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Some properties of para-Kähler–Walker metrics

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Abstract. A Walker 4-manifold is a pseudo-Riemannian manifold (M_4, g) of neutral signature, which admits a field of parallel null 2-planes. We study almost paracomplex structures on 4-dimensional para-Kähler–Walker manifolds. In particular, we obtain conditions under which these almost paracomplex structures are integrable, and the corresponding para-Kähler forms are symplectic. We also show that Petean's example of a nonflat indefinite Kähler-Einstein 4-manifold is a special case of our constructions.

1. Introduction. Let M_{2n} be a Riemannian manifold with a *neutral* metric, i.e., a pseudo-Riemannian metric g of signature (n, n). We denote by $\Im_q^p(M_{2n})$ the set of all tensor fields of type (p, q) on M_{2n}. In this paper, all manifolds, tensor fields and connections are assumed to be differentiable and of class C^∞.

An *almost paracomplex manifold* is an almost product manifold (M_{2n}, φ) with $\varphi^2 = \mathrm{id}$ such that the two eigenbundles T^+M_{2n} and T^-M_{2n} associated respectively with the eigenvalues $+1$ and -1 of φ have the same rank. Note that the dimension of an almost paracomplex manifold is necessarily even. Considering the paracomplex structure φ, we obtain the set of affinors $\{\mathrm{id}, \varphi\}$ with $\varphi^2 = \mathrm{id}$ on M_{2n}. They form a basis of a representation of an algebra of order 2 over the field $\mathbb{R}$ of real numbers. This algebra is called the *algebra of paracomplex* (or *double*) *numbers*, and is denoted by $\mathbb{R}(j) = \{a_0 + ja_1 : j^2 = 1 \text{ and } a_0, a_1 \in \mathbb{R}\}$. Obviously, $\mathbb{R}(j)$ is associative, commutative and with unity, i.e., it admits a principal unit 1. The canonical basis of this algebra is $\{1, j\}$.

Let (M_{2n}, φ) be an almost paracomplex manifold with almost paracomplex structure φ. The integrability of φ is equivalent to the vanishing of the Nijenhuis tensor

$$N_\varphi(X, Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + [X, Y].$$

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This structure is called *integrable* if the matrix $\varphi = (\varphi^i_j)$ is constant in a certain holonomic natural frame in a neighborhood U_x of every point $x \in M_{2n}$. A necessary and sufficient condition for an almost paracomplex structure to be integrable is that there exists a torsion free linear connection such that $\nabla\varphi = 0$.

Let (M_{2n}, φ, g) be an almost para-Hermitian manifold with almost paracomplex structure φ and pseudo-Riemannian metric tensor field g. Then

$$\varphi^2 = I \quad \text{and} \quad g(\varphi X, Y) = -g(X, \varphi Y)$$

for any vector fields X and Y on M_{2n}. If φ is integrable, then (M_{2n}, φ, g) is called a *para-Hermitian manifold.*

If g is a para-Hermitian pseudo-Riemannian metric, then $\omega(X, Y) = g(\varphi X, Y)$ is a 2-form, called the *para-Kähler form* of g. A para-Hermitian pseudo-Riemannian metric g is called *para-Kähler* if its para-Kähler form is closed, i.e., $d\omega = 0$. Here, the triple (g, φ, ω) is called an *almost para-Kähler structure.* Moreover, if φ is integrable, then (g, φ, ω) is called a *para-Kähler structure.* In particular, if g is neither positive nor negative definite, it is called an *indefinite para-Kähler metric.*

The goal of this paper is to study certain almost paracomplex structures φ on four-dimensional Walker manifolds, and their associated opposite almost paracomplex structures φ'. In particular, we are interested in the integrability of φ and φ', and whether the corresponding para-Kähler forms ω, ω' are symplectic or not.

2. Para-Kähler walker metrics in dimension four

2.1. Walker metric g. A neutral metric g on a 4-manifold M_4 is called a *Walker metric* if there exists a 2-dimensional null distribution D on M_4 that is parallel with respect to g. For such metrics, a canonical form was obtained by Walker [W], who showed that there exist coordinates (x^1, x^2, x^3, x^4) in which the metric g is expressed as

$$(g_{ij}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a & c \\ 0 & 1 & c & b \end{pmatrix} \tag{2.1}$$

for some functions a, b and c depending on the coordinates (x^1, x^2, x^3, x^4). Note that $D = \text{span}\{\partial_1, \partial_2\}$ where $\partial_i = \partial/\partial x^i$. We shall use the abbreviation $\partial p(x^1, x^2, x^3, x^4)/\partial x^i = \partial p/\partial x^i = p_i$ for any function p with $i = 1, 2, 3, 4$. We refer to [GT] for an application of such a 4-dimensional Walker metric. We note that Walker 4-manifolds have been intensively studied in the literature—see [BCHM, D–V1, D–V2, GHKM, GT, M1, M2, SIA, SI], for example.

