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# New results on global exponential stability of almost periodic solutions for a delayed Nicholson blowflies model

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**Abstract.** This paper is concerned with a class of Nicholson's blowflies models with multiple time-varying delays, which is defined on the nonnegative function space. Under appropriate conditions, we establish some criteria to ensure that all solutions of this model converge globally exponentially to a positive almost periodic solution. Moreover, we give an example with numerical simulations to illustrate our main results.

**1. Introduction.** In a classic study of population dynamics, Nicholson [N] and Gurney et al. [GBN] proposed the following delay differential equation model:

$$x'(t) = -\delta x(t) + Px(t-\tau)e^{-ax(t-\tau)}, \tag{1.1}$$

where $x(t)$ is the size of the population at time $t$, $P$ is the maximum per capita daily egg production, $1/a$ is the size at which the population reproduces at its maximum rate, $\delta$ is the per capita daily adult death rate, and $\tau$ is the generation time.

In population dynamics of the real world, the selective forces on systems in fluctuating environment and stable environment are distinct. Therefore, the variation of the environment has great impact on the evolutionary theory. This impact has attracted the attention of many researchers and various achievements have been obtained; some of them are presented in [K, Y]. Furthermore, much research has been done on the problem of existence of positive periodic solutions for Nicholson's blowflies equation. In particular, Liu [L1], Saker et al. [S1], Li et al. [LD], Wang [W1], Chen et al. [CW], Hou [HDH] and Chen [C] obtained the existence of positive periodic solutions of Nicholson's blowflies model (1.1) and of analogous equations.

On the other hand, as pointed out by Fink [F] and He [H], compared with periodic effects, almost periodic effects are more frequent in many bi-

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ological and ecological dynamical systems. Recently, Alzabut [A], Chen et al. [CL], Wang et al. [WWC], Long [L2] and Wang [W2] established some criteria for the solutions of this model and its generalizations to converge locally exponentially to a positive almost periodic solution. Unfortunately, all the above mentioned existence and stability results have only been demonstrated in a bounded region. Moreover, Chen et al. [CL] pointed out that the unique exponential stable almost periodic solution of Nicholson's blowflies equation in Alzabut [A] may be zero, and it is difficult to establish criteria to ensure the global exponential stability of positive almost periodic solutions for (1.1).

Motivated by the above discussion, we consider the existence, uniqueness and global exponential stability of positive almost periodic solutions for the following Nicholson blowflies model with multiple time-varying delays:

$$x'(t) = -a(t)x(t) + \sum_{j=1}^{m} \beta_j(t)x(t-\tau_j(t))e^{-\gamma_j(t)x(t-\tau_j(t))}, \tag{1.2}$$

where $a, \beta_j, \gamma_j : \mathbb{R} \to (0,\infty)$ and $\tau_j : \mathbb{R} \to [0,\infty)$ are almost periodic functions, and $j = 1, \ldots, m$. Here, the assumption of almost periodicity of the coefficient functions and the delays in (1.2) is a way of incorporating the time-dependent variability of the environment, especially when the various components of the environment are periodic with not necessarily commensurate periods (e.g. seasonal effects of weather, food supplies, mating habits and harvesting). Obviously, (1.1) is a special case of (1.2) with $m = 1$.

For convenience, we introduce some notation. Given a bounded continuous function $g$ defined on $\mathbb{R}$, let

$$g^+ = \sup_{t\in\mathbb{R}} g(t), \qquad g^- = \inf_{t\in\mathbb{R}} g(t). \tag{1.3}$$

It will be assumed that

$$r = \max_{1\le j\le m} \tau_j^+, \quad a^-, \beta_j^- > 0, \quad \gamma_j^- \ge 1, \quad j = 1, \ldots, m. \tag{1.4}$$

Throughout, let $\mathbb{R}_+$ denote the nonnegative reals, $C = C([-r,0],\mathbb{R})$ be the continuous function space equipped with the usual supremum norm $\|\cdot\|$, and let $C_+ = C([-r,0],\mathbb{R}_+)$. If $x(t)$ is continuous and defined on $[-r+t_0,\sigma)$ with $t_0, \sigma \in \mathbb{R}$, then we define $x_t \in C$ by $x_t(\theta) = x(t+\theta)$ for all $\theta \in [-r,0]$.

It is biologically reasonable to assume that only positive solutions of model (1.2) are meaningful and therefore admissible. Much can be learnt by considering admissible initial conditions

$$x_{t_0} = \varphi, \quad \varphi \in C_+ \quad \text{and} \quad \varphi(0) > 0. \tag{1.5}$$

Define a continuous map $f : \mathbb{R} \times C_+ \to \mathbb{R}$ by setting

$$f(t,\varphi) = -a(t)\varphi(0) + \sum_{j=1}^{m} \beta_j(t)\varphi(-\tau_j(t))e^{-\gamma_j(t)\varphi(-\tau_j(t))}.$$

In view of the inequality

$$\begin{aligned}|se^{-s} - te^{-t}| &= \left|\frac{1-(s+\theta(t-s))}{e^{s+\theta(t-s)}}\right||s-t| \\ &\leq |s-t| \quad \text{where } s,t \in [0,\infty),\ 0<\theta<1,\end{aligned}$$

we have

$$\|f(t,\phi_1) - f(t,\phi_2)\| \leq \left[a^+ + \sum_{j=1}^{m}\left(\frac{\beta_j}{\gamma_j}\right)^+\right]\|\phi_1 - \phi_2\| \quad \text{for all } \phi_1,\phi_2 \in C_+.$$

Thus $f$ is a locally Lipschitz map with respect to $\varphi \in C_+$, which ensures the existence and uniqueness of the solution of (1.2) with admissible initial conditions (1.5).

We denote by $x_t(t_0,\varphi)$ $(x(t;t_0,\varphi))$ an admissible solution of admissible initial value problem (1.2), (1.5). Also, let $[t_0,\eta(\varphi))$ be the maximal right interval of existence of $x_t(t_0,\varphi)$.

Definition 1.1 (see [F, H]). A continuous function $u : \mathbb{R} \to \mathbb{R}$ is said to be *almost periodic* on $\mathbb{R}$ if, for any $\epsilon > 0$, the set $T(u,\epsilon) = \{\delta : |u(t+\delta) - u(t)| < \epsilon \text{ for all } t \in \mathbb{R}\}$ is relatively dense, i.e., for any $\epsilon > 0$, it is possible to find a real number $l = l(\epsilon) > 0$ with the property that, for any interval of length $l(\epsilon)$, there exists a number $\delta = \delta(\epsilon)$ in this interval such that $|u(t+\delta) - u(t)| < \epsilon$ for all $t \in \mathbb{R}$.

