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Admissibly integral manifolds for semilinear evolution equations

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Abstract. We prove the existence of integral (stable, unstable, center) manifolds of admissible classes for the solutions to the semilinear integral equation $u(t) = U(t,s)u(s) + \int_s^t U(t,\xi)f(\xi,u(\xi))\,d\xi$ when the evolution family $(U(t,s))_{t\geq s}$ has an exponential trichotomy on a half-line or on the whole line, and the nonlinear forcing term f satisfies the (local or global) φ-Lipschitz conditions, *i.e.*, $\|f(t,x)-f(t,y)\| \leq \varphi(t)\|x-y\|$ where $\varphi(t)$ belongs to some classes of admissible function spaces. These manifolds are formed by trajectories of the solutions belonging to admissible function spaces which contain wide classes of function spaces like function spaces of L_p type, the Lorentz spaces $L_{p,q}$ and many other function spaces occurring in interpolation theory. Our main methods involve the Lyapunov–Perron method, rescaling procedures, and techniques using the admissibility of function spaces.

1. Introduction and preliminaries. Consider the semilinear evolution equation of the form

$$\frac{dx}{dt} = A(t)x(t) + f(t,x(t)), \qquad t \in \mathbb{J}, \tag{1.1}$$

where $\mathbb{J}$ is a subinterval of the real line $\mathbb{R}$, each $A(t)$ is in general an unbounded linear operator on a Banach space X for every fixed $t \in \mathbb{J}$ and $f : \mathbb{J} \times X \to X$ is a nonlinear operator.

One of important directions of research regarding the asymptotic behavior of solutions to (1.1) is to find conditions for this equation to have an integral manifold (e.g., a stable, unstable, or center manifold). Such results can be traced back to Hadamard [8], Perron [25, 26], Bogoliubov and Mitropolsky [3] for the case of matrix coefficients $A(t)$, to Daletskiĭ and

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Krein [6] for the case of bounded coefficients acting on Banach spaces, and to Henry [10] and Sell and You [29] for the case of unbounded coefficients.

There are two main methods to prove the existence of such integral manifolds: the Hadamard and Perron methods. The Hadamard method has been generalized to the so-called graph transform method which has been used, e.g., in [1, 11, 20] to prove the existence of invariant manifolds. This is a powerful method related to complicated choices of the transforms between graphs representing the manifolds involved. Meanwhile, the Perron method was extended to the well-known Lyapunov–Perron method. This method is related to the construction and use of the so-called Lyapunov–Perron equations (or operators) involving the evolution equations under consideration to show the existence of integral manifolds. It seems to be more natural to use the Lyapunov–Perron method to handle the flows or semiflows which are generated by semilinear evolution equations, since in this case it is convenient to construct such Lyapunov–Perron equations or operators. We refer the reader to [2, 5, 6, 9, 10, 15, 29] and references therein for more information.

To our best knowledge, the most popular conditions for the existence of integral manifolds are the exponential trichotomy (or dichotomy) of the linear part $dx/dt = A(t)x$ and the uniform Lipschitz continuity of the nonlinear part $f(t,x)$ with sufficiently small Lipschitz constants (i.e., $\|f(t,x) - f(t,y)\| \le q\|x-y\|$ for q small enough). However, for equations arising in complicated reaction-diffusion processes, the function f represents the source of material (or population) which, in many contexts, depends on time in diversified manners (see [21, Chapt. 11], [22], [31]). Therefore, sometimes one cannot hope to have the uniform Lipschitz continuity of f. Thus, one tries to extend the conditions on nonlinear parts so that they describe such reaction-diffusion processes more exactly. Moreover, almost all of the manifolds considered in the existing literature are formed by trajectories of solutions bounded on the positive (or negative) half-line. We refer the reader to [1, 2, 9, 10, 11, 13, 20, 29] and references therein for more on this matter.

Recently, we have obtained exciting results in [13] where we have proved the existence of a new class of invariant manifolds, namely, invariant manifolds of $\mathcal{E}$-class for (1.1) (see [13, Theorems 3.7, 4.6]). Such manifolds are formed by trajectories of solutions belonging to the Banach space $\mathcal{E}$ which can be a space of L_p type ($1 \le p \le \infty$) or a Lorentz space $L_{p,q}$ or some function spaces occurring in interpolation theory (see [13, Definitions 3.3, 4.2] and [14]). The methods used in [13] are the Lyapunov–Perron method and the characterization of the exponential dichotomy (obtained in [12]) of evolution equations in admissible spaces of functions defined on the half-line $\mathbb{R}_+$. The use of admissible spaces has helped us to construct invariant manifolds of $\mathcal{E}$-class for (1.1) in the case of dichotomic linear parts without using the smallness of Lipschitz constants of nonlinear forcing terms in the classical sense. Instead, the "smallness" is understood as the sufficient smallness of $\sup_{t\geq 0}\int_t^{t+1}\varphi(\tau)\,d\tau$ (see the conditions in Theorem 3.9 below).

The purpose of the present paper is to establish the existence of stable, unstable, and center manifolds of $\mathcal{E}$-class when the linear part of (1.1) has an exponential trichotomy on the half-line or on the whole line under similar conditions on the nonlinear term $f(t,x)$ to those in [13], that is, nonuniform Lipschitz continuity of f: $\|f(t,x)-f(t,y)\|\leq\varphi(t)\|x-y\|$ for φ being a real and positive function which belongs to an admissible function space as specified in Definition 2.3 below. Under some conditions on φ, we will prove the existence of center-stable manifolds of $\mathcal{E}$-class for (1.1) provided that the linear part $dx/dt = A(t)x$ has an exponential trichotomy on a half-line. Our method is to transform to the case of exponential dichotomy by some rescaling procedures, and then apply the techniques and results of [13]. Moreover, using the same method we can also obtain the existence of unstable and center-unstable manifolds of $\mathcal{E}$-class in the case of dichotomic and trichotomic linear parts (respectively) for evolution equations defined on the whole line. Our main results are contained in Theorems 4.2, 6.5, 6.11, and Corollaries 6.6, 6.12, 6.15. We also illustrate our results in Examples 5.1, 5.2, 5.3, 6.13.

We now recall some notions.

Definition 1.1. Let $\mathbb{J}$ be $\mathbb{R}_+$ or $\mathbb{R}$. A family $\{U(t,s)\}_{t\geq s,\,t,s\in\mathbb{J}}$ of bounded linear operators acting on a Banach space X is a (strongly continuous, exponentially bounded) *evolution family on* $\mathbb{J}$ if:

(i) $U(t,t)=\mathrm{Id}=: I$ and $U(t,r)U(r,s)=U(t,s)$ for all $t\geq r\geq s$ and $t,s,r\in\mathbb{J}$,

(ii) the map $(t,s)\mapsto U(t,s)x$ is continuous on $\Delta := \{(t,s)\in\mathbb{J}\times\mathbb{J} : t\geq s\}$ for every $x\in X$,

(iii) $\|U(t,s)x\|\leq Ke^{\omega(t-s)}\|x\|$ for all $t\geq s$, $t,s\in\mathbb{J}$, $x\in X$, and some fixed constants K,ω.

The notion of an evolution family arises naturally in the theory of well-posed evolution equations. Namely, if the abstract Cauchy problem

$$\begin{cases}\frac{du(t)}{dt}=A(t)u(t), & t\geq s,\ t,s\in\mathbb{J},\\ u(s)=x_s\in X\end{cases}\tag{1.2}$$

is well-posed, then there exists an evolution family $(U(t,s))_{t\geq s,\,t,s\in\mathbb{J}}$ such that the solution of problem (1.2) is given by $u(t)=U(t,s)u(s)$.

For more details on the notion, properties and applications of evolution families we refer the reader to Pazy [24], Henry [10], and Nagel and Nickel [23]. For a given evolution family, we have the following concept of exponential trichotomy on $\mathbb{J}$.