2.2. Almost paracomplex structure φ. Let φ be an almost paracomplex structure on a Walker manifold M_4 which satisfies

(i) $\varphi^2 = I$,
(ii) $g(\varphi X, Y) = -g(X, \varphi Y)$ (Hermitian property),
(iii) $\varphi\partial_1 = \partial_1,\ \varphi\partial_2 = -\partial_2$.

It can easily be seen that these three properties define φ uniquely:

$$\begin{cases} \varphi\partial_1 = \partial_1, \\ \varphi\partial_2 = -\partial_2, \\ \varphi\partial_3 = a\partial_1 - \partial_3, \\ \varphi\partial_4 = -b\partial_2 + \partial_4, \end{cases}$$

and φ has local components

$$(\varphi_j^i) = \begin{pmatrix} 1 & 0 & a & 0 \\ 0 & -1 & 0 & -b \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with respect to the natural frame $\{\partial_1, \partial_2, \partial_3, \partial_4\}$.

In order to simplify our calculations in Section 4, and verification of the example in Section 5, we shall restrict ourselves to almost paracomplex structures with $a = b$,

$$(\varphi_j^i) = \begin{pmatrix} 1 & 0 & a & 0 \\ 0 & -1 & 0 & -a \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.2}$$

In this case, the metric g in (2.1) takes the form

$$(g_{ij}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a & c \\ 0 & 1 & c & a \end{pmatrix}. \tag{2.3}$$

By writing $\varphi\partial_i$ in the form $\varphi\partial_i = \sum_{j=1}^4 \varphi_i^j \partial_j$, and using (2.2), we obtain

$$\varphi_1^1 = -\varphi_2^2 = -\varphi_3^3 = \varphi_4^4 = 1 \quad \text{and} \quad \varphi_3^1 = -\varphi_4^2 = a.$$

The triple (M_4, φ, g) is called an *almost para-Hermitian–Walker manifold*. Following the terminology of [D–V1, D–V2, M1, M2, SI], we call φ a *proper almost paracomplex structure*.

2.3. Para-Kähler–Walker structures. Let (M_4, φ, g) be an almost para-Hermitian–Walker manifold with g given in (2.3) and φ given in (2.2). If $d\omega = 0$, then (M_4, φ, g) is called an *almost para-Kähler–Walker manifold*. Then we can define a para-Kähler structure in terms of g and φ by $\omega(X, Y) = g(\varphi X, Y)$, or explicitly

$$\omega = dx^1 \wedge dx^3 - dx^2 \wedge dx^4 - c dx^3 \wedge dx^4. \tag{2.4}$$

It is clear that ω is independent of the function a. We are interested in the case for which ω is symplectic, i.e., $d\omega = 0$.

THEOREM 2.1. *An almost para-Hermitian–Walker manifold (M_4, φ, g) is an almost para-Kähler–Walker manifold, i.e., $d\omega = 0$, if and only if c is independent of x^1 and x^2. In fact, c then satisfies the following PDEs:*

$$c_1 = c_2 = 0. \tag{2.5}$$

Proof. These conditions follow directly from $d\omega = dc \wedge dx^3 \wedge dx^4$. ∎

If c is independent of x^1 and x^2, i.e., $c = c(x^3, x^4)$, then c satisfies the PDEs in (2.5), and therefore the para-Kähler form becomes

$$\omega = dx^1 \wedge dx^3 - dx^2 \wedge dx^4 - c(x^3, x^4) dx^3 \wedge dx^4,$$

which is clearly closed.

The almost paracomplex structure φ on an almost para-Kähler–Walker manifold is integrable if and only if

$$(N_\varphi)^i_{jk} = \varphi^m_j \partial_m \varphi^i_k - \varphi^m_k \partial_m \varphi^i_j - \varphi^i_m \partial_j \varphi^m_k + \varphi^i_m \partial_k \varphi^m_j = 0. \tag{2.6}$$

By explicit calculations for the proper almost paracomplex structure φ in (2.2), the nonzero components of the Nijenhuis tensor are as follows:

$$N^2_{14} = -N^2_{41} = -2a_1, \quad N^1_{23} = -N^1_{32} = -2a_2,$$
$$N^1_{34} = -N^1_{43} = aa_2, \quad N^2_{34} = -N^2_{43} = -aa_1.$$

Therefore, we can state the following theorem.