From the theory of almost periodic functions in [F, H], it follows that for any $\epsilon > 0$, it is possible to find a real number $l = l(\epsilon) > 0$ such that for any interval of length $l(\epsilon)$, there exists a number $\delta = \delta(\epsilon)$ in this interval such that

$$\begin{cases} |a(t+\delta) - a(t)| < \epsilon, & |\beta_j(t+\delta) - \beta_j(t)| < \epsilon, \\ |\tau_j(t+\delta) - \tau_j(t)| < \epsilon, & |\gamma_j(t+\delta) - \gamma_j(t)| < \epsilon, \end{cases} \tag{1.6}$$

for all $t \in \mathbb{R}$ and $j = 1,\ldots,m$.

Since the function $(1-x)/e^x$ is decreasing on $[0,1]$, it follows easily that there exists a unique $\kappa \in (0,1)$ such that

$$\frac{1-\kappa}{e^\kappa} = \frac{1}{e^2}. \tag{1.7}$$

Obviously,

$$\sup_{x \geq \kappa}\left|\frac{1-x}{e^x}\right| = \frac{1}{e^2}. \tag{1.8}$$

Moreover, since $xe^{-x}$ increases on $[0,1]$ and decreases on $[1,\infty)$, let $\widetilde{\kappa}$ be the unique number in $(1,\infty)$ such that

$$\kappa e^{-\kappa} = \widetilde{\kappa} e^{-\tilde{\kappa}}. \tag{1.9}$$

**2. Preliminary results.** In this section, some lemmas will be presented, which are of importance in proving our main results in Section 3.

LEMMA 2.1. *Suppose that there exists a positive constant $M > \kappa$ such that*

$$\sup_{t\in\mathbb{R}}\left\{-a(t)M + \frac{1}{e}\sum_{j=1}^{m}\frac{\beta_j(t)}{\gamma_j(t)}\right\} < 0, \quad \inf_{t\in\mathbb{R}}\left\{-a(t) + \sum_{j=1}^{m}\frac{\beta_j(t)}{\gamma_j(t)}e^{-\kappa}\right\} > 0, \tag{2.1}$$
$$\max_{1\le j\le m}\gamma_j^+ \le \widetilde{\kappa}/M.$$

*Then, for $\varphi \in C^0 = \{\varphi \in C : \varphi(\theta) \in (\kappa, M) \text{ for all } \theta \in [-r,0]\}$,*

$$\eta(\varphi) = \infty, \qquad x_t(t_0,\varphi) \in C^0 \quad \text{for } t \ge t_0.$$

*Proof.* Let $x(t) = x(t;t_0,\varphi)$, where $\varphi \in C^0$. We first claim that

$$x(t) < M \quad \text{for all } t \in [t_0, \eta(\varphi)). \tag{2.2}$$

Suppose, for contradiction, that there exists $t_1 \in (t_0, \eta(\varphi))$ such that

$$x(t_1) = M, \quad x(t) < M \quad \text{for all } t \in [t_0 - r, t_1). \tag{2.3}$$

Calculating the derivative of $x(t)$, and using the fact that $\sup_{x\in\mathbb{R}} xe^{-x} = 1/e$, (1.2), (2.1) and (2.3), we find that

$$\begin{aligned} 0 &\le x'(t_1) \\ &= -a(t_1)M + \sum_{j=1}^{m}\frac{\beta_j(t_1)}{\gamma_j(t_1)}\gamma_j(t_1)x(t_1 - \tau_j(t_1))e^{-\gamma_j(t_1)x(t_1-\tau_j(t_1))} \\ &\le -a(t_1)M + \frac{1}{e}\sum_{j=1}^{m}\frac{\beta_j(t_1)}{\gamma_j(t_1)} < 0, \end{aligned}$$

which is a contradiction and implies that (2.2) holds.

We next show that

$$x(t) > \kappa \quad \text{for all } t \in [t_0, \eta(\varphi)). \tag{2.4}$$

Assume otherwise. Then there exists $t_2 \in (t_0, \eta(\varphi))$ such that

$$x(t_2) = \kappa \quad \text{and} \quad x(t) > \kappa \quad \text{for all } t \in [t_0 - r, t_2). \tag{2.5}$$

Then $\kappa \le \gamma_j(t_2)x(t_2 - \tau_j(t_2)) \le \gamma_j(t_2)M \le \widetilde{\kappa}$ and hence

$$\gamma_j(t_2)x(t_2 - \tau_j(t_2))e^{-\gamma_j(t_2)x(t_2-\tau_j(t_2))} \ge \min\{\kappa e^{-\kappa}, \widetilde{\kappa}e^{-\widetilde{\kappa}}\} = \kappa e^{-\kappa},$$

for all $j = 1, \dots, m$. It follows from (2.1) and (2.5) that

$$0 \geq x'(t_2) = -a(t_2)\kappa + \sum_{j=1}^{m} \frac{\beta_j(t_2)}{\gamma_j(t_2)} \gamma_j(t_2) x(t_2 - \tau_j(t_2)) e^{-\gamma_j(t_2) x(t_2 - \tau_j(t_2))}$$
$$\geq -a(t_2)\kappa + \sum_{j=1}^{m} \frac{\beta_j(t_2)}{\gamma_j(t_2)} \kappa e^{-\kappa} \geq \kappa \inf_{t\in\mathbb{R}} \Big\{ -a(t) + \sum_{j=1}^{m} \frac{\beta_j(t)}{\gamma_j(t)} e^{-\kappa} \Big\} > 0,$$

which is a contradiction and implies that (2.4) holds. This implies that $x(t)$ is bounded on $[t_0, \eta(\varphi))$. From [HVL, Theorem 2.3.1], we easily obtain $\eta(\varphi) = \infty$. This ends the proof of Lemma 2.1.

