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Foliations making a constant angle with principal directions on ellipsoids

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Abstract. We study the global behavior of foliations of ellipsoids by curves making a constant angle with the lines of curvature.

1. Introduction. Research on differential geometry of quadrics has a pretty long history but is still of some interest. Let us point out the following issues:

(i) already in the late eighteenth century, Gaspar Monge [Mo1, Mo2] described the global behavior of lines of principal curvature on the ellipsoid (probably, this was also the origin of the theory of singular foliations on surfaces);
(ii) a bit later, in the nineteenth century, Jacobi [Ja] studied the geodesic flow on the ellipsoid (see [Kle] and [GS3]);
(iii) quite recently, Marcel Berger [Be] has shown interest in caustics on quadrics.

A major part of this work belongs to conformal geometry.

In [LW1], while studying the conformal geometry of foliations, we made the following observation (p. 147): Given a point p of a surface $S \subset \mathbb{R}^3$ or S^3, the osculating circles at p to the leaves of the foliations $\mathcal{F}_\alpha$ by curves making constant angle α with the lines of curvature form a pencil; in other words, two such circles, $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$, intersect at two points, p and $\tilde{p}$, $\tilde{p}$ being independent of the choice of α and β. A bit later, this observation was used in [LW2], where the following geometric characterization of holomorphic maps of the complex plane was provided: *A* C^2*-diffeomorphism* f *of an open subset* U *of* $\mathbb{C}$ *onto a subset* V *of* $\mathbb{C}$ *is holomorphic if and only if it is orientation preserving and for each* z *in* U *the family of circles osculating at* $f(z)$ *to the*

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images of the circles C_θ of a pencil $\mathcal{B}(z,*)$ with one of its base points at z is also a pencil.

The foliations $\mathcal{F}_\alpha$ belong to the conformal geometry of surfaces: principal directions are preserved under an arbitrary conformal change of a Riemannian metric, therefore are invariant under conformal transformations of either $\mathbb{R}^3$ or S^3, and obviously the notion of angle belongs to conformal geometry.

The above motivates our interest in the foliations $\mathcal{F}_\alpha$ on surfaces of different conformal types. For example, $\mathcal{F}_\alpha$'s are rather simple on *Dupin cyclides*, surfaces which are very special: they are in two different ways envelopes of one-parameter families of spheres (see [Da]).

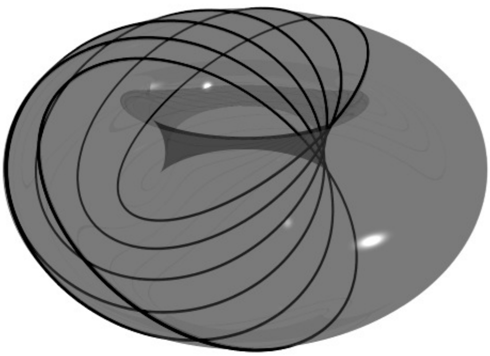

Fig. 1. Foliation of a torus of revolution by Villarceau circles

Dupin cyclides arise as conformal images of tori, cylinders and cones of revolution, therefore there are three types of them and one can choose a nice representative of each class:

(A) the boundary of a tubular neighborhood of a geodesic of $\mathbb{S}^3$; using the complex plane $\mathbb{C}^2$, it admits the equations

$$\mathbb{T}_{a,b} = \{|z_1|^2 = a^2,\, |z_2|^2 = b^2 : z_1, z_2 \in \mathbb{C},\, 0 < a \le b,\, a^2 + b^2 = 1\}$$

(the condition $a^2 + b^2 = 1$ guarantees that it is contained in the unit sphere of $\mathbb{C}^2$);

(B) a cylinder of revolution in $\mathbb{R}^3$;

(C) a cone of revolution in $\mathbb{R}^3$.