Definition 1.2. Let $\mathbb{J}$ be $\mathbb{R}_+$ or $\mathbb{R}$. A given evolution family $(U(t,s))_{t\geq s,\, t,s\in\mathbb{J}}$ on $\mathbb{J}$ is said to have an *exponential trichotomy on* $\mathbb{J}$ if there are three families of projections $(P_j(t))_{t\in\mathbb{J}}$, $j=0,1,2$, and positive constants H,α,β with $\alpha<\beta$ such that:

(i) $\sup_{t\in\mathbb{J}}\|P_j(t)\|<\infty,\ j=0,1,2$,
(ii) $P_0(t)+P_1(t)+P_2(t)=I$ for all $t\in\mathbb{J}$, and $P_j(t)P_i(t)=0$ for all $j\neq i$,
(iii) $P_j(t)U(t,s)=U(t,s)P_j(s)$ for all $t\geq s$, $t,s\in\mathbb{J}$, $j=0,1,2$,
(iv) $U(t,s)|_{\operatorname{Im}P_1(s)}$ and $U(t,s)|_{\operatorname{Im}P_2(s)}$ are homeomorphisms from $\operatorname{Im}P_1(s)$ onto $\operatorname{Im}P_1(t)$ and from $\operatorname{Im}P_2(s)$ onto $\operatorname{Im}P_2(t)$, respectively, for all $t\geq s$, $t,s\in\mathbb{J}$, also we denote the inverse of $U(t,s)|_{\operatorname{Im}P_1(s)}$ by $U(s,t)_|$ (here $s\leq t$),
(v) for all $t\geq s$, $t,s\in\mathbb{J}$, and $x\in X$, we have

$$\|U(t,s)P_0(s)x\|\leq He^{-\beta(t-s)}\|P_0(s)x\|,$$
$$\|U(s,t)_|P_1(t)x\|\leq He^{-\beta(t-s)}\|P_1(t)x\|,$$
$$\|U(t,s)P_2(s)x\|\leq He^{\alpha(t-s)}\|P_2(s)x\|.$$

We then put $N:=H\sup_{t\in\mathbb{J}}\{\|P_j(t)\|:j=0,1,2\}$, and call N,H,α,β the *trichotomy constants* of this exponential trichotomy.

The evolution family is said to have an *exponential dichotomy on* $\mathbb{J}$ if it has an exponential trichotomy for which the family of projections $P_2(t)$ is trivial, i.e., $P_2(t)=0$ for all $t\in\mathbb{J}$. In this case, property (i) is a consequence of the other properties (see [19, Lem. 4.2]). For the dichotomy case, we put $P(t)=P_0(t)$; then $P_1(t)$ is simply $I-P(t)$ for all $t\in\mathbb{J}$.

2. Function spaces and admissibility. We recall some notions on function spaces and refer to Massera and Schäffer [18] and Räbiger and Schnaubelt [27] for concrete applications.

Denote by $\mathcal{B}$ the Borel algebra and by λ the Lebesgue measure on $\mathbb{R}_+$. The space $L_{1,\mathrm{loc}}(\mathbb{R}_+)$ of real-valued locally integrable functions on $\mathbb{R}_+$ (modulo λ-nullfunctions) becomes a Fréchet space for the seminorms $p_n(f):=\int_{J_n}|f(t)|\,dt$, where $J_n=[n,n+1]$ for each $n\in\mathbb{N}$ (see [18, Chapt. 2, §20]).

We can now define Banach function spaces:

Definition 2.1. A vector space E of real-valued Borel-measurable functions on $\mathbb{R}_+$ (modulo λ-nullfunctions) is called a *Banach function space* (over $(\mathbb{R}_+,\mathcal{B},\lambda)$) if:

(1) E is a Banach lattice with respect to a norm $\|\cdot\|_E$, i.e., $(E,\|\cdot\|_E)$ is a Banach space, and if $\varphi\in E$ and ψ is a real-valued Borel-measurable function such that $|\psi(\cdot)|\leq|\varphi(\cdot)|$, λ-a.e., then $\psi\in E$ and $\|\psi\|_E\leq\|\varphi\|_E$,

(2) the characteristic functions χ_A belong to E for all $A \in \mathcal{B}$ of finite measure, and $\sup_{t\geq 0} \|\chi_{[t,t+1]}\|_E < \infty$ and $\inf_{t\geq 0} \|\chi_{[t,t+1]}\|_E > 0$,

(3) $E \hookrightarrow L_{1,\mathrm{loc}}(\mathbb{R}_+)$, i.e., for each seminorm p_n of $L_{1,loc}(\mathbb{R}_+)$ there exists a number $\beta_{p_n} > 0$ such that $p_n(f) \leq \beta_{p_n} \|f\|_E$ for all $f \in E$.

We then define Banach spaces of vector-valued functions corresponding to Banach function spaces:

Definition 2.2. Let E be a Banach function space and X be a Banach space endowed with the norm $\|\cdot\|$. We set

$$\mathcal{E} := \mathcal{E}(\mathbb{R}_+, X) := \{f : \mathbb{R}_+ \to X : f \text{ is strongly measurable and } \|f(\cdot)\| \in E\}$$

(modulo λ-nullfunctions) endowed with the norm $\|f\|_{\mathcal{E}} := \big\|\|f(\cdot)\|\big\|_E$. One can easily see that $\mathcal{E}$ is a Banach space. We call it the *Banach space corresponding to the Banach function space E.*

We now introduce the notion of admissibility:

Definition 2.3. A Banach function space E is called *admissible* if:

(1) there is a constant $M \geq 1$ such that for every compact interval $[a, b] \subset \mathbb{R}_+$ we have

$$\int_a^b |\varphi(t)|\, dt \leq \frac{M(b-a)}{\|\chi_{[a,b]}\|_E} \|\varphi\|_E \quad \text{for all } \varphi \in E, \tag{2.1}$$

(2) for $\varphi \in E$ the function $\Lambda_1\varphi$ defined by $\Lambda_1\varphi(t) := \int_t^{t+1} \varphi(\tau)\, d\tau$ belongs to E,

(3) E is T_τ^+-invariant and T_τ^--invariant, where T_τ^+ and T_τ^- are defined, for $\tau \in \mathbb{R}_+$, by

$$\begin{aligned} T_\tau^+\varphi(t) &:= \begin{cases} \varphi(t-\tau) & \text{for } t \geq \tau \geq 0, \\ 0 & \text{for } 0 \leq t \leq \tau, \end{cases} \\ T_\tau^-\varphi(t) &:= \varphi(t+\tau) \text{ for } t \geq 0; \end{aligned} \tag{2.2}$$

moreover, there are constants N_1, N_2 such that $\|T_\tau^+\| \leq N_1$, $\|T_\tau^-\| \leq N_2$ for all $\tau \in \mathbb{R}_+$.

Example 2.4. Besides the spaces $L_p(\mathbb{R}_+)$, $1 \leq p \leq \infty$, and the space

$$\mathbf{M}(\mathbb{R}_+) := \Big\{ f \in L_{1,\mathrm{loc}}(\mathbb{R}_+) : \sup_{t\geq 0} \int_t^{t+1} |f(\tau)|\, d\tau < \infty \Big\}$$

with the norm $\|f\|_{\mathbf{M}} := \sup_{t\geq 0} \int_t^{t+1} |f(\tau)|\, d\tau$, many other function spaces occurring in interpolation theory, e.g. the Lorentz spaces $L_{p,q}$, $1 < p < \infty$, $1 \leq q < \infty$ (see [4, Thm. 3 and p. 284], [30, 1.18.6, 1.19.3]) and, more generally, the class of rearrangement invariant function spaces over $(\mathbb{R}_+, \mathcal{B}, \lambda)$ (see [16, 2.a]), are admissible.

REMARK 2.5. If E is an admissible Banach function space then $E \hookrightarrow \mathbf{M}(\mathbb{R}_+)$. Indeed, put $\beta := \inf_{t\geq 0} \|\chi_{[t,t+1]}\|_E > 0$ (by Definition 2.1(2)). Then from Definition 2.3(i) we derive

$$\int_t^{t+1} |\varphi(\tau)|\, d\tau \leq \frac{M}{\beta}\|\varphi\|_E \quad \text{for all } t \geq 0 \text{ and } \varphi \in E. \tag{2.3}$$

Therefore, if $\varphi \in E$ then $\varphi \in \mathbf{M}(\mathbb{R}_+)$ and $\|\varphi\|_{\mathbf{M}} \leq \frac{M}{\beta}\|\varphi\|_E$. We thus obtain $E \hookrightarrow \mathbf{M}(\mathbb{R}_+)$.

