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Global exponential stability of almost periodic solutions for a delayed single population model with hereditary effect

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Abstract. This paper is concerned with a delayed single population model with hereditary effect. Under appropriate conditions, we employ a novel argument to establish a criterion of the global exponential stability of positive almost periodic solutions of the model. Moreover, an example and its numerical simulation are given to illustrate the main result.

1. Introduction. As pointed out in [F, H], ecological effects and environmental variability in nature are crucial factors in the study of biomathematical model dynamics. In particular, periodically or almost periodically varying environment is basic in the theory of natural selection. In fact, almost periodically varying environment is more common, and biodynamics under almost periodic conditions has attracted a great deal of attention (see, for example, [Y, K, AS, WL, XL, MC, L2, C, WWC]). Furthermore, time-varying delay is another important factor in the modeling of biological systems which should not be ignored. As Li and Kuang have remarked in [LK], "Naturally, more realistic and interesting models of single or multiple species growth should take into account both the seasonality of the changing environment and the effects of time delays."

Accordingly, in the study of single population dynamics, the following almost periodic model with time-varying delay and hereditary effect was built:

$$x'(t) = x(t)[a(t) - b(t)x(t) - c(t)x(t - \tau(t))], \tag{1.1}$$

where $a, b, c, \tau : \mathbb{R} \to (0, \infty)$ are almost periodic functions, $x(t)$ represents the density of the species at time t, $a(t)$ is the growth rate, $b(t)$ is the self-inhibition rate, $c(t)$ is the hereditary rate, and $\tau(t)$ is the time required to reproduce a generation. Here, the assumption of almost periodicity of the coefficient functions and the delays in (1.1) is a way of incorporating the time-

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variability of the environment, especially when the various components of the environment are periodic with not necessarily commensurate periods (e.g. seasonal effects of weather, food supplies, mating habits, and harvesting).

Model (1.1) with periodic parameters has been extensively studied in [CL1, CL2, FW, C, CXC, L1]. But for the case of almost periodic parameters, only Xie and Li [XL] claimed to obtain the existence of almost periodic solutions by using coincidence degree theory. Unfortunately, Wang and Zhang [WZ] and Ortega [O] found that this theory is not suitable to deduce the existence of almost periodic solutions. The main issue is that we need the compactness of a set of almost periodic functions, which is difficult to get. For example, the mapping N in [XL, Lemma 3.3] is not guaranteed to be L-compact, and hence one cannot deduce the existence of almost periodic solutions for (1.1).

Motivated by the above discussion, in this paper, we develop a new approach to obtain a condition for the global exponential stability of positive almost periodic solutions of (1.1). Moreover, we estimate the exponential convergence rate.

For simplicity of notation, for a bounded continuous function g defined on $\mathbb{R}$, we set

$$g^+ = \sup_{t\in\mathbb{R}} g(t) \quad \text{and} \quad g^- = \inf_{t\in\mathbb{R}} g(t).$$

Throughout this paper, we make the following assumptions for (1.1):

$$\text{(1.2)} \qquad \begin{cases} a^-, b^-, \tau^+ > 0, \quad c(t) > b(t) \quad \text{for all } t \in \mathbb{R}, \\ \Big(\int\limits_{t-\tau(t)}^{t} a(s)\, ds\Big)^+ < \infty, \quad \Big(\int\limits_{t-\tau(t)}^{t} [a(s) - 2Mc(s)]\, ds\Big)^- > -\infty \end{cases}$$

and

$$\text{(1.3)} \qquad \begin{cases} M = \left(\dfrac{a(t)}{c(t)}\right)^+ \exp\Big(\int\limits_{t-\tau(t)}^{t} a(s)\, ds\Big)^+ > \kappa, \\ \kappa = \left(\dfrac{a(t)}{2c(t)}\right)^- \exp\Big(\int\limits_{t-\tau(t)}^{t} [a(s) - 2Mc(s)]\, ds\Big)^- > 0. \end{cases}$$

Then

$$\text{(1.4)} \qquad a(t) - 2Mc(t) < a(t) - 2\frac{a(t)}{c(t)}c(t) = -a(t) < 0 \quad \text{for all } t \in \mathbb{R}.$$

Let $\mathbb{R}_+ = [0,\infty)$, $C = C([-\tau^+, 0], \mathbb{R})$ be the Banach space of all continuous functions on $[-\tau^+, 0])$ equipped with the usual supremum norm $\|\cdot\|$, and $C_+ = C([-\tau^+, 0], \mathbb{R}_+)$. If $x(t)$ is defined on $[t_0 - \tau^+, \sigma)$ with $t_0 < \sigma$, then we define $x_t \in C$ by $x_t(\theta) = x(t+\theta)$ for all $\theta \in [-\tau^+, 0]$.

The initial conditions associated with (1.1) are

$$x_{t_0} = \varphi, \quad \varphi \in C_+,\ \varphi(0) > 0. \tag{1.5}$$

We denote an admissible solution of (1.1) and (1.5) on the maximal right-interval of the existence by $x(t; t_0, \varphi)$. For such a solution, $x_t(t_0, \varphi) \in C$ is defined by $(x_t(t_0, \varphi))(\theta) = x(t + \theta; t_0, \varphi)$ for $\theta \in [-\tau^+, 0]$.

2. Preliminaries and lemmas. Before giving our main results we first present some definitions and lemmas.

DEFINITION 2.1 (see [F, H]). A continuous function $u : \mathbb{R} \to \mathbb{R}$ is said to be *almost periodic* if, for any $\varepsilon > 0$, the set $T(u, \varepsilon) = \{\delta : |u(t+\delta) - u(t)| < \varepsilon \text{ for all } t \in \mathbb{R}\}$ is relatively dense, i.e., for any $\varepsilon > 0$, there exists $l = l(\varepsilon) > 0$ such that every interval $[t_0, t_0 + l(\varepsilon)]$ contains at least one $\delta = \delta(\varepsilon)$ for which $|u(t + \delta) - u(t)| < \varepsilon$ for all $t \in \mathbb{R}$.