Then, in cases (A) and (B) the foliations $\mathcal{F}_\alpha$ are totally geodesic in the Dupin cyclide. In case (A), four of them consist of circles: two foliations by characteristic circles, $\mathcal{F}_0 = \mathcal{F}_0$ and $\mathcal{F}_{\pi/2}$, and two others by Villarceau circles (Figure 1). Recall that the angle α_v of the Villarceau circles with the principal foliations on $\mathbb{T}_{a,b}$ depends on the quotient a/b; it is $\pi/4$ only for the Clifford torus $\mathbb{T}_{\sqrt{2},\sqrt{2}}$.

In case (C), one can develop the cone on a plane. This procedure provides a local isometry outside the apex.

In the plane, one can see that a foliation by curves making a constant angle with rays is a foliation by logarithmic spirals. The picture on the cone can be obtained by rolling the planar foliation back onto the cone.

Dupin cyclides also enjoy the following property: for each point p of a cyclide $\mathcal{D}$, there exist two circles through p, the Villarceau ones, which form the intersection of $\mathcal{D}$ with a sphere Σ. All the quadrics enjoy this property too: any quadric $\mathcal{Q}$ (with umbilics, if any, deleted) can be covered by two families of circles (or lines) such that any two circles, one in each family, lie on the sphere. This is why, in this article, we study in detail the foliations by curves making a constant angle with lines of curvature on ellipsoids.

While considering a surface M with the principal foliations $\mathcal{F}_0$ and $\mathcal{F}_{\pi/2}$ and the umbilical set $\mathcal{U}$, the triple $\mathcal{P} = (\mathcal{F}_0, \mathcal{F}_{\pi/2}, \mathcal{U})$ will be referred to as the *principal configuration* of the surface. Again, for each $\alpha \in (-\pi/2, \pi/2)$ we will denote by $\mathcal{F}_\alpha$ the foliation of $M \setminus U$ whose leaves are curves making constant angle α with the leaves of the principal foliation $\mathcal{F}_0$. Notice that the normal curvature of a leaf of $\mathcal{F}_\alpha$ is precisely

$$k_n(\alpha) = k_1 \cos^2 \alpha + k_2 \sin^2 \alpha$$

and the local behavior of $\mathcal{F}_\alpha$'s is fairly easy to describe: any surface admits a principal chart around any of its non-umbilical points and in that chart the leaves of $\mathcal{F}_\alpha$ intersect the coordinate lines at angle α. Therefore, here we shall focus on the global behavior of $\mathcal{F}_\alpha$'s on ellipsoids. (On hyperboloids and paraboloids, the dynamical study of the foliations $\mathcal{F}_\alpha$ is simpler.)

Our main results are contained in Theorem 3 and Corollary 4, where: (1) we describe the equations of the foliations $\mathcal{F}_\alpha$ in the conformal principal charts, (2) we determine the form of the Poincaré return map on a particular transverse section and (3) we derive a condition on α providing minimality (that is, density of the leaves) of $\mathcal{F}_\alpha$.

This article is closely related to [GLW], where the authors study the dynamics of *Darboux curves*, which are curves Γ on surfaces M in either $\mathbb{R}^3$ or S^3 such that their osculating spheres are tangent to M. In particular, the position of Darboux curves with respect to our foliations $\mathcal{F}_\alpha$ is considered therein.

2. Results. Consider the ellipsoid

$$\mathbb{Q}_{a,b,c} = \{(x, y, z) : x^2/a + y^2/b + z^2/c = 1\}$$

with $a > b > c > 0$. It belongs to the Dupin triple orthogonal system of surfaces defined by

$$q = q_\lambda(x, y, z) = \frac{x^2}{a - \lambda} + \frac{y^2}{b - \lambda} + \frac{z^2}{c - \lambda} - 1$$

(see also [Sp, Vol. 3, Chapter 4] and [St, pp. 99–103]. Recall that the intersection of two orthogonal surfaces of this family is a line of principal curvature for the two surfaces.

Given $p_0 = (x_0, y_0, z_0) \in \mathbb{Q}_{a,b,c}$ with $x_0 y_0 z_0 \neq 0$, consider the equation in λ, $q_\lambda(x_0, y_0, z_0) = 0$.