We now collect some properties of admissible Banach function spaces (see [12, Proposition 2.6] and originally [18, 23.V(1)]).

PROPOSITION 2.6. *Let E be an admissible Banach function space. Then:*

(a) *Let $\varphi \in L_{1,\mathrm{loc}}(\mathbb{R}_+)$ be such that $\varphi \geq 0$ and $\Lambda_1\varphi \in E$, where Λ_1 is as in Definition* 2.3(ii). *For $\sigma > 0$ we define*

$$\Lambda'_\sigma\varphi(t) := \int_0^t e^{-\sigma(t-s)}\varphi(s)\, ds, \qquad \Lambda''_\sigma\varphi(t) := \int_t^\infty e^{-\sigma(s-t)}\varphi(s)\, ds.$$

Then $\Lambda'_\sigma\varphi$ and $\Lambda''_\sigma\varphi$ belong to E. In particular, if $\sup_{t\geq 0}\int_t^{t+1}\varphi(\tau)\, d\tau < \infty$ (this will be satisfied if $\varphi \in E$ (see Remark 2.5)), *then $\Lambda'_\sigma\varphi$ and $\Lambda''_\sigma\varphi$ are bounded. Moreover,*

$$\|\Lambda'_\sigma\varphi\|_E \leq \frac{N_1}{1-e^{-\sigma}}\|\Lambda_1 T_1^+\varphi\|_E \quad \text{and} \quad \|\Lambda''_\sigma\varphi\|_E \leq \frac{N_2}{1-e^{-\sigma}}\|\Lambda_1\varphi\|_E \tag{2.4}$$

for T_1^+ and N_1, N_2 as in Definition 2.3.

(b) *E contains all exponentially decaying functions $\psi(t) = e^{-\alpha t}$ for $t \geq 0$ and any fixed constant $\alpha > 0$.*

(c) *E contains no exponentially growing functions $f(t) := e^{bt}$ for $t \geq 0$ and any fixed constant $b > 0$.*

REMARK 2.7. If we replace $\mathbb{R}_+$ by an infinite (or half-infinite) interval $\mathbb{I}$ (precisely, $\mathbb{I}$ is $\mathbb{R}$, $(-\infty, t_0]$ or $[t_0, \infty)$ for any fixed $t_0 \in \mathbb{R}$), then we have similar notions of admissible spaces on $\mathbb{I}$:

(1) In Definition 2.3, the translation semigroups T_τ^+ and T_τ^- for $\tau \in \mathbb{R}_+$ should be replaced by T_τ^+ and T_τ^- defined for $\tau \in \mathbb{I}$ as

$$\begin{aligned} T_\tau^+\varphi(t) &:= \begin{cases} \varphi(t-\tau) & \text{for } t \text{ and } t-\tau \text{ in } \mathbb{I}, \\ 0 & \text{for } t \in \mathbb{I} \text{ but } t-\tau \notin \mathbb{I}, \end{cases} \\ T_\tau^-\varphi(t) &:= \begin{cases} \varphi(t+\tau) & \text{for } t \text{ and } t+\tau \text{ in } \mathbb{I}, \\ 0 & \text{for } t \in \mathbb{I} \text{ but } t+\tau \notin \mathbb{I}. \end{cases} \end{aligned} \tag{2.5}$$

(2) In Proposition 2.6(a), the functions Λ'_σ and Λ''_σ should be replaced by

$$\Lambda'_\sigma\varphi(t) := \int_t^{t_0} e^{-\sigma|t-s|}\varphi(s)\,ds \quad \text{with } t_0 = \infty \text{ if } \mathbb{I} = \mathbb{R},$$

$$\Lambda''_\sigma\varphi(t) := \begin{cases} \int_{-\infty}^t e^{-\sigma|s-t|}\varphi(s)\,ds & \text{if } \mathbb{I} = \mathbb{R} \text{ or } (-\infty, t_0], \\ \int_t^\infty e^{-\sigma|s-t|}\varphi(s)\,ds & \text{if } \mathbb{I} = [t_0, \infty). \end{cases}$$

(3) In Proposition 2.6(b)&(c) the functions $\psi(t) = e^{-\alpha t}$ ($t \ge 0$, with fixed $\alpha > 0$) should be replaced by $\psi(t) = e^{-\alpha|t|}$, $t \in \mathbb{I}$, with fixed $\alpha > 0$; and the functions $f(t) := e^{bt}$ ($t \ge 0$, with any fixed constant $b > 0$) should be replaced by $f(t) := e^{b|t|}$, $t \in \mathbb{I}$, with fixed $b > 0$.

These notions will be used in Section 6. We denote by $E_\mathbb{I}$ the admissible function space of functions defined on $\mathbb{I}$. If $\mathbb{I} = \mathbb{R}_+$, we simply write $E := E_{\mathbb{R}_+}$. For a function φ defined on the whole line we denote by $\varphi|_\mathbb{I}$ the restriction of φ to $\mathbb{I}$. It is obvious that if $\varphi \in E_\mathbb{R}$, then $\varphi|_\mathbb{I} \in E_\mathbb{I}$.

Similarly to Definition 2.2, for a Banach function space $E_\mathbb{I}$ and a Banach space X with the norm $\|\cdot\|$ we set

$$\mathcal{E}_\mathbb{I} := \mathcal{E}(\mathbb{I}, X) := \{f : \mathbb{I} \to X : f \text{ is strongly measurable and } \|f(\cdot)\| \in E_\mathbb{I}\}$$

(modulo λ-nullfunctions) endowed with the norm

$$\|f\|_{\mathcal{E}_\mathbb{I}} := \big\|\|f(\cdot)\|\big\|_{E_\mathbb{I}}.$$

Then $\mathcal{E}_\mathbb{I}$ is a Banach space called the *Banach space corresponding to the Banach function space* $E_\mathbb{I}$. Also, if $\mathbb{I} = \mathbb{R}_+$ we write simply $\mathcal{E} := \mathcal{E}_{\mathbb{R}_+}$.

Definition 2.8. Let $E_\mathbb{I}$ be an admissible Banach function space and denote by $S(E_\mathbb{I})$ the unit sphere in $E_\mathbb{I}$. Recall that $L_1 = \{g : \imath \to \mathbb{R} : g$ is measurable and $\int_\mathbb{I} |g(t)|\,dt < \infty\}$. Consider the set $E'_\mathbb{I}$ of all measurable real-valued functions ψ on $\mathbb{I}$ such that

$$\varphi\psi \in L_1, \quad \int_\mathbb{I} |\varphi(t)\psi(t)|\,dt \le k \quad \text{for all } \varphi \in S(E_\mathbb{I}),$$

where k depends only on ψ. Then $E'_\mathbb{I}$ is a normed space with the norm given by (see [18, Chapt. 2, 22.M])

$$\|\psi\|_{E'_\mathbb{I}} := \sup\Big\{\int_\mathbb{I} |\varphi(t)\psi(t)|\,dt : \varphi \in S(E_\mathbb{I})\Big\} \quad \text{for } \psi \in E'_\mathbb{I}.$$

We call $E'_\mathbb{I}$ the *associate space* of $E_\mathbb{I}$.

Remark 2.9. Let $E_\mathbb{I}$ be an admissible Banach function space and $E'_\mathbb{I}$ be its associate space. Then, by [18, Chapt. 2, 22.M], the following "Hölder inequality" holds:

$$(2.6) \qquad \int_{\mathbb{I}} |\varphi(t)\psi(t)|\,dt \le \|\varphi\|_{E_{\mathbb{I}}}\|\psi\|_{E'_{\mathbb{I}}} \quad \text{for all } \varphi \in E_{\mathbb{I}},\ \psi \in E'_{\mathbb{I}}.$$

In order to study the integral manifolds of $\mathcal{E}_{\mathbb{I}}$-class for semilinear evolution equations we need some restrictions on admissible Banach function spaces and assume the following hypothesis.