LEMMA 2.2. *Suppose* (2.1) *holds, and*

$$\sup_{t\in\mathbb{R}} \Big\{ -a(t) + \sum_{j=1}^{m} \beta_j(t) \frac{1}{e^2} \Big\} < 0. \tag{2.6}$$

*Moreover, assume that $x(t) = x(t; t_0, \varphi)$ is a solution of equation* (1.2) *with initial condition $\varphi \in C^0$, and $\varphi'$ is bounded continuous on $[-r, 0]$. Then for any $\epsilon > 0$, there exists $l = l(\epsilon) > 0$ such that every interval $[\alpha, \alpha + l]$ contains at least one number $\delta$ for which there exists $N > 0$ satisfying*

$$|x(t+\delta) - x(t)| \leq \epsilon \quad \textit{for all } t > N. \tag{2.7}$$

*Proof.* Define a continuous function $\Gamma(u)$ by setting

$$\Gamma(u) = \sup_{t\in\mathbb{R}} \Big\{ -[a(t) - u] + \sum_{j=1}^{m} \beta_j(t) \frac{1}{e^2} e^{ur} \Big\}, \qquad u \in [0, 1].$$

Then

$$\Gamma(0) = \sup_{t\in\mathbb{R}} \Big\{ -a(t) + \sum_{j=1}^{m} \beta_j(t) \frac{1}{e^2} \Big\} < 0,$$

which implies that there exist $\eta > 0$ and $\lambda \in (0, 1]$ such that

$$\Gamma(\lambda) = \sup_{t\in\mathbb{R}} \Big\{ -[a(t) - \lambda] + \sum_{j=1}^{m} \beta_j(t) \frac{1}{e^2} e^{\lambda r} \Big\} < -\eta < 0. \tag{2.8}$$

For $t \in (-\infty, t_0 - r]$, we define $x(t) \equiv x(t_0 - r)$. Set

$$\begin{aligned} \epsilon(\delta, t) = {}& -[a(t+\delta) - a(t)] x(t+\delta) \\ & + \sum_{j=1}^{m} [\beta_j(t+\delta) - \beta_j(t)] x(t+\delta - \tau_j(t+\delta)) e^{-\gamma_j(t+\delta) x(t+\delta-\tau_j(t+\delta))} \\ & + \sum_{j=1}^{m} \beta_j(t) \big[ x(t+\delta - \tau_j(t+\delta)) e^{-\gamma_j(t+\delta) x(t+\delta-\tau_j(t+\delta))} \\ & \qquad - x(t - \tau_j(t) + \delta) e^{-\gamma_j(t+\delta) x(t-\tau_j(t)+\delta)} \big] \end{aligned} \tag{2.9}$$

$$+\sum_{j=1}^{m}\beta_j(t)\big[x(t-\tau_j(t)+\delta)e^{-\gamma_j(t+\delta)x(t-\tau_j(t)+\delta)} - x(t-\tau_j(t)+\delta)e^{-\gamma_j(t)x(t-\tau_j(t)+\delta)}\big], \qquad t\in\mathbb{R}.$$

By Lemma 2.1, the solution $x(t)$ is bounded and

$$\kappa < x(t) < M \qquad \text{for all } t\in\mathbb{R},$$

which implies that the right side of (1.2) is also bounded, and $x'(t)$ is a bounded function on $[t_0-r,\infty)$. Thus, in view of the fact that $x(t)\equiv x(t_0-r)$ for $t\in(-\infty,t_0-r]$, we see that $x(t)$ is uniformly continuous on $\mathbb{R}$. From (1.6) and (2.9), for any $\epsilon>0$, there exists $l=l(\epsilon)>0$ such that every interval $[\alpha,\alpha+l]$, $\alpha\in\mathbb{R}$, contains a $\delta$ for which

$$|\epsilon(\delta,t)| \le \tfrac{1}{2}\eta\epsilon \qquad \text{for all } t\in\mathbb{R}. \tag{2.10}$$

Let $N_0\ge\max\{t_0,t_0-\delta\}$. For $t\in\mathbb{R}$, denote

$$u(t)=x(t+\delta)-x(t).$$

Then, for all $t\ge N_0$, we get

$$\begin{aligned}\frac{du(t)}{dt} &= -a(t)[x(t+\delta)-x(t)] \\ &\quad +\sum_{j=1}^{m}\beta_j(t)\big[x(t-\tau_j(t)+\delta)e^{-\gamma_j(t)x(t-\tau_j(t)+\delta)} \\ &\quad -x(t-\tau_j(t))e^{-\gamma_j(t)x(t-\tau_j(t))}\big]+\epsilon(\delta,t).\end{aligned}\tag{2.11}$$

From (2.11) and the inequality

$$\begin{aligned}|se^{-s}-te^{-t}| &= \left|\frac{1-(s+\theta(t-s))}{e^{s+\theta(t-s)}}\right||s-t| \\ &\le \frac{1}{e^2}|s-t| \qquad \text{where } s,t\in[\kappa,\infty),\ 0<\theta<1,\end{aligned}\tag{2.12}$$

we obtain

$$\begin{aligned}&D^-(e^{\lambda s}|u(s)|)|_{s=t} \\ &\le \lambda e^{\lambda t}|u(t)| + e^{\lambda t}\Big\{-a(t)|x(t+\delta)-x(t)| \\ &\qquad +\Big|\sum_{j=1}^{m}\beta_j(t)\big[x(t-\tau_j(t)+\delta)e^{-\gamma_j(t)x(t-\tau_j(t)+\delta)} \\ &\qquad - x(t-\tau_j(t))e^{-\gamma_j(t)x(t-\tau_j(t))}\big]+\epsilon(\delta,t)\Big|\Big\} \\ &\le \lambda e^{\lambda t}|u(t)| + e^{\lambda t}\Big\{-a(t)|u(t)| + \sum_{j=1}^{m}\beta_j(t)\frac{1}{e^2}|u(t-\tau_j(t))| + |\epsilon(\delta,t)|\Big\}\end{aligned}\tag{2.13}$$

$$= -[a(t)-\lambda]e^{\lambda t}|u(t)| + e^{\lambda t}|\epsilon(\delta,t)| + \sum_{j=1}^{m} \beta_j(t)\frac{1}{e^2}e^{\lambda\tau_j(t)}e^{\lambda(t-\tau_j(t))}|u(t-\tau_j(t))| \quad \text{for all } t \geq N_0.$$

Let

$$U(t) = \sup_{t_0-r\leq s\leq t}\{e^{\lambda s}|u(s)|\}. \tag{2.14}$$

It is obvious that $e^{\lambda t}|u(t)| \leq U(t)$, and $U(t)$ is nondecreasing.