THEOREM 2.2. *The proper almost paracomplex structure φ given in* (2.2) *is integrable if and only if the following PDEs hold:*

$$a_1 = a_2 = 0. \tag{2.7}$$

From (2.5) and (2.7), we have the following para-Kähler condition:

THEOREM 2.3. *The triple (g, φ, ω) with g in* (2.3) *and φ in* (2.2) *is a para-Kähler–Walker structure if and only if the following PDEs hold:*

$$a_1 = a_2 = c_1 = c_2 = 0. \tag{2.8}$$

COROLLARY 2.4. *If*

$$(\varphi_j^i) = \begin{pmatrix} 1 & 0 & a(x^3, x^4) & 0 \\ 0 & -1 & 0 & -a(x^3, x^4) \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$(g_{ij}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a(x^3, x^4) & c(x^3, x^4) \\ 0 & 1 & c(x^3, x^4) & a(x^3, x^4) \end{pmatrix},$$

then the triple (g, φ, ω) *is always para-Kähler–Walker.*

REMARK 2.5. It follows from Theorem 2.3 that if (g, φ, ω) is a para-Kähler–Walker structure, then we have $a = a(x^3, x^4)$ and $c = c(x^3, x^4)$. In particular, if $c(x^3, x^4) = 0$, then the metric g is of the form

$$(g_{ij}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a(x^3, x^4) & 0 \\ 0 & 1 & 0 & a(x^3, x^4) \end{pmatrix}.$$

This is the same as Petean's example [P] for the paracomplex case (see [M2] for Petean's example for the complex case).

3. Opposite almost paracomplex structure φ' and opposite para-Kähler form ω'. Let (M_4, g) be a four-dimensional manifold of signature $(2, 2)$. Suppose that g is anti-invariant under both φ and φ'. If φ and φ' satisfy

$$\varphi^2 = \varphi'^2 = 1, \qquad \varphi\varphi' = \varphi'\varphi,$$
$$g(\varphi X, \varphi Y) = -g(X, Y), \qquad g(\varphi' X, \varphi' Y) = -g(X, Y)$$

for any $X, Y \in \Im_0^1(M_4)$, then φ is called an *almost paracomplex structure*, and φ' is called an *opposite almost paracomplex structure.*

For a Walker manifold M_4 with a proper almost paracomplex structure φ, the opposite almost paracomplex structure φ' satisfies

$$\varphi'\partial_1 = -\left(\alpha + 2\theta\frac{c}{a}\right)\partial_1 - \theta\partial_2 + 2\theta\frac{1}{a}\partial_4,$$

$$\begin{aligned}
\varphi'\partial_2 &= \theta\partial_1 - \alpha\partial_2 - 2\theta\frac{1}{a}\partial_3,\\
\varphi'\partial_3 &= -(\alpha a + \theta c)\partial_1 - \frac{1}{2\theta}a(1+\theta^2-\alpha^2)\partial_2 + \alpha\partial_3 + \theta\partial_4,\\
\varphi'\partial_4 &= \left(\frac{1}{2\theta}a + \frac{\theta}{2}a + \frac{\alpha^2}{2\theta}a - 2\alpha c - 2\theta\frac{c^2}{a}\right)\partial_1 - a\left(\alpha + \theta\frac{c}{a}\right)\partial_2\\
&\quad - \theta\partial_3 + \left(\alpha + 2\theta\frac{c}{a}\right)\partial_4,
\end{aligned}$$

where α and nonzero θ are two parameters.