STANDING HYPOTHESIS 2.10. We suppose that $E_{\mathbb{I}}$ is an admissible Banach function space such that its associate space $E'_{\mathbb{I}}$ is also an admissible Banach function space. Moreover, we suppose that $E'_{\mathbb{I}}$ contains a ν-*exponentially $E_{\mathbb{I}}$-invariant* function, that is, a function $\varphi \ge 0$ such that, for a fixed $\nu > 0$, the function

$$h_\nu(t) := \|e^{-\nu|t-\cdot|}\varphi(\cdot)\|_{E'_{\mathbb{I}}} \quad \text{for } t \in \mathbb{I}$$

belongs to $E_{\mathbb{I}}$. Also, we denote by e_ν the function $e_\nu(t) = e^{-\nu|t|}$.

EXAMPLE 2.11. $L'_p = L_q$ for $1/p + 1/q = 1,\ 1 < p < \infty$, and $L'_1 = L_\infty$, $L'_\infty = L_1$.

Besides the above function e_ν, the functions $\varphi = c\chi_{[a,b]}$ for any fixed constant $c > 0$ and any finite interval $[a,b] \subset \mathbb{I}$ are also ν-exponentially L_p-invariant for any $\nu > 0$. More examples can be seen in Section 5.

In the rest of our paper we will make use of the following assumption.

ASSUMPTION 2.12. Let the evolution family $(U(t,s))_{t\ge s,\,t,s\in\mathbb{I}}$ have an exponential dichotomy on $\mathbb{I} = \mathbb{R}_+$ or $\mathbb{R}$ with dichotomy projections $(P(t))_{t\in\mathbb{I}}$ and dichotomy constants $N, \beta > 0$. Suppose that $\varphi \in E'_{\mathbb{I}}$ is a β-exponentially $E_{\mathbb{I}}$-invariant function whose existence is guaranteed by Standing Hypothesis 2.10.

In the case of infinite-dimensional phase spaces, instead of equation (1.1), for an evolution family $(U(t,s))_{t\ge s,\,t,s\in\mathbb{I}}$, we consider the integral equation

$$(2.7) \qquad u(t) = U(t,s)u(s) + \int_s^t U(t,\xi)f(\xi,u(\xi))\,d\xi \quad \text{for a.e. } t \ge s,\ t,s \in \mathbb{I}.$$

We note that, if the evolution family $(U(t,s))_{t\ge s,\,t,s\in\mathbb{I}}$ arises from the well-posed Cauchy problem (1.2), then a function u which satisfies (2.7) for some given f is called a *mild solution* of the semilinear problem

$$\begin{cases} \frac{du(t)}{dt} = A(t)u(t) + f(t,u(t)), & t \ge s,\ t,s \in \mathbb{I}, \\ u(s) = x_s \in X. \end{cases}$$

We refer the reader to Pazy [24] for more details on the relation between classical and mild solutions of evolution equations (see also [7, 17, 29]).

To obtain the existence of an admissibly integral manifold for (2.7), besides the exponential dichotomy (or trichotomy) of the evolution family, we also need the (local) φ-Lipschitz properties of the nonlinear term f, according to the following definitions in which we suppose as above that $\mathbb{I}$ is $\mathbb{R}_+$ or $\mathbb{R}$.

DEFINITION 2.13 (Local φ-Lipschitz functions). Let φ be a positive function belonging to $E_{\mathbb{I}}$, and B_ρ be the ball $B_\rho := \{x \in X : \|x\| \le \rho\}$. A function $f : \mathbb{I} \times B_\rho \to X$ is said to belong to the *class* (M, φ, ρ) for some positive constants M, ρ if

(i) $\|f(t,x)\| \le M\varphi(t)$ for a.e. $t \in \mathbb{I}$ and all $x \in B_\rho$, and
(ii) $\|f(t,x_1) - f(t,x_2)\| \le \varphi(t)\|x_1 - x_2\|$ for a.e. $t \in \mathbb{I}$ and all $x_1, x_2 \in B_\rho$.

REMARK 2.14. If $f(t,0) = 0$ then (ii) already implies that f belongs to the class (ρ, φ, ρ).

DEFINITION 2.15 (φ-Lipschitz functions). Let φ be a positive function belonging to $E_{\mathbb{I}}$. A function $f : \mathbb{I} \times X \to X$ is said to be *φ-Lipschitz* if

(i) $f(t,0) = 0$ for a.e. $t \in \mathbb{I}$,
(ii) $\|f(t,x_1) - f(t,x_2)\| \le \varphi(t)\|x_1 - x_2\|$ for a.e. $t \in \mathbb{I}$ and all $x_1, x_2 \in X$.

3. Exponential dichotomy and admissibly stable manifolds on $\mathbb{R}_+$. In this section, we recall preparatory results on $\mathbb{R}_+$ obtained in [13] which will be used in the next sections. In this case, $\mathbb{I} = \mathbb{R}_+$. For an evolution family $(U(t,s))_{t \ge s \ge 0}$ we rewrite the integral equation (2.7) as

$$u(t) = U(t,s)u(s) + \int_s^t U(t,\xi)f(\xi, u(\xi))\,d\xi \quad \text{for a.e. } t \ge s \in \mathbb{R}_+. \tag{3.1}$$

We also denote by $(\mathcal{E}_\infty, \|\cdot\|_{\mathcal{E}_\infty})$ the Banach space

$$\mathcal{E}_\infty := \mathcal{E} \cap L_\infty(\mathbb{R}_+, X) \quad \text{with the norm} \quad \|f\|_{\mathcal{E}_\infty} := \max\{\|f\|_{\mathcal{E}}, \|f\|_\infty\}.$$

We refer the reader to [12] for a detailed discussion on the relation between exponential dichotomy of evolution equations and admissibility of function spaces.

3.1. Local-stable manifolds of $\mathcal{E}$-class on $\mathbb{R}_+$. Throughout this subsection we assume that the evolution family $(U(t,s))_{t \ge s \ge 0}$ has an exponential dichotomy on $\mathbb{R}_+$, and the nonlinear term f is local φ-Lipschitz and in the class (M, φ, ρ) as in Definition 2.13.

DEFINITION 3.1. A set $\mathbf{S} \subset \mathbb{R}_+ \times X$ is said to be a *local-stable manifold of $\mathcal{E}$-class* for the solutions of equation (3.1) if for every $t \in \mathbb{R}_+$ the phase space X splits into a direct sum $X = X_0(t) \oplus X_1(t)$ with positive inclination, i.e.,

$$\inf_{t\in\mathbb{R}_+} \inf_{\substack{x_i\in X_i(t)\\ \|x_i\|=1,\, i=0,1}} \|x_0+x_1\| > 0,$$

and if there exist positive constants ρ, ρ_0, ρ_1 and a family of Lipschitz continuous mappings

$$g_t : B_{\rho_0}\cap X_0(t) \to B_{\rho_1}\cap X_1(t), \qquad t\in\mathbb{R}_+,$$

with Lipschitz constants independent of t, such that:

(i) $\mathbf{S} = \{(t, x+g_t(x)) \in \mathbb{R}_+\times(X_0(t)\oplus X_1(t)) : t\in\mathbb{R}_+,\ x\in B_{\rho_0}\cap X_0(t)\}$,
(ii) $\mathbf{S}_t := \{x+g_t(x) : (t, x+g_t(x))\in\mathbf{S}\}$ is homeomorphic to $B_{\rho_0}\cap X_0(t)$ for all $t\ge 0$,
(iii) to each $x_0\in\mathbf{S}_{t_0}$ there corresponds one and only one solution $u(t)$ of (3.1) on $[t_0,\infty)$ such that $u(t_0)=x_0$ and the function $\chi_{[t_0,\infty)}u(\cdot)$ belongs to the ball $\mathcal{B}_\rho := \{g\in\mathcal{E}_\infty : \|g\|_{\mathcal{E}_\infty}\le\rho\}$.