It can be shown that the three roots of this equation of degree 3 are real; one is zero. One can also check that the other two roots, u and v, satisfy $u \in (b, a)$ and $v \in (c, b)$. Following the roots u, v when the point p_0 moves around the quadric $\mathbb{Q}_{a,b,c}$ gives the foliations $\mathcal{F}_0$ and $\mathcal{F}_{\pi/2}$, therefore provides a chart of the quadric with coordinate curves, the lines of principal curvature near p_0.

The three quadrics satisfy $q_u(p_0) = q_v(p_0) = q_0(p_0) = 0$. Let us show that any two of the three quadrics meet orthogonally along their curve of intersection. We should check for example that $\langle \nabla q_0(p_0), \nabla q_u(p_0)\rangle = 0$.

We have

$$\nabla q_0(p_0) = \left(\frac{2x}{a}, \frac{2y}{b}, \frac{2z}{c}\right) \quad \text{and} \quad \nabla q_u(p_0) = \left(\frac{2x}{a-u}, \frac{2y}{b-u}, \frac{2z}{c-u}\right).$$

Therefore,

$$\langle \nabla q_0(p_0), \nabla q_u(p_0)\rangle = \frac{4x^2}{a(a-u)} + \frac{4y^2}{b(b-u)} + \frac{4z^2}{c(c-u)}.$$

One also has

$$\begin{aligned} 4q_u - 4q_0 &= \frac{4x^2}{a-u} - \frac{4x^2}{a} + \frac{4y^2}{b-u} - \frac{4y^2}{b} + \frac{4z^2}{c-u} - \frac{4z^2}{c} \\ &= u\langle \nabla q_0(p_0), \nabla q_u(p_0)\rangle. \end{aligned}$$

This expression is equal to zero at p_0. The proofs of the other two orthogonality relations are similar.

Solving the linear system $q_0(p_0) = q_u(p_0) = q_v(p_0) = 0$ in the variables x_0^2, y_0^2 and z_0^2 and taking square roots we obtain a parametrization $\beta : [b, a] \times [c, b] \to \mathbb{Q}_{a,b,c}$ given by

(1)
$$\beta(u, v) = \left(\pm\sqrt{\frac{a(u-a)(v-a)}{(b-a)(c-a)}}, \pm\sqrt{\frac{b(u-b)(v-b)}{(b-a)(b-c)}}, \pm\sqrt{\frac{c(u-c)(v-c)}{(c-a)(c-b)}}\right).$$

The first fundamental form of β is given by

(2)
$$I = ds^2 = E du^2 + G dv^2 = \frac{(v-u)u}{4H(u)} du^2 + \frac{(u-v)v}{4H(v)} dv^2.$$

The second fundamental form of β with respect to the normal $N = -(\beta_u \wedge \beta_v)/\|\beta_u \wedge \beta_v\|$ is given by

(3)
$$II = e du^2 + g dv^2 = \frac{v-u}{4H(u)}\sqrt{\frac{abc}{uv}} du^2 + \frac{u-v}{4H(v)}\sqrt{\frac{abc}{uv}} dv^2$$

where $H(t) = (t-a)(t-b)(t-c)$.

Therefore the principal curvatures $k_1 \leq k_2$ are

(4) $$k_1 = \frac{e}{E} = \frac{1}{u}\sqrt{\frac{abc}{uv}}, \qquad k_2 = \frac{g}{G} = \frac{1}{v}\sqrt{\frac{abc}{uv}}.$$

The four umbilical points U_i $(i = 1, \ldots, 4)$ are given by

(5) $$(\pm x_0, 0, \pm z_0) = \left(\pm\sqrt{\frac{a(a-b)}{a-c}}, 0, \pm\sqrt{\frac{c(c-b)}{c-a}}\right).$$

