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A result on the comparison principle for the log canonical threshold of plurisubharmonic functions

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Abstract. We prove a comparison principle for the log canonical threshold of plurisubharmonic functions under an assumption on complex Monge–Ampère measures.

1. Introduction. Let Ω be a domain in $\mathbb{C}^n$, $x\in\Omega$ and u be a plurisubharmonic function on Ω (briefly, psh). Demailly and Kollár [DeKo] introduced and investigated the *log canonical threshold* of u at x, defined as follows:

$$c_u(x) = \sup\{c>0 : e^{-2cu} \text{ is } L^1 \text{ on a neighbourhood of } x\}.$$

Note that $c_u(x)$ measures the singularity of u at x. Demailly and Kollár proved the lower semicontinuity of $x\mapsto c_u(x)$ in the holomorphic Zariski topology and established a relation between the Lelong number $\nu_u(x)$ and $c_u(x)$:

$$\frac{1}{\nu_u(x)} \le c_u(x) \le \frac{n}{\nu_u(x)}$$

(see [DeKo, 1.4]).

Recently, lower estimates and the comparison principle for this quantity have been studied. Namely, Demailly and Hiep Hoang Pham proved that if Ω is a bounded hyperconvex domain in $\mathbb{C}^n$, $0\in\Omega$ and $u\in\widetilde{\mathcal{E}}(\Omega)$ then

$$c_u(0) \ge \sum_{j=0}^{n-1} \frac{e_j(u)}{e_{j+1}(u)},$$

where $e_j(u) = \int_{\{0\}} (dd^c u)^j \wedge (dd^c \log\|z\|)^{n-j}$ is the Lelong number of $(dd^c u)^j$ at 0 (see [DP, Theorem 1.5]) and $\widetilde{\mathcal{E}}(\Omega)$ is the set of psh functions which, on a neighbourhood U of an arbitrary point $x_0\in\Omega$, are equal to a sum $u+v$

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with $u \in \mathcal{E}(U)$ and $v \in C^\infty(U)$. Here $\mathcal{E}(U)$ is the class of psh functions on U introduced and investigated in [Ce1]; it is recalled in Section 2. Moreover, Demailly and Hiep Hoang Pham showed that the above estimate is sharp.

Next, Hiep Hoang Pham [P] gave a comparison principle for c_u. He proved that if Ω is a domain in $\mathbb{C}^n$ and $\{\Omega_j\}_{j\geq 1}$ is a sequence of smooth domains with $\Omega \Supset \Omega_1 \Supset \Omega_2 \Supset \cdots$, $\bigcap_{j=1}^\infty \Omega_j = \{0\}$, and u, v are plurisubharmonic functions on Ω such that $u \geq v$ on $\partial\Omega_j$ for $j \geq 1$, then $c_u(0) \geq c_v(0)$ (see [P, Theorem 1.1]).

Continuing this study, by relying on the solvability of the complex Monge–Ampère equations for measures carrying pluripolar sets, proved in [ACCP], we give the following comparison principle for the log canonical threshold of plurisubharmonic functions:

Main Theorem. *Assume that Ω is a domain in $\mathbb{C}^n$, $0 \in \Omega$ and $u, v \in \mathrm{PSH}^-(\Omega)$ are such that*

$$\int_{\{0\}} (dd^c \max(u, v, \varphi))^n = \int_{\{0\}} (dd^c \max(u, \varphi))^n$$

for every $\varphi \in \mathrm{PSH}^-(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega \setminus \{0\})$. Then $c_u(0) \geq c_v(0)$.

The note is organized as follows. In Section 2 we recall the classes $\mathcal{F}(\Omega)$, $\mathcal{E}(\Omega)$ for Ω being a bounded hyperconvex domain in $\mathbb{C}^n$, and some related results. Section 3 is devoted to the proof of the above comparison principle.

2. Background. The elements of pluripotential theory that will be used throughout this paper can be found in [BT], [Ce1], [Ce2], [De], [Kl], [Ko1], [Ko2], while elements of the theory of log canonical thresholds can be found in [DeKo] and [DP].

In this paper we denote by $\mathrm{PSH}^-(\Omega)$ the set of negative plurisubharmonic functions on a domain Ω in $\mathbb{C}^n$. Now we recall the definition of the classes $\mathcal{F}(\Omega)$, $\mathcal{E}(\Omega)$ and some related results. For more details we refer the readers to the papers of Cegrell [Ce1], [Ce2].