From the theory of almost periodic functions in [F, H], we know that for any $\varepsilon > 0$ there is $l = l(\varepsilon) > 0$ such that any interval $[t_0, t_0 + l(\varepsilon)]$ contains a number $\delta = \delta(\varepsilon)$ for which

$$\begin{cases} |a(t + \delta) - a(t)| < \varepsilon, & |b(t + \delta) - b(t)| < \varepsilon, \\ |c(t + \delta) - c(t)| < \varepsilon, & |\tau(t + \delta) - \tau(t)| < \varepsilon, \end{cases} \tag{2.1}$$

for all $t \in \mathbb{R}$.

LEMMA 2.1. *Every solution $x(t; t_0, \varphi)$ of* (1.1) *and* (1.5) *is positive and bounded on $[t_0, \eta(\varphi))$, and $\eta(\varphi) = \infty$.*

Proof. Since $\varphi \in C_+$, using [S1, Theorem 5.2.1], we find that $x_t(t_0, \varphi)$ is in C_+ for all $t \in [t_0, \eta(\varphi))$. For brevity, let $x(t) = x(t; t_0, \varphi)$.

We first show that $x(t) > 0$ for all $t \in [t_0, \eta(\varphi))$. By way of contradiction, suppose there exists $t_1 \in (t_0, \eta(\varphi))$ such that $x(t_1) = 0$ and $x(t) > 0$ for all $t \in [t_0, t_1)$. It follows from (1.1) that

$$\frac{x'(t)}{x(t)} = a(t) - b(t)x(t) - c(t)x(t - \tau(t)). \tag{2.2}$$

For any $\varepsilon \in (0, t_1 - t_0)$, integrating (2.2) from t_0 to $t_1 - \varepsilon$ produces

$$x(t_1 - \varepsilon) = x(t_0) \exp \int_{t_0}^{t_1-\varepsilon} [a(s) - b(s)x(s) - c(s)x(s - \tau(s))]\, ds.$$

Then

$$\begin{aligned} x(t_1) &= \lim_{\varepsilon \to 0^+} x(t_1 - \varepsilon) \\ &= \varphi(0) \exp \int_{t_0}^{t_1} [a(s) - b(s)x(s) - c(s)x(s - \tau(s))]\, ds > 0, \end{aligned}$$

which contradicts the assumption that $x(t_1) = 0$. Therefore, $x(t)$ is positive on $[t_0, \eta(\varphi))$.

We next show that $x(t)$ is bounded on $[t_0, \eta(\varphi))$. For each $t \in [t_0, \eta(\varphi))$, define

$$M^*(t) = \max\Big\{\xi \le t : x(\xi) = \max_{t_0-\tau^+\le s\le t} x(s)\Big\}.$$

Suppose that $x(t)$ is unbounded on $[t_0, \eta(\varphi))$. Then obviously $M^*(t) \to \eta(\varphi)$ as $t \to \eta(\varphi)$ and

$$\lim_{t\to\eta(\varphi)} x(M^*(t)) = \infty. \tag{2.3}$$

It follows from $x(M^*(t)) = \max_{t_0-\tau^+\le s\le t} x(s)$ that $x'(M^*(t)) \ge 0$ if $M^*(t) \ge t_0$. Then, for t with $M^*(t) \ge t_0$, we have

$$\begin{aligned}0 &\le x'(M^*(t))\\ &= x(M^*(t))\big[a(M^*(t)) - b(M^*(t))x(M^*(t))\\ &\qquad\qquad - c(M^*(t))x(M^*(t) - \tau(M^*(t)))\big]\\ &\le x(M^*(t))[a(M^*(t)) - b(M^*(t))x(M^*(t))].\end{aligned}$$

With the help of (1.2) and (2.3), we get

$$0 \le \lim_{t\to\eta(\varphi)} x(M^*(t))[a(M^*(t)) - b(M^*(t))x(M^*(t))] = -\infty,$$

which is a contradiction. As a result, $x(t)$ is bounded on $[t_0, \eta(\varphi))$.

Finally, as $x(t)$ is bounded on $[t_0, \eta(\varphi))$, we easily see that $\eta(\varphi) = \infty$ by applying [HVL, Theorem 2.3.1]. ∎

LEMMA 2.2. *For every solution $x(t; t_0, \varphi)$ of* (1.1) *and* (1.5), *there exists $t_\varphi > t_0$ such that*

$$\kappa \le x(t; t_0, \varphi) \le M \quad \textit{for all } t \ge t_\varphi.$$

Proof. Again, let $x(t) = x(t; t_0, \varphi)$ and denote

$$L = \limsup_{t\to\infty} x(t) \quad \text{and} \quad l = \liminf_{t\to\infty} x(t).$$

The proof is divided into three steps.

STEP 1. We show that there exists $T_1 > t_0 + \tau^+$ such that

$$x(t) \le M \quad \text{for all } t \ge T_1. \tag{2.4}$$

This is achieved by distinguishing two cases.