Proposition 1. *There exists a conformal parametrization* $\varphi : (-s_1, s_1)\times(-s_2, s_2) \to \mathbb{Q}_{a,b,c} \cap \{(x,y,z) : y \geq 0\}$ *such that the lines of principal curvature are the coordinate curves, and the leaves of* $\mathcal{F}_{\pi/4}$ *and* $\mathcal{F}_{-\pi/4}$ *are defined by* $dV^2 - dU^2 = 0$, *i.e., are images of straight lines making angle* $\pm\pi/4$ *with the coordinate lines; here*

$$s_1 = \int_b^a \sqrt{\frac{-u}{H(u)}}\,du < \infty \quad \text{and} \quad s_2 = \int_c^b \sqrt{\frac{v}{H(v)}}\,dv < \infty,$$

where, as before, $H(t) = (t-a)(t-b)(t-c)$.

Moreover, $\varphi(s_1, s_2) = U_1$, $\varphi(-s_1, s_2) = U_2$, $\varphi(-s_1, -s_2) = U_3$ *and* $\varphi(s_1, -s_2) = U_4$. *By symmetry the same result holds for the region* $\mathbb{Q}_{a,b,c} \cap \{(x,y,z) : y \leq 0\}$ *(see Figure 2).*

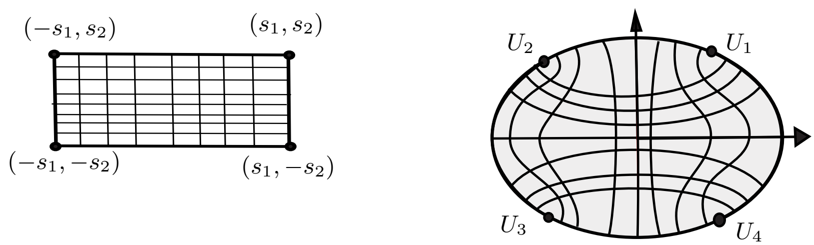

Fig. 2. Lines of principal curvature of the ellipsoid $\mathbb{Q}_{a,b,c}$ represented in a conformal chart

Proof. The differential equation of the foliations $\mathcal{F}_\alpha$ and $\mathcal{F}_{-\alpha}$ is given by

$$k_n(\beta(u,v),\alpha)) = \frac{edu^2 + gdv^2}{Edu^2 + Gdv^2} = k_1\cos^2\alpha + k_2\sin^2\alpha.$$

In the principal chart (u,v) the above equation reads

$$H(u)v\cos^2\alpha\,dv^2 + H(v)u\sin^2\alpha\,du^2 = 0,$$

equivalently,

$$\frac{v}{H(v)}\cos^2\alpha\,dv^2 + \sin^2\alpha\,\frac{u}{H(u)}du^2 = 0.$$

Set $d\tau_1 = \sqrt{-u/H(u)}\,du$, $d\tau_2 = \sqrt{v/H(v)}\,dv$, $s_1 = \int_b^a d\tau_1$ and $s_2 = \int_c^b d\tau_2$. Then the differential equation of $\mathcal{F}_{\pi/4}$ and $\mathcal{F}_{-\pi/4}$ becomes equivalent to $d\tau_2^2 - d\tau_1^2 = 0$ in the rectangle $[0, s_1] \times [0, s_2]$.

Defining the change of coordinates $\phi(u,v) = (U,V)$, $U = s_1 - \int_b^u d\tau_1$ and $V = s_2 - \int_c^v d\tau_2$ it follows that $\varphi(U,V) = (\beta \circ \phi^{-1})(U,V)$ is a conformal parametrization of the first octant of the ellipsoid having the coordinate curves as lines of principal curvature, and the leaves of $\mathcal{F}_{\pi/4}$ and $\mathcal{F}_{-\pi/4}$ correspond to the straight lines in the plane (U,V) making angle $\pm\pi/4$ with the lines of principal curvature. By construction it follows that $\varphi(s_1, s_2) = U_1 = (x_0, 0, z_0)$ is an umbilical point contained in the first octant of the ellipsoid.