Next, we get an explicit form of φ' by fixing $\theta = 1$ and $\alpha = 0$ (only for simplicity), as follows:

$$\begin{aligned}
\varphi'\partial_1 &= -\frac{2c}{a}\partial_1 - \partial_2 + \frac{2}{a}\partial_4,\\
\varphi'\partial_2 &= \partial_1 - \frac{2}{a}\partial_3,\\
\varphi'\partial_3 &= -c\partial_1 - a\partial_2 + \partial_4,\\
\varphi'\partial_4 &= \left(a - \frac{2c^2}{a}\right)\partial_1 - c\partial_2 - \partial_3 + \frac{2c}{a}\partial_4.
\end{aligned} \tag{3.1}$$

Thus, φ' has local components

$$(\varphi'^i_j) = \begin{pmatrix} -2c/a & 1 & -c & a-2c^2/a\\ -1 & 0 & -a & -c\\ 0 & -2/a & 0 & -1\\ 2/a & 0 & 1 & 2c/a \end{pmatrix}. \tag{3.2}$$

3.1. Opposite para-Kähler form ω'. Let (M_4, φ', g) be an opposite almost para-Hermitian–Walker manifold with g of the form (2.3), and φ' of the form (3.2). We can define an opposite para-Kähler form $\omega'(X,Y) = g(\varphi' X, Y)$, whose explicit form is given by

$$\omega' = \frac{2}{a}dx^1\wedge dx^2 + dx^1\wedge dx^4 - dx^2\wedge dx^3 - \frac{2c}{a}dx^2\wedge dx^4. \tag{3.3}$$

From (3.3), we have the following theorem:

THEOREM 3.1. *Let (M_4, φ', g) be an opposite almost para-Hermitian–Walker manifold. Then it is an opposite almost para-Kähler–Walker manifold, i.e., $d\omega' = 0$, if and only if the following PDEs hold:*

$$a_3 = c_3 = 0 \quad \textit{and} \quad a_4 + ac_1 - ca_1 = 0. \tag{3.4}$$

Proof. These PDEs can be obtained from the differential of ω':

$$d\omega' = -\frac{2a_3}{a^2}dx^1\wedge dx^2\wedge dx^3 - \frac{2}{a^2}(a_4 + ac_1 - ca_1)dx^1\wedge dx^2\wedge dx^4 + \frac{2}{a^2}(ac_3 - ca_3)dx^2\wedge dx^3\wedge dx^4. \blacksquare$$

3.2. Integrability of φ'. The opposite almost paracomplex structure φ' in (3.2) is integrable if the PDEs in (2.6) hold for φ'^i_j in (3.2). Then we have the following theorem:

THEOREM 3.2. *The opposite almost paracomplex structure φ' in* (3.2) *is integrable if and only if the following PDEs hold:*

$$\begin{aligned} &a_4 - ca_1 = 0, \quad a_3 + ac_2 = 0,\\ &a_3 - aa_1 = 0, \quad ac_1 - aa_2 - 2c_3 = 0. \end{aligned} \tag{3.5}$$

3.3. Opposite para-Kähler–Walker structure. The triple (g, φ', ω') is called an *opposite para-Kähler structure* if $d\omega' = 0$ and $N_{\varphi'} = 0$. The following theorem follows from (3.4) and (3.5).

THEOREM 3.3. *The triple (g, φ', ω') with g in* (2.3) *and φ' in* (3.2) *is an opposite para-Kähler–Walker structure if and only if the following PDEs hold:*

$$a = \textit{constant} \quad \textit{and} \quad c_1 = c_2 = c_3 = 0.$$

4. Curvatures of a Walker metric for a special case. Let R, r and S be the curvature tensor, Ricci tensor and scalar curvature of the Walker metric g in (2.3), respectively. Then the components of R, r and S are as follows (see [M2, Appendices A–C]):

$$\begin{aligned} &R_{1313} = R_{1414} = -\tfrac{1}{2}a_{11}, \quad R_{1314} = -\tfrac{1}{2}c_{11}, \quad R_{1323} = R_{1424} = -\tfrac{1}{2}a_{12},\\ &R_{1324} = R_{1423} = -\tfrac{1}{2}c_{12}, \quad R_{1334} = \tfrac{1}{2}a_{14} - \tfrac{1}{2}c_{13} - \tfrac{1}{4}a_1a_2 + \tfrac{1}{4}c_1c_2,\\ &R_{1434} = \tfrac{1}{2}c_{14} - \tfrac{1}{2}a_{13} - \tfrac{1}{4}(c_1)^2 + \tfrac{1}{4}(a_1)^2 - \tfrac{1}{4}a_1c_2 + \tfrac{1}{4}a_2c_1,\\ &R_{2323} = R_{2424} = -\tfrac{1}{2}a_{22}, \quad R_{2324} = -\tfrac{1}{2}c_{22},\\ &R_{2334} = \tfrac{1}{2}a_{24} - \tfrac{1}{2}c_{23} - \tfrac{1}{4}a_1c_2 + \tfrac{1}{4}a_2c_1 - \tfrac{1}{4}(a_2)^2 + \tfrac{1}{4}(c_2)^2,\\ &R_{2434} = \tfrac{1}{2}c_{24} - \tfrac{1}{2}a_{23} - \tfrac{1}{4}c_1c_2 + \tfrac{1}{4}a_1a_2,\\ &R_{2334} = c_{34} - \tfrac{1}{2}a_{44} - \tfrac{1}{2}a_{33} - \tfrac{1}{4}a(c_1)^2 + \tfrac{1}{4}a(a_1)^2 + \tfrac{1}{2}ca_1a_2 - \tfrac{1}{2}cc_1c_2\\ &\qquad + \tfrac{1}{2}a_4c_1 + \tfrac{1}{2}a_1c_4 + \tfrac{1}{4}a(a_2)^2 - \tfrac{1}{4}a(c_2)^2 - \tfrac{1}{2}a_3c_2 + \tfrac{1}{2}a_2c_3, \end{aligned} \tag{4.1}$$