CASE 1: *$x'(t) < 0$ for all $t \ge t_0 + \tau^+$.* In this case, x is eventually decreasing and hence

$$L = \limsup_{t\to\infty} x(t) = \lim_{t\to\infty} x(t) = \lim_{t\to\infty} x(t - \tau(t)). \tag{2.5}$$

By the fluctuation lemma [S2, Lemma A.1], there exists a sequence $\{t_p\}$ such that

$$t_p \to \infty, \quad x(t_p) \to L, \quad x'(t_p) \to 0, \quad \text{as } p \to \infty.$$

Since $\{x_{t_p}\}$ is bounded and equicontinuous, by the Ascoli–Arzelà theorem, it has a convergent subsequence (not relabelled), say

$$x_{t_p} \to \overline{\varphi} \quad \text{as } p \to \infty \text{ for some } \bar{\varphi} \in C_+.$$

From (2.5), we get

$$\overline{\varphi}(0) = L = \overline{\varphi}(\theta) \quad \text{for all } \theta \in [-\tau^+, 0).$$

By almost periodicity, without loss of generality, we can also assume that the sequences $\{a(t_p)\}$, $\{b(t_p)\}$, $\{c(t_p)\}$, and $\{\tau(t_p)\}$ converge to $\overline{a}$, $\overline{b}$, $\overline{c}$, and $\overline{\tau}$, respectively. As $L = \overline{\varphi}(-\overline{\tau})$ and $-\overline{\tau} \in [-\tau^+, 0]$, it follows from

$$x'(t_p) = x(t_p)[a(t_p) - b(t_p)x(t_p) - c(t_p)x(t_p - \tau(t_p))]$$

that (taking limits)

$$0 = L[\overline{a} - \overline{b}L - \overline{c}\,\overline{\varphi}(-\overline{\tau})] = L(\overline{a} - \overline{b}L - \overline{c}L),$$

which yields

$$L \le \max\left\{0, \frac{\overline{a}}{\overline{b} + \overline{c}}\right\} \le \left(\frac{a(t)}{b(t) + c(t)}\right)^+ \le M,$$

and hence (2.4) holds.

Case 2: $x'(\rho) \ge 0$ *for some* $\rho \ge t_0 + \tau^+$. Then by (1.1) we have

$$\begin{aligned} 0 \le x'(\rho) &= x(\rho)[a(\rho) - b(\rho)x(\rho) - c(\rho)x(\rho - \tau(\rho))] \\ &\le x(\rho)[a(\rho) - c(\rho)x(\rho - \tau(\rho))], \end{aligned}$$

which implies that $x(\rho - \tau(\rho)) \le \frac{a(\rho)}{c(\rho)} \le \left(\frac{a(t)}{c(t)}\right)^+$. Moreover,

$$x'(t) = x(t)[a(t) - b(t)x(t) - c(t)x(t - \tau(t))] \le x(t)a(t).$$

Integrating the above inequality from $\rho - \tau(\rho)$ to ρ gives

$$\begin{aligned} x(\rho) &\le x(\rho - \tau(\rho)) \exp \int_{\rho-\tau(\rho)}^{\rho} a(s)\,ds \\ &\le \left(\frac{a(t)}{c(t)}\right)^+ \exp\left(\int_{t-\tau(t)}^{t} a(s)\,ds\right)^+ \le M. \end{aligned}$$

Similarly, one can show that $x(t) \le M$ if $x'(t) \ge 0$ and $t > \rho$. Now, suppose that $t > \rho$ and $x'(t) < 0$. Then we can choose $\bar{t} \in [\rho, t)$ such that $x'(\bar{t}) = 0$ and $x'(s) < 0$ for all $s \in (\bar{t}, t]$. It follows that $x(t) < x(\bar{t}) \le M$. Thus (2.4) holds with $T_1 = \rho$. This completes the discussion of Step 1.

STEP 2. We prove that $l > 0$. Otherwise, from Lemma 2.1, we have $l = 0$ and $x(t) > 0$ for all $t \geq t_0$. For each $t \geq t_0$, we define

$$m(t) = \max\Big\{\xi \leq t : x(\xi) = \min_{t_0 \leq s \leq t} x(s)\Big\}.$$

Noting that $m(t) \to \infty$ as $t \to \infty$, we obtain $\lim_{t\to\infty} x(m(t)) = 0$. On the other hand, $x'(m(t)) \leq 0$ for $m(t) \geq t_0$ as $x(m(t)) = \min_{t_0 \leq s \leq t} x(s)$. Then, for t with $m(t) \geq t_0$, we have

$$\begin{aligned}0 \geq x'(m(t)) &= x(m(t))\big[a(m(t)) - b(m(t))x(m(t)) \\ &\qquad - c(m(t))x(m(t) - \tau(m(t)))\big] \\ &\geq x(m(t))[a(m(t)) - 2c(m(t))x(m(t) - \tau(m(t)))],\end{aligned}$$

and hence $x(m(t) - \tau(m(t))) \geq \frac{a(m(t))}{2c(m(t))} \geq \big(\frac{a(t)}{2c(t)}\big)^- > 0$. This, combined with (1.1), (1.2), and (2.4), produces

$$\begin{aligned}x(m(t)) &= x(m(t) - \tau(m(t)))\exp\int_{m(t)-\tau(m(t))}^{m(t)} \big[a(s) - b(s)x(s) \\ &\qquad - c(s)x(s - \tau(s))\big]\, ds \\ &\geq \left(\frac{a(t)}{2c(t)}\right)^- \exp\Big(\int_{t-\tau(t)}^{t} [a(s) - 2Mc(s)]\, ds\Big)^-\end{aligned}$$

for $m(t) \geq T_1 + \tau^+$. Taking limits gives

$$\lim_{t\to\infty} x(m(t)) \geq \left(\frac{a(t)}{2c(t)}\right)^- \exp\Big(\int_{t-\tau(t)}^{t} [a(s) - 2Mc(s)]\, ds\Big)^- > 0,$$

contradicting $\lim_{t\to\infty} x(m(t)) = 0$. This proves that $l > 0$.