Now, using the symmetry of the ellipsoid with respect to the plane coordinates we consider the rectangle $R = (-s_1, s_1) \times (-s_2, s_2)$ and an analytic continuation of φ to obtain a conformal chart (U,V) which maps this rectangle onto the region $\mathbb{Q}_{a,b,c} \cap \{y \geq 0\}$.

By construction, $\varphi(\partial R) = \Sigma$ and the four vertices of the rectangle R are mapped by φ to the four umbilical points U_i given by (5).

By symmetry of the ellipsoid the same result can be obtained in the region $\mathbb{Q}_{a,b,c} \cap \{y \leq 0\}$. ■

Remark 2. On the ellipse $\Sigma = \{(x,y,z) : x^2/a + z^2/c = 1,\ y = 0\}$ the quantities $2s_1 = 2\int_b^a \sqrt{-u/H(u)}\, du < \infty$ and $2s_2 = 2\int_c^b \sqrt{v/H(v)}\, dv < \infty$ can be interpreted as the distances between the umbilical points $U_1 = (x_0, 0, z_0)$ and $U_2 = (-x_0, 0, z_0)$, and $U_1 = (x_0, 0, z_0)$ and $U_4 = (x_0, 0, -z_0)$, respectively.

Theorem 3. *In the conformal chart (U,V), $(U,V) \in (-s_1, s_1) \times (-s_2, s_2)$ given in Proposition 1, the foliation $\mathcal{F}_\alpha$, $\alpha \in (-\pi/2, \pi/2)$, can be defined by the linear differential equation $\cos\alpha\, dV - \sin\alpha\, dU = 0$. Moreover, the ellipse $\Sigma \setminus \{U_1, U_2, U_3, U_4\}$ is a transverse section of $\mathcal{F}_\alpha$ for any $\alpha \in (-\pi/2, \pi/2)$, and the Poincaré return map $\Pi_\alpha : \Sigma \to \Sigma$ coincides with the composition of the involutions Π_α^+ and Π_α^-.*

Proof. Consider the conformal principal coordinates (U,V) with $-s_1 \leq U \leq s_1$ and $-s_2 \leq V \leq s_2$ as in Proposition 1.

The leaves of $\mathcal{F}_\alpha$ and $\mathcal{F}_{-\alpha}$ are given in this chart by the differential equation

$$\cos^2\alpha\, dU^2 - \sin^2\alpha\, dV^2 = 0,$$
$$(\cos\alpha\, dU - \sin\alpha\, dV)(\cos\alpha\, dU + \sin\alpha\, dV) = 0.$$

Also consider a conformal principal chart $(\bar{U}, \bar{V})$ and the parametrization

$$\varphi_1 : (-s_1, s_1) \times (-s_2, s_2) \to \mathbb{Q}_{a,b,c} \cap \{y \leq 0\}$$

having the same properties as φ.

The ellipse Σ is the union of four umbilical points U_i and four principal umbilical separatrices of the principal foliations $\mathcal{F}_0$ and $\mathcal{F}_{\pi/2}$. So $\Sigma \setminus \{U_1, U_2, U_3, U_4\}$ is a transverse section of the foliations $\mathcal{F}_\alpha$ for any $\alpha \in (-\pi/2, \pi/2)$.

Therefore, near the umbilical point $U_1 = (x_0, 0, z_0) = \varphi(s_1, s_2)$ contained in the positive octant, the foliation $\mathcal{F}_\alpha$, with umbilical separatrix of U_1 (the leaf of $\mathcal{F}_\alpha$ whose limit set contains the umbilical point U_1) contained in the region $\{y > 0\}$ defines a return map $\Pi_\alpha^+ : \Sigma \to \Sigma$ reverting the orientation, with $\Pi_\alpha^+(U_2) = U_2$ and $\Pi_\alpha^+(U_4) = U_4$. This follows because in the conformal principal chart (U, V) this return map is defined by the differential equation $\frac{dV}{dU} = \frac{\sin\alpha}{\cos\alpha}$.