2.1. Let Ω be a bounded *hyperconvex* domain in $\mathbb{C}^n$, that is, a bounded domain in $\mathbb{C}^n$ for which there exists a negative plurisubharmonic function ϱ on Ω such that $\Omega_c = \{z \in \Omega : \varrho(z) < c\} \Subset \Omega$ for all $c < 0$. Following [Ce1] we define

$$\mathcal{E}_0 = \mathcal{E}_0(\Omega) = \Big\{\varphi \in \mathrm{PSH}^-(\Omega) \cap L^\infty(\Omega) : \lim_{z\to\xi} \varphi(z) = 0, \\ \forall \xi \in \partial\Omega,\ \int_\Omega (dd^c\varphi)^n < \infty\Big\},$$

$$\mathcal{F} = \mathcal{F}(\Omega) = \Big\{\varphi \in \mathrm{PSH}^-(\Omega) : \exists \mathcal{E}_0 \ni \varphi_j \searrow \varphi,\ \sup_j \int_\Omega (dd^c\varphi_j)^n < \infty\Big\},$$

$$\mathcal{E} = \mathcal{E}(\Omega) = \Big\{\varphi \in \mathrm{PSH}^-(\Omega) : \forall z_0 \in \Omega, \exists \text{ a neighbourhood } \omega \ni z_0, \\ \mathcal{E}_0 \ni \varphi_j \searrow \varphi \text{ on } \omega, \sup_j \int_\Omega (dd^c\varphi_j)^n < \infty\Big\}.$$

The following inclusions are clear: $\mathcal{E}_0 \subset \mathcal{F} \subset \mathcal{E}$.

As in [Ce1], we note that if $u \in \mathcal{F}(\Omega)$ then $(dd^c u)^n$ is a positive Radon measure on Ω and $\int_\Omega (dd^c u)^n < \infty$.

We denote by $\mathcal{F}^a(\Omega)$ the subclass of $u \in \mathcal{F}(\Omega)$ such that $(dd^c u)^n$ vanishes on all pluripolar sets of Ω.

2.2. We recall the following result of [Ce1] on $\mathcal{F}^a(\Omega)$. Assume that μ is a positive Borel measure vanishing on all pluripolar sets of Ω with $\mu(\Omega) < \infty$. Then there exists a unique function $u \in \mathcal{F}^a(\Omega)$ such that $(dd^c u)^n = \mu$ (see [Ce1, Lemma 5.14]).

2.3. Next, we recall the following comparison principle from [Ce1]. Let $u \in \mathcal{F}^a(\Omega)$ and $v \in \mathcal{E}(\Omega)$ with $(dd^c v)^n \geq (dd^c u)^n$. Then $u \geq v$ on Ω (see [Ce1, Theorem 5.15]).

3. Comparison principle for the log canonical threshold. In this section we give the proof of the Main Theorem. For this we need the following auxiliary lemmas.

LEMMA 3.1. *Assume that Ω is a domain in $\mathbb{C}^n$, $0 \in \Omega$ and $u, v, \varphi \in \mathrm{PSH}^-(\Omega)$ with $u \geq v \geq u + \varphi$ and $c_\varphi(0) = \infty$. Then $c_u(0) = c_v(0)$.*

Proof. First we prove that $c_u(0) \leq c_{u+\varphi}(0)$. Indeed, let $c \in (0, c_u(0))$. Choose $q > 1$ such that $qc \in (0, c_u(0))$. Let $p > 1$ be such that $1/q + 1/p = 1$. Take $r > 0$ such that

$$\int_{\mathbb{B}(0,r)} e^{-2cqu}\, dV < \infty \quad \text{and} \quad \int_{\mathbb{B}(0,r)} e^{-2pc\varphi}\, dV < \infty.$$

By the Hölder inequality, we have

$$\int_{\mathbb{B}(0,r)} e^{-2c(u+\varphi)}\, dV \leq \Big(\int_{\mathbb{B}(0,r)} e^{-2cqu}\, dV\Big)^{1/q} \cdot \Big(\int_{\mathbb{B}(0,r)} e^{-2cp\varphi}\, dV\Big)^{1/p} < \infty.$$

Hence, $c \leq c_{u+\varphi}(0)$ and we are done.

