 2.3 with $\omega_1 = \underline{u}^p$, $\omega_2 = \overline{u}^p$, we have

$$
\begin{aligned}
0 &\le \int\limits_{\Omega_{\underline{u},\overline{u}}} \left(\frac{-\Delta_p \omega_1^{1/p}}{\omega_1^{(p-1)/p}} - \frac{-\Delta_p \omega_2^{1/p}}{\omega_2^{(p-1)/p}} \right) (\omega_1 - \omega_2)\, dx \\
&= \int\limits_{\Omega_{\underline{u},\overline{u}}} \left(\frac{-\Delta_p \underline{u}}{\underline{u}^{p-1}} - \frac{-\Delta_p \overline{u}}{\overline{u}^{p-1}} \right) (\underline{u}^{p-1} - \overline{u}^{p-1})\, dx \\
&\le \int\limits_{\Omega_{\underline{u},\overline{u}}} a_1(x) \left[\frac{F_1(x, \underline{u}, \underline{v})}{\underline{u}^{p-1}} - \frac{g_1(\overline{u})}{\overline{u}^{p-1}} \right] (\underline{u}^{p-1} - \overline{u}^{p-1})\, dx \\
&\le \int\limits_{\Omega_{\underline{u},\overline{u}}} a_1(x) \left[\frac{g_1(\underline{u})}{\underline{u}^{p-1}} - \frac{g_1(\overline{u})}{\overline{u}^{p-1}} \right] (\underline{u}^{p-1} - \overline{u}^{p-1})\, dx < 0,
\end{aligned}
$$

which is a contradiction. Thus $\Omega_{\underline{u},\overline{u}} = \emptyset$. On the other hand, $\underline{u} = \overline{u} = 0$ on $\partial\Omega$. Thus, we have $\underline{u} \le \overline{u}$ for all $x \in \overline{\Omega}$. Similarly, $\underline{v} \le \overline{v}$ for all $x \in \overline{\Omega}$.

By Lemma 2.2, there exists a function (u, v) solving (1.1) with $\underline{u} \le u \le \overline{u}$ on $\overline{\Omega}$ and $\underline{v} \le v \le \overline{v}$ on $\overline{\Omega}$. Thus, the proof of Theorem 1.1 is finished.

4. Proof of Theorem 1.2. Consider the system

$$
\begin{cases}
-\Delta_p u = a_1(x) F_1(x, u, v) & \text{in } B_n, \\
-\Delta_q v = a_2(x) F_2(x, u, v) & \text{in } B_n, \\
u, v > 0 & \text{in } B_n, \\
u = v = 0 & \text{on } \partial B_n,
\end{cases}
\tag{4.1}
$$

where B_n is the open ball of radius n centered at the origin. By Theorem 1.1, we know (4.1) has a solution, say (u^n, v^n). Next, we construct an upper bound for this sequence. Similar to the proof of Theorem 1.1, we define $\overline{u}(\cdot) : [0, \infty) \to (0, \infty)$ implicitly by

$$
w_A(r) = \frac{1}{C_1} \int\limits_0^{\overline{u}(r)} \left(\frac{s^{p-1}}{s^{p-1} \overline{g}_1(s) + 1} \right)^{\frac{1}{p-1}} ds
$$

where C_1, $\overline{g}_1(s)$ are defined in Theorem 1.1, and $w_A(\cdot)$ is a positive bounded radially symmetric solution of the problem

$$
\begin{cases}
-\Delta_p w(r) = A(r), & 0 \le r < \infty, \\
w(r) > 0, & 0 \le r < \infty, \\
w'(0) = 0, \quad \lim_{r \to \infty} w(r) = 0.
\end{cases}
$$

Then, by direct computation similar to the one in the proof of Theorem 1.1, we obtain

$$
-\Delta_p \overline{u}(r) \ge A(r) g_1(\overline{u}(r)) \ge a_1(x) g_1(\overline{u}(r)) \ge a_1(x) F_1(x, \overline{u}(r), \overline{v}(r))
$$

for all $x \in \mathbb{R}^N$. Using the same method, we can find a function $\overline{v}(\cdot) : [0, \infty) \to (0, \infty)$ satisfying

$$-\Delta_q \overline{v}(r) \geq A(r) g_2(\overline{v}(r)) \geq a_2(x) g_2(\overline{v}(r)) \geq a_2(x) F_2(x, \overline{u}(r), \overline{v}(r))$$