By analytic continuation, it follows that Π_α^+ is an involution, i.e. $\Pi_\alpha^+ \circ \Pi_\alpha^+ = \mathrm{id}$, with two fixed points $\{U_2,\ U_4\}$.

Also, one can define a return map Π_α^-, another involution in the region $y < 0$. It has two umbilics, U_1 and U_3, as fixed points, and in the conformal principal chart $(\bar U, \bar V)$ it is defined by the differential equation $\frac{d\bar V}{d\bar U} = -\frac{\sin\alpha}{\cos\alpha}$.

Therefore, the Poincaré return map $\Pi_\alpha : \Sigma \to \Sigma$ is the composition of the involutions Π_α^+ and Π_α^- (see Figure 3). ∎

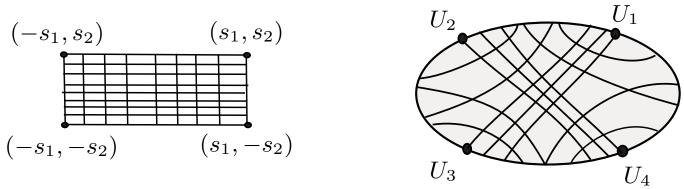

Fig. 3. The foliations $\mathcal{F}_\alpha$ and $\mathcal{F}_{-\alpha}$ of the ellipsoid $\mathbb{Q}_{a,b,c}$ represented in a conformal chart

Corollary 4. *Consider the ellipsoid $\mathbb{Q}_{a,b,c}$, $0 < c < b < a$, and let $\rho(\alpha) = \frac{s_1 \sin\alpha}{s_2 \cos\alpha}$. If $\rho(\alpha) \in \mathbb{R} \setminus \mathbb{Q}$ (resp. $\rho(\alpha) \in \mathbb{Q}$) then all the leaves of $\mathcal{F}_\alpha$ and $\mathcal{F}_{-\alpha}$ are dense (resp. all but the umbilical separatrices are closed). See Figure 4.*

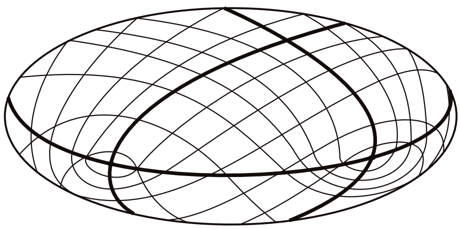

Fig. 4. The foliations $\mathcal{F}_\alpha$ and $\mathcal{F}_{-\alpha}$ of the ellipsoid $\mathbb{Q}_{a,b,c}$

Proof. The result follows from rotation number theory for circle diffeomorphisms (see [MP]).

To compute the rotation number of Π_α consider two lattices of $\mathbb{R}^2$ having the rectangle $R = [-s_1, s_1] \times [-s_2, s_2]$ and the square $Q = [-1, 1] \times [-1, 1]$

as fundamental domains. Now, observe that the linear equation $dV/dU = \sin\alpha/\cos\alpha$ defined in R is equivalent to $\frac{dy}{dx} = \frac{s_1 \sin\alpha}{s_2 \cos\alpha}$ in Q.

Considering the ellipsoid parametrized by the two conformal charts (U, V) and $(\bar{U}, \bar{V})$, we see that the Poincaré map Π_α is conjugate to the rotation of the circle with rotation number $\rho(\alpha) = \frac{s_1 \sin\alpha}{s_2 \cos\alpha}$, which is the return map of the induced flow on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ defined by the linear differential equation $\frac{dy}{dx} = \frac{s_1 \sin\alpha}{s_2 \cos\alpha}$. ■

3. Final remarks. 1. The special case $\alpha = \pi/4$ was studied previously in [GS1]. A more general framework of implicit differential equations, unifying various families of geometric curves, was studied in [GS2] (see also [GS3]).