Now, we distinguish two cases to finish the proof.

Case 1. Suppose

$$U(t) > e^{\lambda t}|u(t)| \quad \text{for all } t \geq N_0. \tag{2.15}$$

We claim that

$$U(t) \equiv U(N_0) \text{ is a constant for all } t \geq N_0. \tag{2.16}$$

Assume, for contradiction, that (2.16) does not hold. Then there exists $\widetilde{t_1} > N_0$ such that $U(\widetilde{t_1}) > U(N_0)$. Since

$$e^{\lambda t}|u(t)| \leq U(N_0) \quad \text{for all } t \leq N_0.$$

There must exist $\beta \in (N_0, \widetilde{t_1})$ such that

$$e^{\lambda\beta}|u(\beta)| = U(\widetilde{t_1}) \geq U(\beta),$$

which contradicts (2.15). This contradiction implies that (2.16) holds. It follows that there exists $\widetilde{t_2} > N_0$ such that

$$|u(t)| \leq e^{-\lambda t}U(t) = e^{-\lambda t}U(N_0) < \epsilon \quad \text{for all } t \geq \widetilde{t_2}. \tag{2.17}$$

Case 2. Suppose there is a $t_0^* \geq N_0$ such that $U(t_0^*) = e^{\lambda t_0^*}|u(t_0^*)|$. Then, in view of (2.8) and (2.13), we get

$$\begin{aligned}
0 &\leq D^-(e^{\lambda s}|u(s)|)|_{s=t_0^*} \\
&\leq -[a(t_0^*)-\lambda]e^{\lambda t_0^*}|u(t_0^*)| \\
&\quad + \sum_{j=1}^{m}\beta_j(t_0^*)\frac{1}{e^2}e^{\lambda\tau_j(t_0^*)}e^{\lambda(t_0^*-\tau_j(t_0^*))}|u(t_0^*-\tau_j(t_0^*))| + e^{\lambda t_0^*}|\epsilon(\delta,t_0^*)| \\
&\leq \left\{-[a(t_0^*)-\lambda] + \sum_{j=1}^{m}\beta_j(t_0^*)\frac{1}{e^2}e^{\lambda r}\right\}U(t_0^*) + \frac{1}{2}\eta\epsilon e^{\lambda t_0^*} \\
&< -\eta U(t_0^*) + \eta\epsilon e^{\lambda t_0^*},
\end{aligned}$$

which yields

$$e^{\lambda t_0^*}|u(t_0^*)| = U(t_0^*) < \epsilon e^{\lambda t_0^*} \quad \text{and} \quad |u(t_0^*)| < \epsilon. \tag{2.18}$$

For any $t > t_0^*$, with the same approach as in deriving (2.18), we can show that

$$e^{\lambda t}|u(t)| < \epsilon e^{\lambda t} \quad \text{and} \quad |u(t)| < \epsilon, \tag{2.19}$$

if $U(t) = e^{\lambda t}|u(t)|$.

On the other hand, if $U(t) > e^{\lambda t}|u(t)|$ and $t > t_0^*$, then we can choose $t_0^* \le t_3 < t$ such that

$$U(t_3) = e^{\lambda t_3}|u(t_3)| \quad \text{and} \quad U(s) > e^{\lambda s}|u(s)| \quad \text{for all } s \in (t_3, t],$$

which, together with (2.19), yields

$$|u(t_3)| < \epsilon.$$

With a similar argument to that in Case 1, we can show that

$$U(s) \equiv U(t_3) \text{ is a constant for all } s \in (t_3, t],$$

which implies that

$$|u(t)| < e^{-\lambda t}U(t) = e^{-\lambda t}U(t_3) = |u(t_3)|e^{-\lambda(t-t_3)} < \epsilon.$$

In summary, there must exist $N > \max\{t_0^*, N_0, \widetilde{t_2}\}$ such that $|u(t)| \le \epsilon$ for all $t > N$. The proof of Lemma 2.2 is now complete.

**3. Main results.** In this section, we establish sufficient conditions for the existence and exponential stability of positive almost periodic solutions of (1.2).

THEOREM 3.1. *Under the assumptions of Lemma* 2.2*, equation* (1.2) *has a positive almost periodic solution.*

*Proof.* Let $v(t) = v(t; t_0, \varphi)$ be a solution of (1.2) with initial conditions satisfying the assumptions in Lemma 2.2. We also add the definition $v(t) \equiv v(t_0 - r)$ for all $t \in (-\infty, t_0 - r]$. Set

$$\begin{aligned} \epsilon(k,t) = {}& -[a(t+t_k) - a(t)]v(t+t_k) \\ & + \sum_{j=1}^{m}[\beta_j(t+t_k) - \beta_j(t)]v(t+t_k-\tau_j(t+t_k))e^{-\gamma_j(t+t_k)v(t+t_k-\tau_j(t+t_k))} \\ & + \sum_{j=1}^{m}\beta_j(t)\big[v(t+t_k-\tau_j(t+t_k))e^{-\gamma_j(t+t_k)v(t+t_k-\tau_j(t+t_k))} \\ & \qquad\qquad - v(t-\tau_j(t)+t_k)e^{-\gamma_j(t+t_k)v(t-\tau_j(t)+t_k)}\big] \\ & + \sum_{j=1}^{m}\beta_j(t)\big[v(t-\tau_j(t)+t_k)e^{-\gamma_j(t+t_k)v(t-\tau_j(t)+t_k)} \\ & \qquad\qquad - v(t-\tau_j(t)+t_k)e^{-\gamma_j(t)v(t-\tau_j(t)+t_k)}\big], \quad t \in \mathbb{R}, \end{aligned} \tag{3.1}$$

where $\{t_k\}$ is any sequence of real numbers. By Lemma 2.1, the solution $v(t)$ is bounded and

$$\kappa < v(t) < M \quad \text{for all } t \in \mathbb{R}, \tag{3.2}$$

which implies that the right side of (1.2) is also bounded, and $v'(t)$ is a bounded function on $[t_0-r, \infty)$. Thus, in view of the fact that $v(t) \equiv v(t_0-r)$ for $t \in (-\infty,\ t_0-r]$, we find that $v(t)$ is uniformly continuous on $\mathbb{R}$. Then, from the almost periodicity of $a, \tau_j, \gamma_j$ and $\beta_j$, we can select a sequence $t_k \to \infty$ such that

$$\begin{aligned} &|a(t+t_k) - a(t)| \le 1/k, && |\tau_j(t+t_k) - \tau_j(t)| \le 1/k,\\ &|\beta_j(t+t_k) - \beta_j(t)| \le 1/k, && |\gamma_j(t+t_k) - \gamma_j(t)| \le 1/k,\\ &|\epsilon(k,t)| \le 1/k, \quad \text{for all } j,\ k,\ t. \end{aligned} \tag{3.3}$$