for all $x \in \mathbb{R}^N$. Lemma 2.2 implies that $0 < u^n(x) \leq \overline{u}(r)$ and $0 < v^n(x) \leq \overline{v}(r)$ for all $x \in \overline{B}_n$, that is, $\{u^n(x)\}_{n=1}^{\infty}$ and $\{v^n(x)\}_{n=1}^{\infty}$ are bounded in $\overline{B}_n \subset \mathbb{R}^N$. By (F_1), (4.1) and the continuity of a_i, we can easily deduce that $\Delta_p u^n(x)$ and $\Delta_q v^n(x)$ are bounded in $\overline{B}_n$, which implies that $|\nabla u^n(x)| \leq M$ and $|\nabla v^n(x)| \leq M$ for some $M > 0$. Thus, by the Arzelà–Ascoli theorem, $\{u^n(x)\}_{n=1}^{\infty}$ and $\{v^n(x)\}_{n=1}^{\infty}$ have subsequences (still denoted by $\{u^n(x)\}_{n=1}^{\infty}$ and $\{v^n(x)\}_{n=1}^{\infty}$) converging uniformly to $u(x)$ and $v(x)$. Moreover, we have

$$u(x) \leq \overline{u}(r), \quad v(x) \leq \overline{v}(r), \quad \forall x \in \mathbb{R}^N.$$

Therefore, (u, v) is a solution of

$$\begin{cases} -\Delta_p u = a_1(x) F_1(x, u, v) & \text{in } \mathbb{R}^N, \\ -\Delta_q v = a_2(x) F_2(x, u, v) & \text{in } \mathbb{R}^N, \\ u(x), v(x) > 0 & \text{in } \mathbb{R}^N, \\ \lim_{|x|\to\infty} u(x) = \lim_{|x|\to\infty} v(x) = 0. \end{cases} \tag{4.2}$$

This finishes the proof of Theorem 1.2.

5. Proof of Theorem 1.3. Arguing by contradiction, we suppose that (u, v) is a radial positive bounded solution for problem (4.2). Then (u, v) satisfies

$$\begin{cases} -(r^{N-1}\Phi(u))' = r^{N-1} a_1(r) F_1(r, u(r), v(r)), & 0 \leq r < \infty, \\ -(r^{N-1}\Psi(v))' = r^{N-1} a_2(r) F_2(r, u(r), v(r)), & 0 \leq r < \infty, \\ u(r), v(r) > 0, \quad 0 \leq r < \infty, \\ \lim_{r\to\infty} u(r) = \lim_{r\to\infty} v(r) = 0, \end{cases} \tag{5.1}$$

where $\Phi(u) = |u'|^{p-2}u'$ and $\Psi(v) = |v'|^{q-2}v'$.

It is easy to see that $u'(r) < 0$ and $v'(r) < 0$, which implies that $u(r)$ and $v(r)$ are decreasing. Summing in (5.1), we obtain

$$r(\Phi'(u(r)) + \Psi'(v(r))) + (N-1)[\Phi(u(r)) + \Psi(v(r))] = -r \sum_{i=1}^{2} a_i(r) F_i(r, u(r), v(r)) \leq -rB(r).$$

Let

$$\varphi(r) = \int_0^r [\Phi(u(t)) + \Psi(v(t))]\, dt.$$

Then

$$r\varphi'(r) - r_0\varphi'(r_0) = \int_{r_0}^{r} (t\varphi'(t))'\,dt = \int_{r_0}^{r} [t\varphi''(t) + \varphi'(t)]\,dt$$
$$\le -\int_{r_0}^{r} tB(t)\,dt + (2-N)\int_{r_0}^{r} \varphi'(t)\,dt \to -\infty$$

as $r \to \infty$. This implies that there exists a constant $C > 0$ satisfying

$$-r\varphi'(r) > C \quad \text{for } r > r_0 > 0,$$

that is,

$$-\varphi'(r) > Cr^{-1} \quad \text{for } r > r_0 > 0.$$

Thus, we have

$$\varphi(r_0) - \varphi(r) = -\int_{r_0}^{r} \varphi'(t)\,dt > \int_{r_0}^{r} Ct^{-1}\,dt = C\ln r - C\ln r_0 \to \infty$$

as $r \to \infty$, which is a contradiction. Hence, the proof of Theorem 1.3 is completed.

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