STEP 3. We show that there exists $T_2 > T_1 + \tau^+$ such that

$$x(t) \geq \kappa \quad \text{for all } t \geq T_2.$$

First suppose that $x'(t) > 0$ for all $t \geq T_1 + \tau^+$. Then $l > 0$. Again, by the fluctuation lemma [S2, Lemma A.1], there exists a sequence $\{t_q\}$ such that $t_q \to \infty$, $x(t_q) \to l$, and $x'(t_q) \to 0$, as $q \to \infty$. Similar arguments to those in Case 1 of Step 1 tell us that we can assume that $x_{t_q} \to l$, $a(t_q) \to a^*$, $b(t_q) \to b^*$, $c(t_q) \to c^*$, and $\tau(t_q) \to \tau^*$, as $q \to \infty$. Then taking limits in

$$x'(t_q) = x(t_q)[a(t_q) - b(t_q)x(t_q) - c(t_q)x(t_q - \tau(t_q))]$$

gives

$$0 = l[a^* - b^*l - c^*\varphi^*(-\tau^*)] = l(a^* - b^*l - c^*l).$$

This implies

$$l = \frac{a^*}{b^* + c^*} \geq \left(\frac{a(t)}{b(t) + c(t)}\right)^- > \kappa.$$

Now, suppose that there exists $\rho^* \geq T_1 + \tau^+$ such that $x'(\rho^*) \leq 0$. Using (1.2), we deduce

$$(2.6)\qquad \begin{aligned} 0 \geq x'(\rho^*) &= x(\rho^*)[a(\rho^*) - b(\rho^*)x(\rho^*) - c(\rho^*)x(\rho^* - \tau(\rho^*))] \\ &\geq x(\rho^*)[a(\rho^*) - c(\rho^*)(x(\rho^*) + x(\rho^* - \tau(\rho^*)))]. \end{aligned}$$

If $x(\rho^*) \geq x(\rho^* - \tau(\rho^*))$, then (2.6) leads to

$$x(\rho^*) \geq \frac{a(\rho^*)}{2c(\rho^*)} \geq \left(\frac{a(t)}{2c(t)}\right)^- \geq \kappa.$$

Otherwise,

$$x(\rho^* - \tau(\rho^*)) \geq \frac{a(\rho^*)}{2c(\rho^*)} \geq \left(\frac{a(t)}{2c(t)}\right)^-,$$

which together with (1.1) and (2.4) implies that

$$x'(t) = x(t)[a(t) - b(t)x(t) - c(t)x(t - \tau(t))] \geq x(t)[a(t) - 2c(t)M]$$

for all $t \geq T_1 + \tau^+$. This gives

$$\begin{aligned} x(\rho^*) &\geq x(\rho^* - \tau(\rho^*)) \exp \int\limits_{\rho^* - \tau(\rho^*)}^{\rho^*} [a(s) - 2Mc(s)]\, ds \\ &\geq \left(\frac{a(t)}{2c(t)}\right)^- \exp\Big(\int\limits_{t-\tau(t)}^{t} [a(s) - 2Mc(s)]\, ds\Big)^- \geq \kappa. \end{aligned}$$

In summary, we have proved that $x(\rho^*) \geq \kappa$. Now, for $t > \rho^*$, if $x'(t) \leq 0$ then the arguments above give $x(t) \geq \kappa$. Suppose that $x'(t) > 0$. Then we can choose $\rho^* \leq \widehat{t} < t$ such that $x'(\widehat{t}) = 0$ and $x'(s) > 0$ for all $s \in (\widehat{t}, t]$. It follows that $x(t) > x(\widehat{t}) \geq \kappa$. This finishes Step 3 and hence completes the proof. ■

LEMMA 2.3. *Let*

$$\sup_{t \in \mathbb{R}} \{-2\kappa[b(t) + c(t)] + c(t)M\tau(t)[2(b^+ + c^+)M + a^+] + a^+\} < 0.$$

Moreover, assume that $x(t) = x(t; t_0, \varphi)$ is a solution of equation (1.1) *with initial condition* (1.5), *and φ' is bounded and continuous on $[-\tau^+, 0]$. Then, for any $\varepsilon > 0$, there exists $l = l(\varepsilon) > 0$ such that every interval $[\alpha, \alpha + l]$ contains at least one number δ for which there exists $N > 0$ satisfying*

$$|x(t + \delta) - x(t)| \leq \varepsilon \quad \text{for all } t > N.$$

Proof. Define a continuous function Γ on $[0, 1]$ by

$$(2.1)\qquad \begin{aligned} \Gamma(\mu) = \sup_{t \in \mathbb{R}} \big\{ -[2\kappa(b(t) + c(t)) - \mu] + c(t)M\tau(t)\big[(2b^+ + c^+)Me^{\mu\tau^+} \\ + c^+Me^{2\mu\tau^+} + a^+e^{\mu\tau^+}\big] + a^+\big\} \end{aligned}$$

for all $\mu \in [0,1]$. Then $\Gamma(0) < 0$. It follows that there exist two constants $\eta > 0$ and $\lambda \in (0,1]$ such that

$$\Gamma(\lambda) < -\eta < 0. \tag{2.7}$$

We trivially extend $x(t)$ to $\mathbb{R}$ by letting $x(t) = x(t_0 - \tau^+)$ for t in $(-\infty, t_0 - \tau^+]$. Set

$$\begin{aligned}
\epsilon(\delta, t) = &-[b(t+\delta) - b(t)]x^2(t+\delta) \\
&- [c(t+\delta) - c(t)]x(t+\delta-\tau(t+\delta))x(t+\delta) \\
&- c(t)[x(t+\delta-\tau(t+\delta)) - x(t+\delta-\tau(t))]x(t+\delta) \\
&+ [a(t+\delta) - a(t)]x(t+\delta).
\end{aligned}$$