Since $\{v(t+t_k)\}_{k=1}^{\infty}$ is uniformly bounded and equicontinuous, by the Arzelà–Ascoli Lemma and diagonal selection principle, we can choose a subsequence $\{t_{k_j}\}$ such that $v(t+t_{k_j})$ (for convenience, still denoted by $v(t+t_k)$) uniformly converges to a continuous function $x^*(t)$ on any compact subset of $\mathbb{R}$, and

$$\kappa \le x^*(t) \le M \quad \text{for all } t \in \mathbb{R}. \tag{3.4}$$

Now, we prove that $x^*(t)$ is a solution of (1.2). In fact, for any $t \ge t_0$ and $\Delta t \in \mathbb{R}$, from (3.3), we have

$$\begin{aligned} x^*(t+\Delta t) - x^*(t) &= \lim_{k\to\infty} [v(t+\Delta t+t_k) - v(t+t_k)] \\ &= \lim_{k\to\infty} \int_t^{t+\Delta t} \Big\{ -a(\mu+t_k)v(\mu+t_k) \\ &\quad + \sum_{j=1}^{m} \beta_j(\mu+t_k) v(\mu+t_k-\tau_j(\mu+t_k)) e^{-\gamma_j(\mu+t_k)v(\mu+t_k-\tau_j(\mu+t_k))} \Big\}\, d\mu \\ &= \lim_{k\to\infty} \int_t^{t+\Delta t} \Big\{ -a(\mu)v(\mu+t_k) \\ &\quad + \sum_{j=1}^{m} \beta_j(\mu) v(\mu+t_k-\tau_j(\mu)) e^{-\gamma_j(\mu)v(\mu+t_k-\tau_j(\mu))} + \epsilon(k,\mu) \Big\}\, d\mu \\ &= \int_t^{t+\Delta t} \Big\{ -a(\mu)x^*(\mu) \\ &\quad + \sum_{j=1}^{m} \beta_j(\mu) x^*(\mu-\tau_j(\mu)) e^{-\gamma_j(\mu)x^*(\mu-\tau_j(\mu))} \Big\}\, d\mu + \lim_{k\to\infty} \int_t^{t+\Delta t} \epsilon(k,\mu)\, d\mu \\ &= \int_t^{t+\Delta t} \Big\{ -a(\mu)x^*(\mu) + \sum_{j=1}^{m} \beta_j(\mu) x^*(\mu-\tau_j(\mu)) e^{-\gamma_j(\mu)x^*(\mu-\tau_j(\mu))} \Big\}\, d\mu, \end{aligned} \tag{3.5}$$

where $t + \Delta t \geq t_0$. Consequently, (3.5) implies that

$$\frac{d}{dt}\{x^*(t)\} = -a(t)x^*(t) + \sum_{j=1}^{m} \beta_j(t)x^*(t-\tau_j(t))e^{-\gamma_j(t)x^*(t-\tau_j(t))}. \tag{3.6}$$

Therefore, $x^*(t)$ is a solution of (1.2).

Secondly, we prove that $x^*(t)$ is an almost periodic solution of (1.2). From Lemma 2.2, for any $\varepsilon > 0$, there exists $l = l(\varepsilon) > 0$ such that every interval $[\alpha, \alpha+l]$ contains at least one number $\delta$ for which there exists $N > 0$ satisfying

$$|v(t+\delta) - v(t)| \leq \varepsilon \quad \text{for all } t > N. \tag{3.7}$$

Then, for any fixed $s \in \mathbb{R}$, we can find a sufficiently large positive integer $N_1 > N$ such that for any $k > N_1$,

$$s + t_k > N, \quad |v(s+t_k+\delta) - v(s+t_k)| \leq \varepsilon. \tag{3.8}$$

Letting $k \to \infty$, we obtain

$$|x^*(s+\delta) - x^*(s)| \leq \varepsilon, \tag{3.9}$$

which implies that $x^*(t)$ is an almost periodic solution of (1.2). The proof of Theorem 3.1 is now complete.

THEOREM 3.2. *Suppose that all conditions in Theorem* 3.1 *are satisfied. Let $x^*(t)$ be a positive almost periodic solution of* (1.2). *Then $x^*(t)$ is globally exponentially stable, i.e., the solution $x(t;t_0,\varphi)$ of* (1.2) *with admissible initial conditions* (1.5) *converges exponentially to $x^*(t)$ as $t \to \infty$.*

*Proof.* Since $\varphi \in C_+$, by [S2, Theorem 5.2.1, p. 81], $x_t(t_0,\varphi) \in C_+$ for all $t \in [t_0, \eta(\varphi))$. Let $x(t) = x(t;t_0,\varphi)$. In view of $x(t_0) = \varphi(0) > 0$, integrating (1.2) from $t_0$ to $t$, we obtain

$$\begin{aligned} x(t) &\geq e^{-\int_{t_0}^{t} a(u)\,du} x(t_0) \\ &\quad + e^{-\int_{t_0}^{t} a(u)\,du} \int_{t_0}^{t} e^{\int_{t_0}^{s} a(v)\,dv} \sum_{j=1}^{m} \beta_j(s)x(s-\tau_j(s))e^{-\gamma_j(s)x(s-\tau_j(s))}\,ds \\ &> 0 \quad \text{for all } t \in [t_0, \eta(\varphi)). \end{aligned} \tag{3.10}$$

We next show that there exists $t_\varphi \in [t_0, \eta(\varphi))$ such that

$$\kappa < x(t) < M \quad \text{for all } t \in [t_\varphi, \eta(\varphi)), \quad \eta(\varphi) = \infty. \tag{3.11}$$