(4.2)

$$r_{13} = \tfrac{1}{2}a_{11} + \tfrac{1}{2}c_{12}, \quad r_{14} = \tfrac{1}{2}a_{12} + \tfrac{1}{2}c_{11},$$
$$r_{23} = \tfrac{1}{2}a_{12} + \tfrac{1}{2}c_{22}, \quad r_{24} = \tfrac{1}{2}a_{22} + \tfrac{1}{2}c_{12},$$
$$r_{33} = \tfrac{1}{2}aa_{11} + ca_{12} + \tfrac{1}{2}aa_{22} - a_{24} + c_{23} - \tfrac{1}{2}a_2c_1 + \tfrac{1}{2}a_1c_2 + \tfrac{1}{2}(a_2)^2 - \tfrac{1}{2}(c_2)^2,$$
$$r_{34} = \tfrac{1}{2}ac_{11} + cc_{12} + \tfrac{1}{2}a_{14} - \tfrac{1}{2}c_{13} - \tfrac{1}{2}a_1a_2 + \tfrac{1}{2}c_1c_2 + \tfrac{1}{2}ac_{22} - \tfrac{1}{2}c_{24} + \tfrac{1}{2}a_{23},$$
$$r_{44} = \tfrac{1}{2}aa_{11} + ca_{12} + c_{14} - a_{13} - \tfrac{1}{2}(c_1)^2 + \tfrac{1}{2}(a_1)^2 - \tfrac{1}{2}a_1c_2 + \tfrac{1}{2}a_2c_1 + \tfrac{1}{2}aa_{22},$$

and

$$S = a_{11} + a_{22} + 2c_{12}. \tag{4.3}$$

It follows from (2.8) that all components of S and r in (4.2) and (4.3) vanish. Taking (2.8) and (4.1) into account, we have the following theorem:

THEOREM 4.1. *If M_4 is a Walker 4-manifold which consists of a para-Kähler–Walker structure (g, φ, ω) with g in (2.3) and φ in (2.2), then M_4 is both scalar flat and Ricci flat. Moreover, M_4 is flat if and only if the following PDE holds:*

$$2c_{34} - a_{33} - a_{44} = 0.$$

From (2.8) and (4.1), we have:

THEOREM 4.2. *Let M_4 be a Walker 4-manifold which consists of an opposite para-Kähler–Walker structure (g, φ', ω') with g in (2.3) and φ' in (3.2). Then M_4 is flat.*

5. Examples of indefinite Ricci flat almost para-Kähler non-para-Kähler 4-manifolds. Let (M_{2n}, φ, g) be an almost para-Hermitian manifold. We can now state the almost para-Hermitian version of Goldberg conjecture [M3]: if (GC1) M_{2n} is compact, (GC2) g is Einstein, and (GC3) the fundamental 2-form ω is closed, then φ must be integrable. In this section, we construct an example of a noncompact indefinite Ricci flat almost para-Kähler non-para-Kähler 4-manifold. This is an indefinite para-Kähler version of the example given in [NP]. For this purpose, we consider the metric g given by

$$(g_{ij}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a & 0 \\ 0 & 1 & 0 & a \end{pmatrix}. \tag{5.1}$$

This metric is defined by choosing $a = b$ and $c = 0$ in the generic form (2.1). From (2.4), we have $\omega = dx^1 \wedge dx^3 - dx^2 \wedge dx^4$. Thus ω is symplectic.