By Lemma 2.2, the solution $x(t)$ is bounded and $\kappa \le x(t) \le M$ for all $t \ge t_\varphi$, which implies that the right-hand side of (1.1) is also bounded, and $x'(t)$ is a bounded function on $[t_0 - \tau^+, \infty)$. Thus, in view of the fact that $x(t) \equiv x(t_0 - \tau^+)$ for $t \in (-\infty, t_0 - \tau^+]$, we deduce that $x(t)$ is uniformly continuous on $\mathbb{R}$. From (2.1), for any $\varepsilon > 0$, there exists $l = l(\varepsilon) > 0$ such that every interval $[\alpha, \alpha + l]$ contains a δ for which

$$|\epsilon(\delta, t)| \le \frac{1}{2}\frac{\eta\varepsilon}{1+\tau^+} \quad \text{for all } t \in \mathbb{R}. \tag{2.8}$$

Pick $N_0 \ge \max\{t_0, t_0 - \delta, t_\varphi + \tau^+, t_\varphi + \tau^+ - \delta\}$. For $t \in \mathbb{R}$, denote $u(t) = x(t+\delta) - x(t)$. Then, for all $t \ge N_0$, we get

$$\begin{aligned}
u'(t) = &-b(t)[x^2(t+\delta) - x^2(t)] - c(t)x(t+\delta)[x(t+\delta-\tau(t)) - x(t-\tau(t))] \\
&- c(t)x(t-\tau(t))[x(t+\delta) - x(t)] + a(t)[x(t+\delta) - x(t)] + \epsilon(\delta, t) \\
= &-b(t)[x(t+\delta) + x(t)]u(t) - c(t)x(t+\delta)u(t-\tau(t)) \\
&- c(t)x(t-\tau(t))u(t) + a(t)u(t) + \epsilon(\delta, t) \\
= &-[b(t)(x(t+\delta) + x(t)) + c(t)x(t+\delta) + c(t)x(t-\tau(t))]u(t) \\
&+ c(t)x(t+\delta)\int_{t-\tau(t)}^{t} u'(s)\,ds + a(t)u(t) + \epsilon(\delta, t) \\
= &-[b(t)(x(t+\delta) + x(t)) + c(t)x(t+\delta) + c(t)x(t-\tau(t))]u(t) \\
&+ c(t)x(t+\delta)\int_{t-\tau(t)}^{t} \big\{-b(s)[x(s+\delta) + x(s)]u(s) \\
&\qquad - c(s)x(s+\delta)u(s-\tau(s)) - c(s)x(s-\tau(s))u(s) \\
&\qquad + a(s)u(s) + \epsilon(\delta, s)\big\}\,ds + a(t)u(t) + \epsilon(\delta, t)
\end{aligned}$$

and

$$
\begin{aligned}
(2.9)\quad & D^-(e^{\lambda t}|u(t)|) \\
&\le \lambda e^{\lambda t}|u(t)| + e^{\lambda t}\Big\{-[b(t)(x(t+\delta)+x(t)) + c(t)x(t+\delta) \\
&\quad + c(t)x(t-\tau(t))]|u(t)| \\
&\quad + c(t)x(t+\delta)\int_{t-\tau(t)}^{t} \big|-b(s)[x(s+\delta)+x(s)]u(s) \\
&\quad - c(s)x(s+\delta)u(s-\tau(s)) - c(s)x(s-\tau(s))u(s) + a(s)u(s) + \epsilon(\delta,s)\big|\,ds \\
&\quad + a(t)|u(t)| + |\epsilon(\delta,t)|\Big\} \\
&\le -[(b(t)+c(t))2\kappa - \lambda]e^{\lambda t}|u(t)| + a^+e^{\lambda t}|u(t)| \\
&\quad + c(t)M\int_{t-\tau(t)}^{t} \big[(2b^+ + c^+)Me^{\lambda(t-s)}e^{\lambda s}|u(s)| \\
&\quad + c^+Me^{\lambda(t-s+\tau(s))}e^{\lambda(s-\tau(s))}|u(s-\tau(s))| \\
&\quad + a^+e^{\lambda(t-s)}e^{\lambda(s)}|u(s)| + e^{\lambda t}|\epsilon(\delta,s)|\big]\,ds + e^{\lambda t}|\epsilon(\delta,t)| \\
&\le -[(b(t)+c(t))2\kappa - \lambda]e^{\lambda t}|u(t)| \\
&\quad + c(t)M\int_{t-\tau(t)}^{t} \big[(2b^+ + c^+)Me^{\lambda(t-s)}e^{\lambda s}|u(s)| \\
&\quad + c^+Me^{\lambda(t-s+\tau(s))}e^{\lambda(s-\tau(s))}|u(s-\tau(s))| + a^+e^{\lambda(t-s)}e^{\lambda s}|u(s)|\big]\,ds \\
&\quad + e^{\lambda t}\tau^+\frac{1}{2}\frac{\eta\varepsilon}{1+\tau^+} + a^+e^{\lambda t}|u(t)| + e^{\lambda t}\frac{1}{2}\frac{\eta\varepsilon}{1+\tau^+}.
\end{aligned}
$$

Set

$$U(t) = \sup_{-\infty<s\le t}\{e^{\lambda t}|u(s)|\}.$$

It is obvious that $e^{\lambda t}|u(t)| \le U(t)$ and $U(t)$ is non-decreasing. We distinguish two cases to finish the proof.