2. It follows from Corollary 4 that all the foliations $\mathcal{F}_\alpha^\pm$ have four singularities of topological index 1/2, and each has a unique separatrix and one hyperbolic sector as in the case of the principal foliations $\mathcal{F}_0$ and $\mathcal{F}_{\pi/2}$.

3. The Dupin cyclides mentioned in the Introduction provide examples of *canal surfaces*, obtained as envelopes of one-parameter families of spheres. The foliations $\mathcal{F}_\alpha$ on such surfaces fairly are easy to describe. For instance, one of the principal foliations, say $\mathcal{F}_0$, consists of circles, intersections of the spheres defining the canal with the corresponding members of the derived family. All the $\mathcal{F}_\alpha$'s can be easily pictured on *special canal surfaces* which appear as conformal images of surfaces of revolution, cones and cylinders over planar (or rather spherical) curves, and were studied in [BLW]. Note that canal surfaces are of some interest in computer graphics (see, for example, [D&al] and the bibliography therein).

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References

[BLW] A. Bartoszek, R. Langevin and P. G. Walczak, *Special canal surfaces of* $\mathbb{S}^3$, Bull. Braz. Math. Soc. (N.S.) 42 (2011), 301–320.

[Be] M. Berger, *Seules les quadriques admettent des caustiques*, Bull. Soc. Math. France 123 (1995), 107–116.

[Da] G. Darboux, *Leçons sur la théorie générale des surfaces*, Gauthier-Villars, Paris, 1887.

[D&al] L. Druoton, L. Garnier, R. Langevin, H. Marcellier et R. Besnard, *Les cyclides de Dupin et l'espace des sphères*, Revue Electr. Franc. Informatique Graphique 5 (2011), 41–59

[GLW] R. Garcia, R. Langevin and P. G. Walczak, *Darboux curves on surfaces*, arXiv: 0912.3749 (2009).

[GS1] R. Garcia and J. Sotomayor, *Structurally stable configurations of lines of mean curvature and umbilic points on surfaces immersed in* $\mathbb{R}^3$, Publ. Mat. 45 (2001), 431–466.

[GS2] R. Garcia and J. Sotomayor, *Lines of mean curvature on surfaces immersed in* $\mathbb{R}^3$, Qualit. Theory Dynam. Systems 5 (2004), 137–183.

[GS3] R. Garcia and J. Sotomayor, *Differential Equations of Classical Geometry, a Qualitative Theory*, Publicações Matemáticas, 27° Colóquio Brasileiro de Matemática, IMPA, 2009.

[Ja] C. G. J. Jacobi, *Note von der geodätischen Linie auf einem Ellipsoid und den verschiedenen Anwendungen einer merkwürdigen analytischen Substitution*, J. Reine Angew. Math. 19 (1839), 309–313.

[Kle] F. Klein, *Lectures on Mathematics*, Macmillan, 1894; reprinted by AMS Chelsea, Providence, RI, 2000 and 2011.

[LW1] R. Langevin and P. G. Walczak, *Conformal geometry of foliations*, Geom. Dedicata 132 (2008), 135–178.

[LW2] R. Langevin and P. G. Walczak, *Holomorphic maps and pencils of circles*, Amer. Math. Monthly 116 (2008), 690–700.

[MP] W. Melo and J. Palis, *Geometric Theory of Dynamical Systems*, Springer, New York, 1982.

[Mo1] G. Monge, *Sur les lignes de courbure de la surface de l'ellipsoïde*, Journal de l'École Polytechnique II^ème^ cahier, cours de Floréal an III (around 1795), 145.

[Mo2] G. Monge, *Application de l'analyse à la géométrie, Première Partie*, Hachette, 1807, 1–57.

[Sp] M. Spivak, *A Comprehensive Introduction to Differential Geometry*, Vol. III, Publish or Perish, Berkeley, 1979.

[St] D. Struik, *Lectures on Classical Differential Geometry*, Addison-Wesley, 1950; reprinted by Dover Collections, 1988.

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