Let $(U(t,s))_{t\ge s\ge 0}$ have an exponential dichotomy with projections $P(t)$, $t\ge 0$, and dichotomy constants $N,\beta>0$. We can then define the *Green function* on the half-line:

$$G(t,\tau) := \begin{cases} P(t)U(t,\tau) & \text{for } t\ge\tau\ge 0,\\ -U(t,\tau)_{|}[I-P(\tau)] & \text{for } 0\le t<\tau. \end{cases} \tag{3.2}$$

Thus, we have

$$\|G(t,\tau)\| \le Ne^{-\beta|t-\tau|} \qquad \text{for all } t\ne\tau\ge 0. \tag{3.3}$$

The following lemma taken from [13, Lemma 3.4] gives the form of bounded solutions of (3.1).

LEMMA 3.2. *Let Standing Hypothesis* 2.10 *and Assumption* 2.12 *be satisfied with $\mathbb{I}=\mathbb{R}_+$ and let $f:\mathbb{R}_+\times B_\rho\to X$ belong to the class (M,φ,ρ). Let $u(t)$ be a solution to* (3.1) *such that, for fixed $t_0\ge 0$, the function $\chi_{[t_0,\infty)}u(\cdot)$ belongs to $\mathcal{B}_\rho$. Then for $t\ge t_0$,*

$$u(t) = U(t,t_0)v_0 + \int_{t_0}^{\infty} G(t,\tau)f(\tau,u(\tau))\,d\tau \tag{3.4}$$

for some $v_0\in X_0(t_0)=P(t_0)X$, where $G(t,\tau)$ is the Green function defined in (3.2).

Moreover, the structure of certain solutions of (3.1) is given in the following theorem taken from [13, Thm. 3.7].

THEOREM 3.3. *Let the assumptions of Lemma* 3.2 *be satisfied and put*

$$k := \frac{N}{1-e^{-\beta}}(N_1\|\Lambda_1T_1^+\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty). \tag{3.5}$$

Then for any positive numbers ρ and M, if the function f belongs to the class (M,φ,ρ) such that

$$k<\min\left\{1,\frac{\rho}{2M}\right\} \quad \textit{and} \quad N\|h_\beta\|_E<1,$$

then for $r=\rho/\max\{2N,2NN_1\|e_\beta\|_E\}$ and $t_0\geq 0$, there corresponds to each $v_0\in B_r\cap X_0(t_0)$ one and only one solution $u(t)$ of (3.1) *on $[t_0,\infty)$ such that $P(t_0)u(t_0)=v_0$ and the function $\chi_{[t_0,\infty)}u(\cdot)$ belongs to $\mathcal{B}_\rho$. Moreover, for any two solutions $u_1(t),u_2(t)$ corresponding to different values $v_1,v_2\in B_r\cap X_0(t_0)$ we have*

$$\|u_1(t)-u_2(t)\|\leq C_\mu e^{-\mu(t-t_0)}\|v_1-v_2\| \quad \textit{for } t\geq t_0, \tag{3.6}$$

where $0<\mu<\beta+\ln(1-k(1-e^{-\beta}))$, and $C_\mu=N/\big(1-\frac{k(1-e^{-\beta})}{1-e^{-(\beta-\mu)}}\big)$.

We now recall the first main result about the existence of a local-stable manifold of $\mathcal{E}$-class obtained in [13, Thm. 3.7].

Theorem 3.4. *Under the assumptions of Theorem* 3.3*, for any $\rho>0$ and $M>0$, if f belongs to the class (M,φ,ρ) such that*

$$k<\min\left\{1,\frac{\rho}{2M}\right\} \quad \textit{and} \quad N\|h_\beta\|_E<1,$$

where k is defined as in (3.5)*, then there exists a local-stable manifold* $\mathbf{S}$ *of $\mathcal{E}$-class for the solutions of* (3.1)*. Moreover, any two solutions $u_1(t),u_2(t)$ on the manifold* $\mathbf{S}$ *attract each other exponentially in the sense that there exist positive constants μ and C_μ independent of $t_0\geq 0$ such that*

$$\begin{aligned}&\|u_1(t)-u_2(t)\|\\ &\qquad\leq C_\mu e^{-\mu(t-t_0)}\|P(t_0)u_1(t_0)-P(t_0)u_2(t_0)\| \quad \textit{for } t\geq t_0.\end{aligned} \tag{3.7}$$

3.2. Invariant-stable manifolds of $\mathcal{E}$-class on $\mathbb{R}_+$. In this subsection, we recall the results on the existence of an invariant-stable manifold of $\mathcal{E}$-class obtained in [13, Thm 4.6] under the conditions that the evolution family $(U(t,s))_{t\geq s\geq 0}$ has an exponential dichotomy and the nonlinear function f is φ-Lipschitz as in Definition 2.15.

We now give the definition of an invariant-stable manifold of $\mathcal{E}$-class for the solutions of the integral equation (3.1).

Definition 3.5. A set $\mathbf{S}\subset\mathbb{R}_+\times X$ is said to be an *invariant-stable manifold of $\mathcal{E}$-class* for the solutions of equation (3.1) if for every $t\in\mathbb{R}_+$ the phase space X splits into a direct sum $X=X_0(t)\oplus X_1(t)$ with positive inclination, and if there exists a family of Lipschitz continuous mappings

$$g_t: X_0(t)\to X_1(t), \quad t\in\mathbb{R}_+,$$

with Lipschitz constants independent of t, such that:

(i) $\mathbf{S} = \{(t, x + g_t(x)) \in \mathbb{R}_+ \times (X_0(t) \oplus X_1(t)) : t \in \mathbb{R}_+,\ x \in X_0(t)\}$,

(ii) $\mathbf{S}_t := \{x + g_t(x) : (t, x + g_t(x)) \in \mathbf{S}\}$ is homeomorphic to $X_0(t)$ for all $t \geq 0$,

(iii) to each $x_0 \in \mathbf{S}_{t_0}$ there corresponds one and only one solution $u(t)$ of (3.1) on $[t_0, \infty)$ such that $u(t_0) = x_0$ and the function $\chi_{[t_0,\infty)} u(\cdot)$ belongs to $\mathcal{E}$,

(iv) $\mathbf{S}$ is invariant under equation (3.1) in the sense that, if $u(\cdot)$ is a solution of (3.1) on $[t_0, \infty)$ such that $u(t_0) = u_0 \in \mathbf{S}_{t_0}$ and $\chi_{[t_0,\infty)} u(\cdot) \in \mathcal{E}$, then $u(s) \in \mathbf{S}_s$ for all $s \geq t_0$.

Note that if we identify $X_0(t) \oplus X_1(t)$ with $X_0(t) \times X_1(t)$ then we can write $\mathbf{S}_t = \operatorname{graph}(g_t)$.

Next, we recall from [13] some related results for later use.

Lemma 3.6 ([13, Lem. 4.3]). *Let Assumption* 2.12 *and Standing Hypothesis* 2.10 *be satisfied with* $\mathbb{I} = \mathbb{R}_+$. *Suppose that* $f : \mathbb{R}_+ \times X \to X$ *is* φ*-Lipschitz. Let* $u(t)$ *be a solution to* (3.1) *such that, for fixed* $t_0 \geq 0$, *the function* $\chi_{[t_0,\infty)} u(\cdot)$ *belongs to* $\mathcal{E}$. *Then for* $t \geq t_0$,

$$u(t) = U(t, t_0) v_0 + \int_{t_0}^{\infty} G(t, \tau) f(\tau, u(\tau))\, d\tau \tag{3.8}$$

for some $v_0 \in X_0(t_0) = P(t_0)X$, *where* $G(t, \tau)$ *is the Green function defined by* (3.2).