CASE 1: $U(t) > e^{\lambda t}|u(t)|$ *for all* $t \ge N_0$. We claim that $U(t) \equiv U(N_0)$ for all $t \ge N_0$. Assume, by way of contradiction, that there exists $t_1 > N_0$ such that $U(t_1) > U(N_0)$. Since $e^{\lambda t}|u(t)| \le U(N_0)$ for all $t \le N_0$, there must exist $\beta \in (N_0, t_1)$ such that $e^{\lambda\beta}|u(\beta)| = U(t_1) \ge U(\beta)$, a contradiction. This proves the claim. Then there exists $t_2 > N_0$ such that

$$|u(t)| \le e^{-\lambda t}U(t) = e^{-\lambda t}U(N_0) < \varepsilon \quad \text{for all } t \ge t_2.$$

Case 2: *There is a* $t_0^* \ge N_0$ *such that* $U(t_0^*) = e^{\lambda t_0^*}|u(t_0^*)|$. Then, in view of (2.7)–(2.9), we get

$$\begin{aligned}
0 &\le D^-(e^{\lambda t}|u(t)|)|_{t=t_0^*} \\
&\le -[(b(t_0^*)+c(t_0^*))2\kappa-\lambda]e^{\lambda t_0^*}|u(t_0^*)| \\
&\quad + c(t_0^*)M \int\limits_{t_0^*-\tau(t_0^*)}^{t_0^*} \big[(2b^+ + c^+)Me^{\lambda(t_0^*-s)}e^{\lambda s}|u(s)| \\
&\quad + c^+Me^{\lambda(t_0^*-s+\tau(s))}e^{\lambda(s-\tau(s))}|u(s-\tau(s))| + a^+e^{\lambda(t_0^*-s)}e^{\lambda s}|u(s)|\big]\, ds \\
&\quad + e^{\lambda t_0^*}\tau^+ \frac{1}{2}\frac{\eta\varepsilon}{1+\tau^+} + a^+e^{\lambda t_0^*}|u(t_0^*)| + e^{\lambda t_0^*}\frac{1}{2}\frac{\eta\varepsilon}{1+\tau^+} \\
&\le \big\{-[(b(t_0^*)+c(t_0^*))2\kappa-\lambda] + c(t_0^*)M\tau(t_0^*)\big[(2b^+ + c^+)Me^{\lambda\tau^+} \\
&\quad + c^+Me^{2\lambda\tau^+} + a^+e^{\lambda\tau^+}\big] + a^+\big\}U(t_0^*) + e^{\lambda t_0^*}\eta\varepsilon \\
&\le -\eta U(t_0^*) + e^{\lambda t_0^*}\eta\varepsilon,
\end{aligned}$$

which yields

$$e^{\lambda t_0^*}|u(t_0^*)| = U(t_0^*) < \varepsilon e^{\lambda t_0^*} \quad \text{and} \quad |u(t_0^*)| < \varepsilon. \tag{2.10}$$

For any $t > t_0^*$, with the same approach as in deriving (2.10), we can show

$$e^{\lambda t}|u(t)| < \varepsilon e^{\lambda t} \text{ and } |u(t)| < \varepsilon \quad \text{if } U(t) = e^{\lambda t}|u(t)|. \tag{2.11}$$

On the other hand, if $U(t) > e^{\lambda t}|u(t)|$ and $t > t_0^*$, then we can choose $t_0^* \le t_3 < t$ such that

$$U(t_3) = e^{\lambda t_3}|u(t_3)| \text{ and } U(s) > e^{\lambda s}|u(s)| \quad \text{for all } s \in (t_3, t].$$

This, together with (2.11), leads to $|u(t_3)| < \varepsilon$. With a similar argument to that in the proof of Case 1, we can show that $U(s) \equiv U(t_3)$ for all $s \in (t_3, t]$, which implies

$$|u(t)| < e^{-\lambda t}U(t) = e^{-\lambda t}U(t_3) = |u(t_3)|e^{-\lambda(t-t_3)} < \varepsilon.$$

In summary, there must exist $N > \max\{t_0^*, N_0, t_2\}$ such that $|u(t)| \le \varepsilon$ for all $t > N$. ∎

3. Main results. In this section, we establish sufficient conditions for the existence and global exponential stability of almost periodic solutions of (1.1).

Theorem 3.1. *Under the assumptions of Lemma* 2.3, *equation* (1.1) *has at least one positive almost periodic solution* $x^*(t)$. *Moreover,* $x^*(t)$ *is globally exponentially stable, that is, there exist constants* K_{φ,x^*} *and* t_{φ,x^*} *such that*

$$|x(t;t_0,\varphi) - x^*(t)| < K_{\varphi,x^*}e^{-\lambda t} \quad \textit{for all } t > t_{\varphi,x^*}.$$

Proof. Let $v(t) = v(t; t_0, \varphi^v)$ be a solution of equation (1.1) with initial conditions satisfying the assumptions in Lemma 2.3. We also trivially extend $v(t)$ to $\mathbb{R}$ as before. Set

$$\begin{aligned}\widehat{\epsilon}(k,t) = &-[b(t+t_k) - b(t)]v^2(t+t_k)\\ &-[c(t+t_k) - c(t)]v(t+t_k-\tau(t))v(t+t_k)\\ &-c(t+t_k)[v(t+t_k-\tau(t+t_k)) - v(t+t_k-\tau(t))]v(t+t_k)\\ &+[a(t+t_k)-a(t)]v(t+t_k),\end{aligned}$$

where $\{t_k\}$ is any sequence of real numbers. By Lemma 2.2, the solution $v(t)$ is bounded and $\kappa \le v(t) \le M$ for all $t > t_{\varphi^v}$, which implies that the right-hand side of (1.1) is also bounded and $v'(t)$ is a bounded function on $[t_0 - \tau^+, \infty)$. Thus, since $v(t) \equiv v(t_0 - \tau^+)$ for $(-\infty, t_0 - \tau^+]$, we see that $v(t)$ is uniformly continuous on $\mathbb{R}$. Then, from the almost periodicity of a, b, c, and τ, we can select a sequence $t_k \to \infty$ such that