We first prove that there exists $t_4 \in [t_0, \eta(\varphi))$ such that

$$x(t_4) < M. \tag{3.12}$$

Otherwise,

$$x(t) \geq M \quad \text{for all } t \in [t_0, \eta(\varphi)), \tag{3.13}$$

which, together with (2.1), implies that

$$x'(t) = -a(t)x(t) + \sum_{j=1}^{m} \frac{\beta_j(t)}{\gamma_j(t)} \gamma_j(t) x(t-\tau_j(t)) e^{-\gamma_j(t)x(t-\tau_j(t))} \tag{3.14}$$
$$\le -a(t)M + \frac{1}{e} \sum_{j=1}^{m} \frac{\beta_j(t)}{\gamma_j(t)} < 0 \quad \text{for all } t \in [t_0, \eta(\varphi)).$$

This implies that $x(t)$ is bounded and decreasing on $[t_0, \eta(\varphi))$. Again from [HVL, Theorem 2.3.1], we easily obtain $\eta(\varphi) = \infty$. Then (3.14) leads to

$$x(t) = x(t_0) + \int_{t_0}^{t} x'(s)\,ds$$
$$\le x(t_0) + \sup_{t\in\mathbb{R}} \left\{ -a(t)M + \frac{1}{e} \sum_{j=1}^{m} \frac{\beta_j(t)}{\gamma_j(t)} \right\} (t - t_0), \qquad \forall t \ge t_0,$$

and

$$\lim_{t\to\infty} x(t) = -\infty,$$

which contradicts (3.10). Hence, (3.12) holds. We claim that

$$x(t) < M \quad \text{for all } t \in [t_4, \eta(\varphi)), \quad \text{and} \quad \eta(\varphi) = \infty. \tag{3.15}$$

Suppose, for contradiction, that there exists $t_5 \in (t_4, \eta(\varphi))$ such that

$$x(t_5) = M, \qquad x(t) < M \qquad \text{for all } t \in [t_4, t_5). \tag{3.16}$$

Calculating the derivative of $x(t)$, using (1.2), the fact that $\sup_{x\in\mathbb{R}} xe^{-x} = 1/e$, (2.1) and (3.16), we find that

$$0 \le x'(t_5) = -a(t_5)M + \sum_{j=1}^{m} \frac{\beta_j(t_5)}{\gamma_j(t_5)} \gamma_j(t_5) x(t_5 - \tau_j(t_5)) e^{-\gamma_j(t_5)x(t_5-\tau_j(t_5))}$$
$$\le -a(t_5)M + \frac{1}{e} \sum_{j=1}^{m} \frac{\beta_j(t_5)}{\gamma_j(t_5)} < 0,$$

which is a contradiction and implies that (3.15) holds.

Furthermore, we prove that there exists a positive constant $l$ such that

$$\liminf_{t\to\infty} x(t) = l. \tag{3.17}$$

Otherwise, we assume that $\liminf_{t\to\infty} x(t) = 0$. For each $t \ge t_0$, we define

$$m(t) = \max\left\{ \xi : \xi \le t,\, x(\xi) = \min_{t_0 \le s \le t} x(s) \right\}.$$

Observe that $m(t) \to \infty$ as $t \to \infty$ and

$$\lim_{t\to\infty} x(m(t)) = 0. \tag{3.18}$$

However, $x(m(t)) = \min_{t_0 \le s \le t} x(s)$, and so $x'(m(t)) \le 0$ for all $m(t) > t_0$.

According to (1.2), we have

$$\begin{aligned}0 \geq x'(m(t)) = &-a(m(t))x(m(t)) \\ &+ \sum_{j=1}^{m} \beta_j(m(t))x\big(m(t)-\tau_j(m(t))\big)e^{-\gamma_j(m(t))x(m(t)-\tau_j(m(t)))},\end{aligned}$$

and consequently

$$\begin{aligned}(3.19)\qquad a(m(t))x(m(t)) &\geq \sum_{j=1}^{m} \beta_j(m(t))x\big(m(t)-\tau_j(m(t))\big)e^{-\gamma_j(m(t))x(m(t)-\tau_j(m(t)))} \\ &\geq \beta_j(m(t))x\big(m(t)-\tau_j(m(t))\big)e^{-\gamma_j(m(t))x(m(t)-\tau_j(m(t)))},\end{aligned}$$

where $m(t) > t_0$, $j = 1, \ldots, m$. This, together with (3.18), implies that

$$(3.20)\qquad \lim_{t\to\infty} x\big(m(t)-\tau_j(m(t))\big) = 0, \qquad j = 1, \ldots, m.$$

Since $a(t)$ and $\beta_j(t)$ are continuous and bounded, we can select a sequence $\{t_n\}_{n=1}^{\infty}$ such that

$$(3.21)\qquad \lim_{n\to\infty} t_n = \infty, \qquad \lim_{n\to\infty} x(m(t_n)) = 0, \qquad \lim_{n\to\infty} \frac{\beta_j(m(t_n))}{a(m(t_n))} = a_j^*$$

for $j = 1, \ldots, m$.

In view of (3.19), we get

$$\begin{aligned}a(m(t_n)) &\geq \sum_{j=1}^{m} \beta_j(m(t_n)) \frac{x(m(t_n)-\tau_j(m(t_n)))e^{-\gamma_j(m(t_n))x(m(t_n)-\tau_j(m(t_n)))}}{x(m(t_n))} \\ &\geq \sum_{j=1}^{m} \beta_j(m(t_n)) \frac{x(m(t_n)-\tau_j(m(t_n)))e^{-\gamma_j^+ x(m(t_n)-\tau_j(m(t_n)))}}{x(m(t_n)-\tau_j(m(t_n)))} \\ &= \sum_{j=1}^{m} \beta_j(m(t_n))e^{-\gamma_j^+ x(m(t_n)-\tau_j(m(t_n)))},\end{aligned}$$

which leads to

$$(3.22)\qquad 1 \geq \sum_{j=1}^{m} \frac{\beta_j(m(t_n))}{a(m(t_n))} e^{-\gamma_j^+ x(m(t_n)-\tau_j(m(t_n)))}.$$

Letting $n \to \infty$ in (3.20)–(3.22) implies that

$$\begin{aligned}(3.23)\qquad 1 &\geq \sum_{j=1}^{m} \lim_{n\to\infty} \frac{\beta_j(m(t_n))}{a(m(t_n))} \lim_{n\to\infty} e^{-\gamma_j^+ x(m(t_n)-\tau_j(m(t_n)))} \\ &= \lim_{n\to\infty} \sum_{j=1}^{m} \frac{\beta_j(m(t_n))}{a(m(t_n))} \geq \liminf_{t\to\infty} \sum_{j=1}^{m} \frac{\beta_j(t)}{a(t)}.\end{aligned}$$