Remark 3.7. Formula (3.8) is called the *Lyapunov–Perron equation.* By computing directly, we can see that the converse of Lemma 3.6 is also true. Hence all solutions of (3.8) satisfy (3.1) for $t \geq t_0$. Indeed, putting $y(t) = \int_{t_0}^{\infty} G(t, \tau) f(\tau, u(\tau))\, d\tau$ we then have

$$\begin{aligned}
y(t) &= -\int_t^{\infty} U(t, \tau)_| (I - P(\tau)) f(\tau, u(\tau))\, d\tau + \int_{t_0}^{t} U(t, \tau) P(\tau) f(\tau, u(\tau))\, d\tau \\
&= U(t, t_0)\Big(-\int_{t_0}^{\infty} U(t_0, \tau)_| (I - P(\tau)) f(\tau, u(\tau))\, d\tau\Big) \\
&\quad + \int_{t_0}^{t} U(t, \tau)(I - P(\tau)) f(\tau, u(\tau))\, d\tau + \int_{t_0}^{t} U(t, \tau) P(\tau) f(\tau, u(\tau))\, d\tau \\
&= U(t, t_0) y(t_0) + \int_{t_0}^{t} U(t, \tau) f(\tau, u(\tau))\, d\tau.
\end{aligned}$$

It follows that

$$y(t) = U(t, t_0) y(t_0) + \int_{t_0}^{t} U(t, \tau) f(\tau, u(\tau))\, d\tau \quad \text{for } t \geq t_0.$$

Let now $u(t)$ be a solution of (3.8). Then $u(t) = U(t,t_0)v_0 + y(t)$. Thus,

$$\begin{aligned} u(t) &= U(t,t_0)v_0 + U(t,t_0)y(t_0) + \int_{t_0}^{t} U(t,\tau)f(\tau,u(\tau))\,d\tau \\ &= U(t,s)(U(s,t_0)v_0 + y(s)) - U(t,s)y(s) + U(t,t_0)y(t_0) \\ &\quad + \int_{t_0}^{t} U(t,\tau)f(\tau,u(\tau))\,d\tau \\ &= U(t,s)u(s) - U(t,s)\Big(U(s,t_0)y(t_0) + \int_{t_0}^{s} U(s,\tau)f(\tau,u(\tau))\,d\tau\Big) \\ &\quad + U(t,t_0)y(t_0) + \int_{t_0}^{t} U(t,\tau)f(\tau,u(\tau))\,d\tau \\ &= U(t,s)u(s) + \int_{s}^{t} U(t,\tau)f(\tau,u(\tau))\,d\tau. \end{aligned}$$

Hence, $u(t)$ satisfies (3.1) for $t \ge s \ge t_0$.

THEOREM 3.8 ([13, Thm. 4.5]). *Under the assumptions of Lemma* 3.6, *if f is φ-Lipschitz with $N\|h_\beta\|_E < 1$, then there corresponds to each $v_0 \in X_0(t_0)$ one and only one solution $u(t)$ of* (3.1) *on* $[t_0,\infty)$ *such that $P(t_0)u(t_0) = v_0$ and $\chi_{[t_0,\infty)}u(\cdot) \in \mathcal{E}$. Moreover, there exist positive constants μ and C_μ independent of t_0 such that for any two solutions $u_1(t), u_2(t)$ corresponding to different values $v_1, v_2 \in X_0(t_0)$ we have*

$$\|u_1(t) - u_2(t)\| \le C_\mu e^{-\mu(t-t_0)}\|v_1 - v_2\| \quad \text{for } t \ge t_0. \tag{3.9}$$

THEOREM 3.9 ([13, Thm. 4.6]). *Under the assumptions of Theorem* 3.8, *if f is φ-Lipschitz for φ satisfying*

$$N^2N_1\|e_\beta\|_E\|\varphi\|_{E'} + N\|h_\beta\|_E < 1,$$

then there exists an invariant-stable manifold $\mathbf{S}$ *of $\mathcal{E}$-class for the solutions of* (3.1). *Moreover, any two solutions $u_1(t), u_2(t)$ on* $\mathbf{S}$ *attract each other exponentially in the sense that there exist positive constants μ and C_μ independent of $t_0 \ge 0$ such that*

$$\|u_1(t) - u_2(t)\| \le C_\mu e^{-\mu(t-t_0)}\|P(t_0)u_1(t_0) - P(t_0)u_2(t_0)\| \quad \text{for } t \ge t_0. \tag{3.10}$$

4. Exponential trichotomy and admissibly center-stable manifolds on $\mathbb{R}_+$. In this section, we will generalize Theorem 3.9 to the case where $(U(t,s))_{t\ge s\ge 0}$ has an exponential trichotomy on $\mathbb{R}_+$. To do this, we first make the following assumption.

Assumption 4.1. Let the evolution family $(U(t,s))_{t\geq s,\, t,s\in\mathbb{I}}$ have an exponential trichotomy on $\mathbb{I}$ with constants N, H, α, β $(\alpha < \beta)$, and projections $(P_j(t))_{t\in\mathbb{I}}$, $j = 0,1,2$, as in Definition 1.2, and let $\varphi \in E'_{\mathbb{I}}$ be a β'-exponentially $E_{\mathbb{I}}$-invariant function where $\beta' := (\beta-\alpha)/2$.

In this case, we will prove that there exists an invariant-center-stable manifold of $\mathcal{E}$-class for the solutions of (3.1) if f is φ-Lipschitz.

Theorem 4.2. *Let Standing Hypothesis* 2.10 *and Assumption* 4.1 *be satisfied with* $\mathbb{I} = \mathbb{R}_+$. *Suppose that* $f : \mathbb{R}_+ \times X \to X$ *is* φ*-Lipschitz such that*

$$N^2 N_1 \|e_{\beta'}\|_E \|\varphi\|_{E'} + N\|h_{\beta'}\|_E < 1.$$

Then there exists an invariant-center-stable manifold $\mathbf{C} = \{(t, \mathbf{C}_t) : t \in \mathbb{R}_+$ and $\mathbf{C}_t \subset X\}$ *of* $\mathcal{E}$*-class for the solutions of* (3.1), *with the family* $(\mathbf{C}_t)_{t\geq 0}$ *being the graphs of the family of Lipschitz continuous mappings* $(g_t)_{t\geq 0}$ (*i.e.*, $\mathbf{C}_t := \operatorname{graph}(g_t) = \{x + g_t x : x \in \operatorname{Im}(P_1(t) + P_3(t))\}$ *for each* $t \geq 0$) *where* $g_t : \operatorname{Im}(P_0(t) + P_2(t)) \to \operatorname{Im} P_1(t)$ *has Lipschitz constant* $l = \frac{N^2 N_1 \|e_{\beta'}\|_E \|\varphi\|_{E'}}{1 - N\|h_{\beta'}\|_E}$ *independent of* t, *such that:*

- (i) *To each* $x_0 \in \mathbf{C}_{t_0}$ *there corresponds one and only one solution* $u(t)$ *of* (3.1) *on* $[t_0,\infty)$ *such that* $u(t_0) = x_0$ *and the function* $\chi_{[t_0,\infty)} e^{-\gamma\cdot} u(\cdot)$ *belongs to* $\mathcal{E}$, *where* $\gamma := (\alpha+\beta)/2$.
- (ii) $\mathbf{C}_t$ *is homeomorphic to* $X_0(t) \oplus X_2(t)$ for all $t \geq 0$, *where* $X_0(t) = P_0(t)X$ *and* $X_2(t) = P_2(t)X$.
- (iii) $\mathbf{C}$ *is invariant under* (3.1) *in the sense that, if* $u(t)$ *is the solution of* (3.1) *such that* $u(t_0) = x_0 \in \mathbf{C}_{t_0}$ *and the function* $\chi_{[t_0,\infty)} e^{-\gamma\cdot} u(\cdot)$ *belongs to* $\mathcal{E}$, *then* $u(s) \in \mathbf{C}_s$ *for all* $s \geq t_0$.
- (iv) *For any two solutions* $u_1(t), u_2(t)$ *on the center-stable manifold* $\mathbf{C}$ *there exist positive constants* μ *and* C_μ *independent of* $t_0 \geq 0$ *such that*

$$\begin{aligned}\|x(t) - y(t)\| \leq{}& C_\mu e^{(\gamma-\mu)(t-t_0)} \\ &\times \|(P_0(t_0) + P_2(t_0))x(t_0) - (P_0(t_0) + P_2(t_0))y(t_0)\|\end{aligned}$$

for all $t \geq t_0$.

Proof. Set $P(t) := P_0(t) + P_2(t)$ and $Q(t) := P_1(t) = I - P(t)$. We consider the rescaled evolution family

$$\tilde{U}(t,s)x := e^{-\gamma(t-s)} U(t,s)x \qquad \text{for all } t \geq s \geq 0,\ x \in X,$$

where $\gamma := (\alpha+\beta)/2$.