$$\text{(3.1)}\quad \begin{cases} |a(t+t_k) - a(t)| \le 1/k, \quad |b(t+t_k) - b(t)| \le 1/k,\\ |\tau(t+t_k) - \tau(t)| < \epsilon, \quad |\widehat{\epsilon}(k,t)| \le 1/k, \quad |c(t+t_k) - c(t)| \le 1/k,\end{cases}$$

for all $t \in \mathbb{R}$. Since $\{v(t+t_k)\}$ is uniformly bounded and equicontinuous, by the Ascoli–Arzelà theorem and the diagonal selection principle, we can choose a subsequence of $\{t_k\}$ (not relabelled) such that $\{v(t+t_k)\}$ uniformly converges to a continuous function $x^*(t)$ on any compact subset of $\mathbb{R}$ and $\kappa \le x^*(t) \le M$ for all $t \in \mathbb{R}$.

To complete the proof, we first prove that $x^*(t)$ is a solution of (1.1). In fact, for any $t \ge t_0$ and $\triangle t \in \mathbb{R}$, from (3.1), we have

$$\begin{aligned}\text{(3.2)}\quad x^*(t+\triangle t) - x^*(t) &= \lim_{k\to\infty}[v(t+\triangle t+t_k) - v(t+t_k)]\\ &= \lim_{k\to\infty}\int_t^{t+\triangle t} v(\mu+t_k)\big[a(\mu+t_k) - b(\mu+t_k)v(\mu+t_k)\\ &\qquad -c(\mu+t_k)v(\mu+t_k-\tau(\mu+t_k))\big]\,d\mu\\ &= \lim_{k\to\infty}\int_t^{t+\triangle t} v(\mu+t_k)\big[a(\mu) - b(\mu)v(\mu+t_k)\\ &\qquad -c(\mu)v(\mu+t_k-\tau(\mu))\big]\,d\mu + \lim_{k\to\infty}\int_t^{t+\triangle t}\widehat{\epsilon}(k,\mu)\,d\mu\\ &= \int_t^{t+\triangle t} x^*(\mu)[a(\mu) - b(\mu)x^*(\mu) - c(\mu)x^*(\mu-\tau(\mu))]\,d\mu,\end{aligned}$$

where $t + \triangle t \ge t_0$. Consequently, (3.2) implies that

$$\frac{d}{dt}x^*(t) = x^*(t)[a(t) - b(t)x^*(t) - c(t)x^*(t-\tau(t))],$$

in other words, $x^*(t)$ is a solution of (1.1).

Next, we show that $x^*(t)$ is almost periodic. From Lemma 2.3, for any $\varepsilon > 0$, there exists $l = l(\varepsilon) > 0$ such that every interval $[\alpha, \alpha + l]$ contains at least one number δ for which there exists $N > 0$ satisfying $|v(t + \delta) - v(t)| \le \varepsilon$ for all $t > N$. Then, for any fixed $s \in \mathbb{R}$, we can find a sufficiently large positive integer $N_1 > N$ such that for any $k > N_1$, $s + t_k > N$ and $|v(s+t_k+\delta)-v(s+t_k)| \le \varepsilon$. Letting $k \to \infty$, we obtain $|x^*(s+\delta)-x^*(t)| \le \varepsilon$. This tells us that $x^*(t)$ is an almost periodic solution.

Finally, we prove that $x^*(t)$ is globally exponentially stable. For an arbitrary solution $x(t; t_0, \varphi)$, denote $y(t) = x(t) - x^*(t)$, where $t \in [t_0 - \tau^+, \infty)$. Then

$$\begin{aligned}(3.3)\quad y'(t) &= -b(t)(x(t) + x^*(t))y(t) - c(t)x^*(t)y(t - \tau(t))\\ &\quad - c(t)x(t - \tau(t))y(t) + a(t)y(t)\\ &= -[b(t)(x(t) + x^*(t)) + c(t)(x^*(t) + x(t - \tau(t)))]y(t)\\ &\quad + a(t)y(t) + c(t)x^*(t) \int_{t-\tau(t)}^{t} \big[-b(s)(x(s) + x^*(s))y(s)\\ &\quad - c(s)x^*(s)y(s - \tau(s)) - c(s)x(s - \tau(s))y(s) + a(s)y(s)\big]\, ds,\end{aligned}$$

where $t \ge t_0+\tau^+$. It follows from Lemma 2.2 that there exists $t_{\varphi,x^*} > t_0+\tau^+$ such that $\kappa \le x(t), x^*(t) \le M$ for all $t \in [t_{\varphi,x^*} - \tau^+, \infty)$. Consider the Lyapunov functional $V(t) = |y(t)|e^{\lambda t}$. Calculating the upper left derivative of $V(t)$ along the solution $y(t)$ of (3.3), we obtain

$$\begin{aligned}(3.4)\quad & D^-(V(t))\\ &\le \lambda e^{\lambda t}|y(t)| + e^{\lambda t}\Big\{-[b(t)(x(t)+x^*(t))+c(t)(x^*(t)+x(t-\tau(t)))]y(t)\\ &\quad + c(t)x^*(t) \int_{t-\tau(t)}^{t} \big|-b(s)(x(s) + x^*(s))y(s) - c(s)x^*(s)y(s - \tau(s))\\ &\quad - c(s)x(s - \tau(s))y(s) + a(s)y(s)\big|\, ds + a(t)y(t)\Big\}\\ &\le -[2\kappa(b(t) + c(t)) - \lambda]|y(t)|e^{\lambda t} + c(t)M \int_{t-\tau(t)}^{t} \big[2Mb^+|y(s)|e^{\lambda s}e^{\lambda(t-s)}\\ &\quad + Mc^+|y(s - \tau(s))|e^{\lambda(s-\tau(s))}e^{\lambda(t-s+\tau(s))} + Mc^+|y(s)|e^{\lambda s}e^{\lambda(t-s)}\\ &\quad + a^+|y(s)|e^{\lambda s}e^{\lambda(t-s)}\big]\, ds + a^+|y(t)|e^{\lambda t}\end{aligned}$$