From (2.1), we get

$$0 < \inf_{t\in\mathbb{R}}\Big\{-a(t)+\sum_{j=1}^{m}\frac{\beta_j(t)}{\gamma_j(t)}e^{-\kappa}\Big\} \le -a(t)+\sum_{j=1}^{m}\frac{\beta_j(t)}{\gamma_j(t)}$$
$$\le -a(t)+\sum_{j=1}^{m}\beta_j(t) \quad \text{for all } t\in\mathbb{R},$$

and

$$1 < \inf_{t\in\mathbb{R}}\Big\{\sum_{j=1}^{m}\frac{\beta_j(t)}{a(t)}\Big\},$$

which contradicts (3.23). Hence, (3.17) holds.

To prove (3.11), it is sufficient to show $l > \kappa$. Assume that $l \le \kappa$.

By the fluctuation lemma [S1, Lemma A.1], there exists a sequence $\{t_k\}_{k=1}^{\infty}$ such that $t_k \to \infty$ and

$$x(t_k;t_0,\varphi)\to l, \quad x'(t_k;t_0,\varphi)=f(t_k,x_{t_k}(t_0,\varphi))\to 0, \tag{3.24}$$

as $k\to\infty$. Since $\{x_{t_k}(t_0,\varphi)\}_{k=1}^{\infty}$ is bounded and equicontinuous, by the Arzelà–Ascoli theorem, for a subsequence, still denoted by $\{x_{t_k}(t_0,\varphi)\}_{k=1}^{\infty}$, we have

$$x_{t_k}(t_0,\varphi)\to\varphi^* \quad \text{for some } \varphi^*\in C([-r,0],(0,\infty)).$$

From (3.15), we get

$$\varphi^*(0)=l\le\varphi^*(s)\le M \quad \text{for } s\in[-r,0]. \tag{3.25}$$

By the boundedness of $\{\tau_j(t_k)\}_{k=1}^{\infty}$, there is a subsequence of $\{t_k\}_{k=1}^{\infty}$, still denoted by $\{t_k\}_{k=1}^{\infty}$, which converges to a point $\tau_j^*\in[\tau_j^-,\tau_j^+]$, for all $j=1,\dots,m$. Similarly, we can also suppose that

$$\begin{cases} \lim\limits_{k\to\infty} a(t_k)=a^*\in[a^-,a^+],\\ \lim\limits_{k\to\infty}\beta_j=\beta_j^*\in[\beta_j^-,\beta_j^+],\\ \lim\limits_{k\to\infty}\gamma_j=\gamma_j^*\in[\gamma_j^-,\ \gamma_j^+], \quad j=1,\dots,m.\end{cases}$$

Hence,

$$f(t_k,x_{t_k}(t_0,\varphi))\to\Lambda \quad \text{as } k\to\infty, \tag{3.26}$$

with

$$\Lambda=-a^*\varphi^*(0)+\sum_{j=1}^{m}\beta_j^*\varphi^*(-\tau_j^*)e^{-\gamma_j^*\varphi^*(-\tau_j^*)}. \tag{3.27}$$

According to (1.8), (1.9), (2.1) and the fact that

$$0<l\le\kappa, \quad l\le\gamma_j^*\varphi^*(-\tau_j^*)\le\gamma_j^+M\le\widetilde{\kappa}, \quad j=1,\dots,m,$$

we obtain

$$\begin{aligned}\varLambda &= -a^*l + \sum_{j=1}^{m} \frac{\beta_j^*}{\gamma_j^*}\gamma_j^*\varphi^*(-\tau_j^*)e^{-\gamma_j^*\varphi^*(-\tau_j^*)} \\ &\geq -a^*l + \sum_{j=1}^{m} \frac{\beta_j^*}{\gamma_j^*} le^{-l} = l\Big[-a^* + \sum_{j=1}^{m} \frac{\beta_j^*}{\gamma_j^*}e^{-l}\Big] \\ &\geq l\Big[-a^* + \sum_{j=1}^{m} \frac{\beta_j^*}{\gamma_j^*}e^{-\kappa}\Big] \geq l \min_{t\in[0,T]}\Big\{-a(t) + \sum_{j=1}^{m} \frac{\beta_j(t)}{\gamma_j(t)}e^{-\kappa}\Big\} > 0,\end{aligned}$$

which contradicts (3.24) and implies $l > \kappa$.

Finally, we prove that $x^*(t)$ is globally exponentially stable.

Set $y(t) = x(t) - x^*(t)$, where $t \in [t_0 - r, \infty)$. Then

$$\begin{aligned}y'(t) = -a(t)y(t) + \sum_{j=1}^{m} \beta_j(t)\big[&x(t-\tau_j(t))e^{-\gamma_j(t)x(t-\tau_j(t))} \\ &- x^*(t-\tau_j(t))e^{-\gamma_j(t)x^*(t-\tau_j(t))}\big] \quad \text{for all } t \geq t_0.\end{aligned} \tag{3.28}$$

We consider the Lyapunov functional

$$V(t) = |y(t)|e^{\lambda t}. \tag{3.29}$$

Calculating the upper left derivative of $V(t)$ along the solution $y(t)$ of (3.28), we get

$$\begin{aligned}D^-(V(t)) \leq{} & -a(t)|y(t)|e^{\lambda t} \\ & + \sum_{j=1}^{m} \beta_j(t)\big|x(t-\tau_j(t))e^{-\gamma_j(t)x(t-\tau_j(t))} \\ & \qquad - x^*(t-\tau_j(t))e^{-\gamma_j(t)x^*(t-\tau_j(t))}\big|e^{\lambda t} + \lambda|y(t)|e^{\lambda t}\end{aligned} \tag{3.30}$$

for all $t \geq t_0$. We claim that

$$\begin{aligned}V(t) &= |y(t)|e^{\lambda t} = |x(t) - x^*(t)|e^{\lambda t} \\ &< e^{\lambda(t_\varphi + r)}\Big(\max_{t\in[t_0-r,t_\varphi+r]} |x(t) - x^*(t)| + 1\Big) =: K\end{aligned} \tag{3.31}$$

for all $t > t_\varphi + r$. Indeed, otherwise there must exist $t_* > t_\varphi + r$ such that

$$V(t_*) = K, \qquad V(t) < K \quad \text{for all } t \in [t_0 - r, t_*). \tag{3.32}$$