It is easy to check that $(\tilde{U}(t,s))_{t\geq s\geq 0}$ is an evolution family on X. We claim that $(\tilde{U}(t,s))_{t\geq s\geq 0}$ has an exponential dichotomy with projections $(P(t))_{t\geq 0}$ on the half-line. Indeed, it suffices to verify the estimates in Definition 1.2. By the definition of exponential trichotomy we have

$$\|\tilde{U}(s,t)_{|}Q(t)x\| \le He^{-(\beta-\gamma)(t-s)}\|Q(t)x\| = He^{-(\beta-\alpha)(t-s)/2}\|Q(t)x\|$$

for all $t \ge s \ge 0$ and $x \in X$. On the other hand,

$$\begin{aligned}\|\tilde{U}(t,s)P(s)x\| &= e^{-\gamma(t-s)}\|U(t,s)[P_0(s)+P_2(s)]x\| \\ &\le He^{-(\gamma+\alpha)(t-s)}\|P_0(s)x\| + He^{-(\gamma-\alpha)(t-s)}\|P_2(s)x\| \\ &\le He^{-(\beta-\alpha)/2(t-s)}(\|P_0(s)x\| + \|P_2(s)x\|) \\ &= He^{-(\beta-\alpha)(t-s)/2}(\|P_0(s)(P_0(s)+P_2(s))x\| \\ &\quad + \|P_2(s)(P_0(s)+P_2(s))x\|) \\ &\le Ne^{-\frac{(\beta-\alpha)(t-s)}{2}}(\|(P_0(s)+P_2(s))x\| + \|(P_0(s)+P_2(s))x\|) \\ &= 2Ne^{-(\beta-\alpha)(t-s)/2}\|P(s)x\|\end{aligned}$$

for all $t \ge s \ge 0$ and $x \in X$ (here we use the fact that $N := H\sup_{t\ge 0}\{\|P_0(t)\|, \|P_1(t)\|, \|P_2(t)\|\} < \infty$).

We finally obtain the estimate

$$\|\tilde{U}(t,s)P(s)x\| \le 2Ne^{-(\beta-\alpha)(t-s)/2}\|P(s)x\| \quad \text{for all } t \ge s \ge 0,\ x \in X.$$

Therefore, $(\tilde{U}(t,s))_{t\ge s\ge 0}$ has an exponential dichotomy with projections $(P(t))_{t\ge 0}$ and dichotomy constants $N' := \max\{H, 2N\}$, $\beta' := (\beta-\alpha)/2 > 0$.

Put $\tilde{x}(t) := e^{-\gamma t}x(t)$, and define

$$F : \mathbb{R}_+ \times X \to X, \quad F(t,x) = e^{-\gamma t}f(t, e^{\gamma t}x) \quad \text{for all } t \ge 0,\ x \in X.$$

We can easily verify that F is also φ-Lipschitz. Thus, we can rewrite (3.1) as

$$\tilde{x}(t) = \tilde{U}(t,s)\tilde{x}(s) + \int_s^t \tilde{U}(t,\xi)F(\xi,\tilde{x}(\xi))\,d\xi \quad \text{for a.e. } t \ge s \ge 0. \tag{4.1}$$

Hence, by Theorem 3.9, if

$$N^2N_1\|e_{\beta'}\|_E\|\varphi\|_{E'} + N\|h_{\beta'}\|_E < 1$$

then there exists an invariant-stable manifold $\mathbf{C}$ of $\mathcal{E}_{\mathbb{R}}$-class for the solutions of (4.1). Returning to (3.1), by using the relation $x(t) := e^{\gamma t}\tilde{x}(t)$, we can easily verify the properties of $\mathbf{C}$ stated in (i)–(iv). ∎

REMARK 4.3. In case the evolution family has an exponential trichotomy and the nonlinear term f satisfies the local φ-Lipschitz condition (i.e., f is of class (M,φ,ρ)) with $f(t,0) = 0$ and the positive function $\varphi \in E$ satisfying $k < \min\{\frac{\rho}{2M}, 1\}$ and $N\|h_{\beta'}\|_E < 1$ (here k is defined as in (3.5)), then in a similar way, using the results in Subsection 3.1, we can obtain the existence of a *local-center-stable manifold* of $\mathcal{E}$-class for the solutions of (3.1), that is, a set $\mathbf{C} \subset \mathbb{R}_+ \times X$ such that there exist positive constants ρ, ρ_0, ρ_1 and a family of Lipschitz continuous mappings

$$g_t : B_{\rho_0} \cap \operatorname{Im}(P_0(t) + P_2(t)) \to B_{\rho_1} \cap \operatorname{Im} P_1(t), \qquad t \in \mathbb{R}_+,$$

with Lipschitz constants independent of t, satisfying:

(i) $\mathbf{C} = \{(t, x + g_t(x)) \in \mathbb{R}_+ \times (\operatorname{Im}(P_0(t) + P_2(t)) \oplus \operatorname{Im} P_1(t)) : t \in \mathbb{R}_+, x \in B_{\rho_0} \cap \operatorname{Im}(P_0(t) + P_2(t))\}$,

(ii) $\mathbf{C}_t := \{x + g_t(x) : (t, x + g_t(x)) \in \mathbf{C}\}$ is homeomorphic to $B_{\rho_0} \cap \operatorname{Im}(P_0(t) + P_2(t))$ for all $t \geq 0$,

(iii) to each $x_0 \in \mathbf{C}_{t_0}$ there corresponds one and only one solution $u(t)$ of (3.1) on $[t_0, \infty)$ such that $u(t_0) = x_0$ and $\chi_{[t_0,\infty)} e^{-\gamma \cdot} u(\cdot) \in \mathcal{E}$, where $\gamma := (\alpha + \beta)/2$.

(iv) for any two solutions $u_1(t), u_2(t)$ on the local-center-stable manifold $\mathbf{C}$ there exist positive constants μ and C_μ independent of $t_0 \geq 0$ such that

$$\|x(t) - y(t)\| \leq C_\mu e^{(\gamma-\mu)(t-t_0)} \times \|(P_0(t_0) + P_2(t_0))x(t_0) - (P_0(t_0) + P_2(t_0))y(t_0)\| \tag{4.2}$$

for all $t \geq t_0$.

5. Examples. In this section, we give some concrete examples to illustrate our abstract results. Let us start from a semilinear equation that has an autonomous linear part perturbed by a nonautonomous nonlinear forcing term, to show that, even in this simple case, our result is new.

Example 5.1. Consider the evolution equation

$$\frac{dx(t)}{dt} = Ax(t) + f(t, x), \tag{5.1}$$

where A is a sectorial operator whose spectrum $\sigma(A)$ decomposes into the disjoint sets $\{\lambda \in \sigma(A) : \operatorname{Re} \lambda < 0\}$, $\{\lambda \in \sigma(A) : \operatorname{Re} \lambda > 0\}$, and $\{\lambda \in \sigma(A) : \operatorname{Re}\lambda = 0\}$ such that $\sigma(A) \cap i\mathbb{R}$ consists of finitely many points. Then A is the generator of an analytic semigroup $(T(t))_{t\geq 0}$. We define the evolution family $U(t, s) := T(t - s)$ for all $t \geq s \geq 0$. We claim that it has an exponential trichotomy with appropriate projections. By the spectral mapping theorem for analytic semigroups, for fixed t_0, the spectrum of $T(t_0)$ splits into disjoint sets $\sigma_0, \sigma_1, \sigma_2$, where $\sigma_0 \subset \{|z| < 1\}$, $\sigma_1 \subset \{|z| > 1\}$, $\sigma_2 \subset \{|z| = 1\}$ with σ_2 consisting of finitely many points. Next, we let $P_0 = P_0(t_0)$, $P_1 = P_1(t_0)$, $P_2 = P_2(t_0)$ be the Riesz projections corresponding to the spectral sets $\sigma_0, \sigma_1, \sigma_2$, respectively. Clearly, P_0, P_1, P_2 commute with $T(t)$ for all $t \geq 0$.