Now, since $u \geq v \geq u + \varphi$ we have $c_u(0) \geq c_v(0) \geq c_{u+\varphi}(0) \geq c_u(0)$, and the desired conclusion follows. ∎

Next, by using some results of [ACCP] on the solvability of the complex Monge–Ampère equations for measures carrying pluripolar sets we will obtain the following.

Lemma 3.2. *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$, $0 \in \Omega$ and $u \in \mathcal{F}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega \setminus \{0\})$. Then there exist $\widetilde{u} \in \mathcal{F}(\Omega)$ and $\phi \in \mathcal{F}^a(\Omega)$ such that $u \le \phi$, $\widetilde{u} + \phi \le u \le \widetilde{u}$, $(dd^c\widetilde{u})^n = 1_{\{0\}}(dd^c u)^n$ and $(dd^c\phi)^n = 1_{\Omega\setminus\{0\}}(dd^c u)^n$.*

Proof. We can assume that $\int_{\{0\}}(dd^c u)^n > 0$. Put

$$\widetilde{u} = (\sup\{\varphi \in \mathrm{PSH}^-(\Omega) : \varphi = u \text{ on a neighbourhood } U \text{ of } 0\})^*.$$

By [ACCP, Lemma 4.3] we have $\widetilde{u} \in \mathcal{F}(\Omega)$, $u \le \widetilde{u}$ and $(dd^c\widetilde{u})^n = 1_{\{0\}}(dd^c u)^n$. On the other hand, by hypothesis the measure $\mu = 1_{\Omega\setminus\{0\}}(dd^c u)^n$ vanishes on all pluripolar sets of Ω and $\mu(\Omega) \le (dd^c u)^n(\Omega) < \infty$, so [Ce1, Lemma 5.14] implies that there exists $\phi \in \mathcal{F}^a(\Omega)$ such that $(dd^c\phi)^n = 1_{\Omega\setminus\{0\}}(dd^c u)^n$. By the comparison principle, we deduce that $u \le \phi$.

We now prove that $\widetilde{u} + \phi \le u$. From the definition of $\widetilde{u}$ we can choose $\varepsilon_j > 0$ and $u_j \in \mathcal{F}(\Omega)$ such that $\varepsilon_j \searrow 0$, $u_j \nearrow \widetilde{u}$ and $u_j = u$ on $\mathbb{B}(0, \varepsilon_j)$. Let $0 < \delta_j < \varepsilon_j$. Set $a_j = \inf_{\mathbb{B}(0,\varepsilon_j)\setminus\mathbb{B}(0,\delta_j)} u$ and

$$v_j = \begin{cases} \max(u, a_j) & \text{on } \mathbb{B}(0, \varepsilon_j), \\ u & \text{on } \Omega \setminus \mathbb{B}(0, \varepsilon_j). \end{cases}$$

We have $v_j \in \mathcal{F}^a(\Omega)$ and $v_j = u$ on $\Omega \setminus \mathbb{B}(0, \delta_j)$. Hence,

$$(dd^c(\phi + \max(u_j, v_j)))^n \ge (dd^c\phi)^n + (dd^c \max(u_j, v_j))^n \ge (dd^c v_j)^n.$$

Again by the comparison principle, we get $\phi + u_j \le \phi + \max(u_j, v_j) \le v_j$. It follows that $\phi + u_j \le u$ on $\Omega \setminus \mathbb{B}(0, \varepsilon_k)$ for every $j \ge k$. Letting $j \to \infty$ and then $k \to \infty$ we infer that $\phi + \widetilde{u} \le u$ on Ω, as desired. ∎

Proof of Main Theorem. We consider two cases.