Since $x(t) \geq \kappa$ and $x^*(t) \geq \kappa$ for all $t \geq t_\varphi$, combining (2.14), (3.30) and (3.32), we obtain

$$
\begin{aligned}
&0 \le D^-(V(t_*)) \\
&\quad \le -a(t_*)|y(t_*)|e^{\lambda t_*} + \sum_{j=1}^{m} \beta_j(t_*)\big|x(t_* - \tau_j(t_*))e^{-\gamma_j(t_*)x(t_*-\tau_j(t_*))} \\
&\qquad\qquad - x^*(t_* - \tau_j(t_*))e^{-\gamma_j(t_*)x^*(t_*-\tau_j(t_*))}\big|e^{\lambda t_*} + \lambda|y(t_*)|e^{\lambda t_*} \\
&\quad = -[a(t_*) - \lambda]|y(t_*)|e^{\lambda t_*} + \sum_{j=1}^{m} \frac{\beta_j(t_*)}{\gamma_j(t_*)}\big|\gamma_j(t_*)x(t_* - \tau_j(t_*))e^{-\gamma_j(t_*)x(t_*-\tau_j(t_*))} \\
&\qquad\qquad - \gamma_j(t_*)x^*(t_* - \tau_j(t_*))e^{-\gamma_j(t_*)x^*(t_*-\tau_j(t_*))}\big|e^{\lambda t_*} \\
&\quad \le -[a(t_*) - \lambda]|y(t_*)|e^{\lambda t_*} + \sum_{j=1}^{m} \beta_j(t_*)\frac{1}{e^2}|y(t_* - \tau_j(t_*))|e^{\lambda(t_*-\tau_j(t_*))}e^{\lambda\tau_j(t_*)} \\
&\quad \le \left\{-[a(t_*) - \lambda] + \sum_{j=1}^{m} \beta_j(t_*)\frac{1}{e^2}e^{\lambda r}\right\}K.
\end{aligned}
$$

Thus,

$$0 \le -[a(t_*) - \lambda] + \sum_{j=1}^{m} \beta_j(t_*)\frac{1}{e^2}e^{\lambda r},$$

which contradicts (2.8). Hence, (3.31) holds. It follows that $|y(t)| < Ke^{-\lambda t}$ for all $t > t_\varphi + r$. This completes the proof.

**4. Example.** In this section, we present an example to check the validity of our results obtained in the previous sections.

Example 4.1. Consider the following Nicholson blowflies model with a nonlinear density-dependent mortality term:

$$x'(t) = -0.4040326x(t) + \frac{100 + \cos\sqrt{3}t}{100 + \sin\sqrt{7}t}x(t - 2e^{\sin^4 t})e^{-x(t-2e^{\sin^4 t})}. \tag{4.1}$$

Obviously,

$$r = 2e, \quad a = 0.4040326, \quad \beta_1^- = \frac{99}{101}, \quad \beta_j^+ = \frac{101}{99}, \quad \gamma_1^- = \gamma_1^+ = 1.$$

From (1.7)–(1.9), we obtain

$$\kappa \approx 0.7215355, \qquad \tilde{\kappa} \approx 1.342276.$$

Letting $M = 1.203432$, we get

$$
\begin{aligned}
aM &= 0.4040326 \times 1.203432 \approx 0.4862258, \\
\frac{\beta_1^+}{\gamma_1^-}\frac{1}{e} &= \frac{101}{99}\frac{1}{e} \approx 0.3753113,
\end{aligned}
$$

$$\frac{\beta_1^-}{\gamma_1^+}e^{-\kappa} = \frac{99}{101}e^{-\kappa} \approx \frac{99}{101}e^{-0.7215355} \approx 0.4763816,$$
$$\beta_1^+\frac{1}{e^2} = \frac{101}{99}\frac{1}{e^2} \approx 0.1380693,$$

which implies that the Nicholson blowflies model (4.1) satisfies (2.1) and (2.6). Hence, from Theorem 3.2, equation (4.1) has exactly one positive almost periodic solution $x^*(t)$. Moreover, $x^*(t)$ is globally exponentially stable. This fact is verified by the numerical simulation in Figure 1.

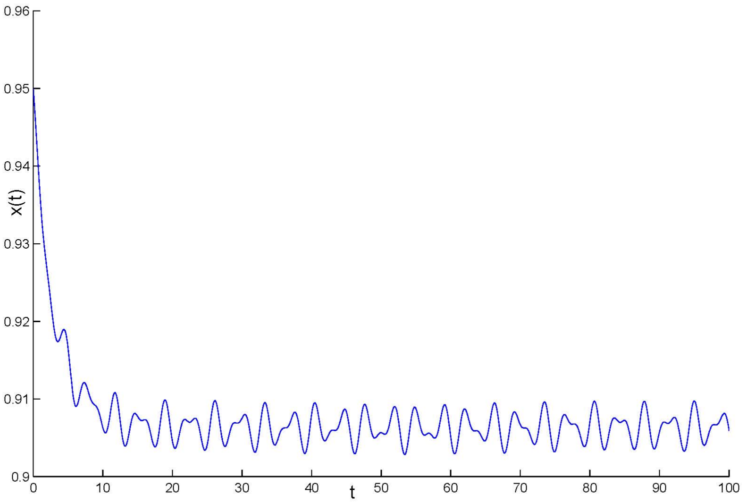


Fig. 1. Numerical solution $x(t)$ of equation (4.1) for the initial value $\varphi(s) \equiv 0.95$, $s \in [-2e, 0]$

Remark 4.2. For all we know, there is no research on the global exponential stability of positive almost periodic solutions of Nicholson's blowflies model with multiple time-varying delays. Thus, the results in [C, CW, HDH, LD, L1, S1, W1] and [A, CL, H, L2, W2, WWC] cannot be applicable to prove that all solutions of (4.1) converge exponentially to the positive almost periodic solution. Moreover, in this present paper, we employ a novel proof to establish some criteria to guarantee the global exponential stability of almost periodic solutions for delayed Nicholson's blowflies model.

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