for all $t > t_{\varphi,x^*}$. We claim that

$$V(t) = |y(t)|e^{\lambda t} < e^{\lambda t_{\varphi,x^*}}\Big(\max_{t_0-\tau^+\le t\le t_{\varphi,x^*}} |x(t) - x^*(t)| + 1\Big) := K_{\varphi,x^*}$$

for all $t > t_{\varphi,x^*}$. Suppose that the claim is not true. Then there must exist

$t_* > t_{\varphi,x^*}$ such that

$$V(t_*) = K_{\varphi,x^*} \text{ and } V(t) < K_{\varphi,x^*} \quad \text{for all } t \in [t_0 - \tau^+, t_*).$$

Combining this with (3.4) gives

$$\begin{aligned}
0 \le D^-(V(t))|_{t=t_*} &\le -[2\kappa(b(t_*)+c(t_*)) - \lambda]|y(t_*)|e^{\lambda t_*} \\
&\quad + c(t_*)M \int_{t_*-\tau(t_*)}^{t_*} \big[2Mb^+|y(s)|e^{\lambda s}e^{\lambda(t_*-s)} \\
&\quad + Mc^+|y(s-\tau(s))|e^{\lambda(s-\tau(s))}e^{\lambda(t_*-s+\tau(s))} + Mc^+|y(s)|e^{\lambda s}e^{\lambda(t_*-s)} \\
&\quad + a^+|y(s)|e^{\lambda s}e^{\lambda(t_*-s)}\big]\,ds + a^+|y(t_*)|e^{\lambda t_*} \\
&\le \big\{-[2\kappa(b(t_*)+c(t_*)) - \lambda] + c(t_*)M\tau(t_*)\big[(2b^+ + c^+)Me^{\lambda\tau^+} \\
&\quad + c^+Me^{2\lambda\tau^+} + a^+e^{\lambda\tau^+}\big] + a^+\big\}K_{\varphi,x^*}.
\end{aligned}$$

Thus

$$\begin{aligned}
0 \le &-[2\kappa(b(t_*)+c(t_*)) - \lambda] \\
&+ c(t_*)M\tau(t_*)[(2b^+ + c^+)Me^{\lambda\tau^+} + c^+Me^{2\lambda\tau^+} + a^+e^{\lambda\tau^+}] + a^+,
\end{aligned}$$

which contradicts (2.7). This proves the claim. It follows from the claim that $|y(t)| < K_{\varphi,x^*}e^{-\lambda t}$ for all $t > t_{\varphi,x^*}$, and this completes the proof. ∎

4. An example. In this section, we present an example to check the validity of the main results obtained in Section 3.

Example 4.1. Consider the following single population model with hereditary effects:

$$\begin{aligned}
x'(t) = x(t)\Big[2 + |\cos\sqrt{2}\,t| - \tfrac{4}{5}(2+|\cos t|)x(t) \\
- (2+|\cos\sqrt{2}\,t|)x\big(t - \tfrac{1}{100}\sin^2 t\big)\Big].
\end{aligned} \tag{4.1}$$

Obviously, $a(t) = c(t) = 2 + |\cos\sqrt{2}t|$, $b(t) = \frac{4}{5}(2+|\cos t|)$, and $\tau(t) = \frac{\sin^2 t}{100}$. After a calculation, we see that $M \approx 1.03$, $\kappa \approx 0.47925$, and

$$\sup_{t\in\mathbb{R}}\{-2\kappa[b(t)+c(t)] + c(t)M\tau(t)[2(b^+ + c^+)M + a^+] + a^+\} < -0.02228 < 0.$$

Therefore, (4.1) satisfies the assumptions of Theorem 3.1. It follows immediately that equation (4.1) has a unique positive almost periodic solution $x(t)$, which is globally exponentially stable with exponential convergence rate $\lambda \approx 0.001$. Figure 1 strongly supports the conclusion.

Remark 4.1. We mention that no results in [XL] and [CL1, CL2, FW, C, CXC, L2] are applicable to (4.1) with initial values (1.5) to obtain existence and stability of positive almost periodic solutions.

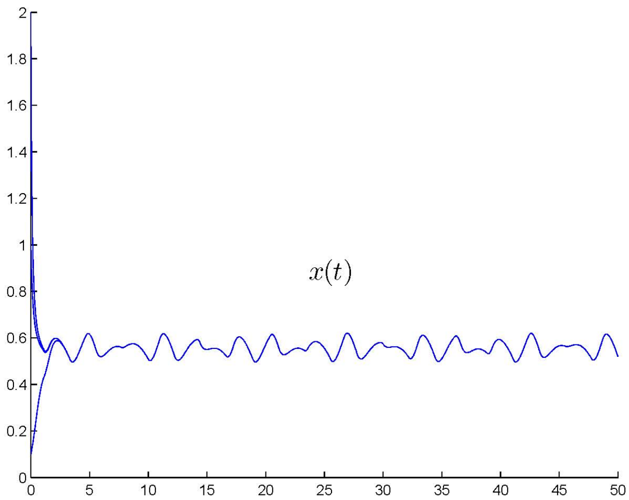

Fig. 1. Numerical solutions $x(t)$ of (4.1) for the initial values $\varphi(t) \equiv 0.1, 1, 2$

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