Case 1: Ω is a bounded hyperconvex domain, $0 \in \Omega$ and $u, v \in \mathcal{F}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega \setminus \{0\})$. Put $\varphi = u + v$. Then $\varphi \in \mathrm{PSH}^-(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega \setminus \{0\})$, so by hypothesis

$$\int_{\{0\}} (dd^c \max(u, v))^n = \int_{\{0\}} (dd^c u)^n.$$

By Lemma 3.2 there exist $\widetilde{u} \in \mathcal{F}(\Omega)$ and $\phi \in \mathcal{F}^a(\Omega)$ such that $u \le \phi$, $\widetilde{u} + \phi \le u \le \widetilde{u}$, $(dd^c\widetilde{u})^n = 1_{\{0\}}(dd^c u)^n$ and $(dd^c\phi)^n = 1_{\Omega\setminus\{0\}}(dd^c u)^n$. Since $\int_{\{0\}}(dd^c\phi)^n = 0$, by [Ce1, Corollary 5.7] we get $c_\phi(0) = \infty$. Therefore, Lemma 3.1 implies that

$$c_u(0) = c_{\widetilde{u}}(0).$$

Now, since $|\max(\widetilde{u}, v) - \max(u, v)| \le -\phi$, by [ACCP, Lemma 4.12] we get

$$\int_{\{0\}} (dd^c \max(\widetilde{u}, v))^n = \int_{\{0\}} (dd^c \max(u, v))^n = \int_{\{0\}} (dd^c u)^n = \int_{\{0\}} (dd^c\widetilde{u})^n.$$

Moreover, again by Lemma 3.2 there exist $\widetilde{v} \in \mathcal{F}(\Omega)$ and $\psi \in \mathcal{F}^a(\Omega)$ such that $v \le \psi$, $\widetilde{v} + \psi \le v \le \widetilde{v}$, $(dd^c\widetilde{v})^n = 1_{\{0\}}(dd^c v)^n$ and $(dd^c\psi)^n =$

$1_{\Omega\setminus\{0\}}(dd^cv)^n$. Similarly, we also have

$$c_v(0) = c_{\widetilde{v}}(0).$$

Now, since $|\max(\widetilde{u},\widetilde{v}) - \max(\widetilde{u},v)| \le -\psi$, again by [ACCP, Lemma 4.12] we get

$$\int_{\{0\}} (dd^c \max(\widetilde{u},\widetilde{v}))^n = \int_{\{0\}} (dd^c \max(\widetilde{u},v))^n = \int_{\{0\}} (dd^c\widetilde{u})^n.$$

Since

$$\begin{aligned}\int_{\Omega}(dd^c\widetilde{u})^n = \int_{\{0\}} (dd^c\widetilde{u})^n &= \int_{\{0\}} (dd^c \max(\widetilde{u},\widetilde{v}))^n \\ &\le \int_{\Omega}(dd^c \max(\widetilde{u},\widetilde{v}))^n \le \int_{\Omega}(dd^c\widetilde{u})^n,\end{aligned}$$

and $(dd^c\widetilde{u})^n \le (dd^c \max(\widetilde{u},\widetilde{v}))^n$, it follows that $(dd^c\widetilde{u})^n = (dd^c \max(\widetilde{u},\widetilde{v}))^n$. Hence, [Ce2, Theorem 3.15] implies that $\max(\widetilde{u},\widetilde{v}) = \widetilde{u}$. Thus, $\widetilde{v} \le \widetilde{u}$, and hence

$$c_u(0) = c_{\widetilde{u}}(0) \ge c_{\widetilde{v}}(0) = c_v(0).$$

Case 2: Ω is a domain in $\mathbb{C}^n$, $0 \in \Omega$ and $u, v \in \mathrm{PSH}^-(\Omega)$. Without loss of generality we can assume that $\Omega = \mathbb{B}(0,1)$. Let $\varphi_j = j\log|z| \in \mathcal{F}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega\setminus\{0\})$. Put $u_j = \max(u,\varphi_j)$, $v_j = \max(v,\varphi_j)$. It is easy to see that $u_j, v_j \in \mathcal{F}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega\setminus\{0\})$ and

$$\int_{\{0\}} (dd^c \max(u_j,v_j,\varphi))^n = \int_{\{0\}} (dd^c \max(u_j,\varphi))^n$$

for every $\varphi \in \mathrm{PSH}^-(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega\setminus\{0\})$, so by Case 1 we have $c_{u_j}(0) \ge c_{v_j}(0)$. Now, [P, Lemma 2.1] implies that

$$c_u(0) = \lim_{j\to\infty} c_{u_j}(0) \ge \lim_{j\to\infty} c_{v_j}(0) = c_v(0). \ \blacksquare$$

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