We now turn to the Einstein conditions for the restricted Walker metric g in (5.1). Let r and S denote the Ricci tensor and the scalar curvature of the metric g in (5.1), respectively. The Einstein tensor is defined by $G_{ij} = r_{ij} - \frac{1}{4}Sg_{ij}$ and has the following nonzero components (see [M2, Appendix D]):

$$
\begin{aligned}
G_{13} &= -G_{24} = \tfrac{1}{4}a_{11} - \tfrac{1}{4}a_{22},\\
G_{14} &= G_{23} = \tfrac{1}{2}a_{12},\\
G_{33} &= \tfrac{1}{4}aa_{11} + \tfrac{1}{4}aa_{22} - a_{24} + \tfrac{1}{2}(a_2)^2,\\
G_{34} &= \tfrac{1}{2}a_{14} - \tfrac{1}{2}a_1a_2 + \tfrac{1}{2}a_{23},\\
G_{44} &= \tfrac{1}{4}aa_{11} - a_{13} + \tfrac{1}{2}(a_1)^2 + \tfrac{1}{4}aa_{22}.
\end{aligned}
\tag{5.2}
$$

It follows from (5.2) that the Einstein condition ($G_{ij} = 0$) consists of the following PDEs:

$$
\begin{aligned}
&a_{11} - a_{22} = 0, \quad a_{12} = 0, \quad aa_{11} - 2a_{24} + (a_2)^2 = 0,\\
&a_{14} - a_1a_2 + a_{23} = 0, \quad aa_{11} - 2a_{13} + (a_1)^2 = 0.
\end{aligned}
$$

If a is independent of x^2 and x^4, and contains x^1 only linearly, the first four PDEs hold trivially, and the last one reduces to $2a_{13} - (a_1)^2 = 0$. Then it follows that $a = -2x^1/x^3$ is a solution to the PDE, and therefore the metric

$$
g = (g_{ij}) = \begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & -2x^1/x^3 & 0\\ 0 & 1 & 0 & -2x^1/x^3 \end{pmatrix}
\tag{5.3}
$$

is Einstein on the coordinate patch $x^3 > 0$ (or $x^3 < 0$). Moreover, the domain $x^3 > 0$ (or $x^3 < 0$) in M_4 must be noncompact. It follows from (4.2) that g is Ricci flat. In particular, this metric admits the following proper almost paracomplex structure:

$$
\varphi\partial_1 = \partial_1, \quad \varphi\partial_2 = -\partial_2, \quad \varphi\partial_3 = a\partial_1 - \partial_3, \quad \varphi\partial_4 = -a\partial_2 + \partial_4.
\tag{5.4}
$$

For the Einstein metric (5.3), the proper almost paracomplex structure φ in (5.4) becomes

$$
\varphi\partial_1 = \partial_1, \quad \varphi\partial_2 = -\partial_2, \quad \varphi\partial_3 = -\frac{2x^1}{x^3}\partial_1 - \partial_3, \quad \varphi\partial_4 = \frac{2x^1}{x^3}\partial_2 + \partial_4.
$$

Then, the integrability condition, given in Theorem 2.2, is not satisfied:

$$
a_1 = -2/x^3 \neq 0 \quad \text{and} \quad a_2 = 0.
$$

Thus, φ cannot be integrable. The condition (3.4) for ω' to be symplectic in Theorem 3.1 also fails:

$$
a_3 = 2x^1/(x^3)^2 \neq 0, \qquad a_4 + ac_1 - ca_1 = 0, \qquad c_3 = 0.
$$

Therefore, ω' is not symplectic.

Similarly, the opposite almost paracomplex structure φ' in (3.1) has the form

$$\varphi'\partial_1 = -\partial_2 - \frac{x^3}{x^1}\partial_4, \quad \varphi'\partial_2 = \partial_1 + \frac{x^3}{x^1}\partial_3,$$
$$\varphi'\partial_3 = \frac{2x^1}{x^3}\partial_2 + \partial_4, \quad \varphi'\partial_4 = -\frac{2x^1}{x^3}\partial_1 - \partial_3.$$

Then the φ'-integrability condition (3.5) in Theorem 3.2 does not hold either:

$$a_4 - ca_1 = 0, \quad a_3 + ac_2 = a_3 = -2x^1/(x^3)^2 \neq 0,$$
$$a_3 - aa_1 = -6x^1/(x^3)^2 \neq 0, \quad ac_1 - aa_2 - 2c_3 = 0.$$

Thus, φ' is not integrable.

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