Obviously, $P_0 + P_1 + P_2 = I$ and $P_i P_j = 0$ for $i \neq j$, and there are positive constants M, δ such that $\|T(t)|_{P_0 X}\| \leq M e^{-\delta t}$ for all $t \geq 0$. Furthermore, let $Q := P_1 + P_2 = I - P_0$ and consider the strongly continuous semigroup $(T_Q(t))_{t\geq 0}$ on $\operatorname{Im} Q$, where $T_Q(t) := T(t)Q$. Since $\sigma_1 \cup \sigma_2 = \sigma(T_Q(t_0))$, we can extend $(T_Q(t))_{t\geq 0}$ to a group $(T_Q(t))_{t\in\mathbb{R}}$ in $\operatorname{Im} Q$. As is well-known in semigroup theory, there are positive constants K, α, γ (with α as small as we wish; we may let $\alpha < \gamma$) such that

$$\|T_Q(-t)|_{P_1X}\| = \|(T_Q(t)|_{P_1X})^{-1}\| \leq Ke^{-\gamma t} \text{ for all } t \geq 0,$$
$$\|T_Q(t)|_{P_2X}\| \leq Ke^{\alpha|t|} \text{ for all } t \in \mathbb{R}.$$

Summing up, the evolution family $(U(t,s))_{t\geq s\geq 0}$ has an exponential trichotomy with projections P_j, $j = 0,1,2$, and positive constants N, α, β, where

$$\beta := \min\{\delta, \gamma\}, \qquad N := \max\{K, M\}.$$

Putting $\beta' := (\beta - \alpha)/2$ we find that, if f is φ-Lipschitz for some positive function $\varphi \in E'$ which is β'-exponentially E-invariant and

$$N^2 N_1 \|e_{\beta'}\|_E \|\varphi\|_{E'} + N\|h_{\beta'}\|_E < 1,$$

then there exists an admissibly center-stable manifold of $\mathcal{E}$-class for mild solutions of (5.1), i.e., of

$$x(t) = T(t-s)x(s) + \int_s^t T(t-\xi)f(\xi, x(\xi))\, d\xi \qquad \text{for all } t \geq s \geq 0.$$

The next examples give concrete samples of φ.

EXAMPLE 5.2. For a fixed $n \in \mathbb{N}^*$, consider the equation

$$\begin{aligned} &w_t(x,t) = w_{xx}(x,t) + n^2 w(x,t) + \varphi(t)\sin(w(x,t)), \\ &\qquad\qquad\qquad\qquad\qquad\qquad 0 < x < \pi,\ t \geq 0, \\ &w(0,t) = w(\pi,t) = 0, \qquad t \geq 0, \end{aligned} \tag{5.2}$$

where the step function $\varphi(t)$ is defined for fixed constants $c > 1$ and $b > 0$ by

$$\varphi(t) = \begin{cases} bm & \text{if } t \in \left[\frac{2m+1}{2} - \frac{1}{e^{cm}}, \frac{2m+1}{2} + \frac{1}{e^{cm}}\right] \quad \text{for } m = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases} \tag{5.3}$$

Here, the values of φ can be very large but we still have $\varphi \in L_q$ (for $1 \leq q < c$) since we can estimate $\|\varphi\|_{L_q}$ as follows:

$$\begin{aligned} \|\varphi\|_{L_q} &= \Big(\sum_{m=1}^{\infty} \int_{(2m+1)/2-1/e^{cm}}^{(2m+1)/2+1/e^{cm}} b^q m^q\, d\tau\Big)^{1/q} = \Big(\sum_{m=1}^{\infty} \frac{2b^q m^q}{e^{cm}}\Big)^{1/q} \\ &\leq \Big(\sum_{m=1}^{\infty} 2b^q e^{(q-c)m}\Big)^{1/q} = \frac{2^{1/q} b}{(1-e^{q-c})^{1/q}}. \end{aligned}$$

Note that in this case, $N_1 = N_2 = 1$.

We define $X := L_2[0,\pi]$, and let $A : X \supset D(A) \to X$ be defined by $A(y) = y'' + n^2 y$, with

$$D(A) = \{y \in X : y \text{ and } y' \text{ are absolutely continuous, } y' \in X,\ y(0) = y(\pi) = 0\}.$$

Equation (5.2) can now be rewritten as

$$\frac{du}{dt} = Au + f(t,u) \quad \text{for } u(t) = w(\cdot,t)$$

where $f : \mathbb{R}_+ \times X \to X$ with $f(t,u) = \varphi(t)\sin(u)$ for φ as in (5.3).

It can be seen [7] that A is the generator of an analytic semigroup $(T(t))_{t\ge 0}$.

Since $\sigma(A) = \{-1+n^2, -4+n^2, \dots, 0, -(1+n)^2+n^2, \dots\}$, applying the spectral mapping theorem for analytic semigroups we get

$$\sigma(T(t)) = e^{t\sigma(A)} = \{e^{t(n^2-1)}, e^{t(n^2-4)}, \dots, e^{t((n-1)^2-n^2)}\} \cup \{1\} \cup \{e^{-t((1+n)^2-n^2)}, e^{-t((2+n)^2-n^2)}, \dots\}.$$

One can easily see that the nonlinear forcing term f is φ-Lipschitz. Note that we may choose $\delta = \gamma = 1$ independent of n and $\alpha < 1$. Then $\beta = 1$, and $\beta' = (1-\alpha)/2$ is small, so that our next estimates are valid. We now compute $h_{\beta'}$ and estimate its norm. By Standing Hypothesis 2.10,

$$h_{\beta'}(t) = \Big(\int_0^\infty e^{-q\beta'|t-\tau|}\varphi^q(\tau)\,d\tau\Big)^{1/q} \tag{5.4}$$
$$= \Big(\sum_{m=1}^{[t]} \int_{(2m+1)/2-1/e^{cm}}^{(2m+1)/2+1/e^{cm}} e^{-q\beta'(t-\tau)} b^q m^q\,d\tau + \sum_{m=[t]+1}^{\infty} \int_{(2m+1)/2-1/e^{cm}}^{(2m+1)/2+1/e^{cm}} e^{-q\beta'(\tau-t)} b^q m^q\,d\tau\Big)^{1/q},$$

where $[t]$ is the integer part of t. Using the facts that $1 \le e^x$ for $x \ge 0$, and $e^X - 1 \le 9X$ for $0 \le X \le 2$, we can estimate the first sum in (5.4) as follows:

$$\sum_{m=1}^{[t]} \int_{(2m+1)/2-1/e^{cm}}^{(2m+1)/2+1/e^{cm}} e^{-q\beta'(t-\tau)} b^q m^q\,d\tau$$
$$= \frac{b^q e^{-q\beta' t}}{q\beta'} \sum_{m=1}^{[t]} \frac{e^{q\beta'(2m+1)/2}(e^{2q\beta'/e^{cm}} - 1)m^q}{e^{q\beta'/e^{cm}}}$$
$$\le \frac{b^q e^{-q\beta' t}}{q\beta'} \sum_{m=1}^{[t]} \frac{18q\beta' e^{q\beta'(2m+1)/2+mq}}{e^{cm}} = \frac{18 b^q e^{q\beta'/2} e^{-q\beta' t}}{1 - e^{q+q\beta'-c}}.$$

The second sum in (5.4) can be estimated by

$$\sum_{m=[t]+1}^{\infty}\int\limits_{(2m+1)/2-1/e^{cm}}^{(2m+1)/2+1/e^{cm}} e^{-q\beta'(\tau-t)}b^q m^q\,d\tau$$
$$=\frac{b^q}{q\beta'}\sum_{m=[t]+1}^{\infty}\frac{e^{-q\beta' t}e^{2q\beta' t}e^{-q\beta'(2m+1)/2}(e^{2q\beta'/e^{cm}}-1)m^q}{e^{q\beta'/e^{cm}}}$$
$$\le\frac{b^q e^{-q\beta'/2}e^{-q\beta' t}}{q\beta'}\sum_{m=[t]+1}^{\infty}\frac{18q\beta' e^{q\beta' m+mq}}{e^{cm}}=\frac{18b^q e^{-q\beta'/2}e^{-q\beta' t}}{1-e^{q+q\beta'-c}}.$$

Therefore,

$$h_{\beta'}(t)\le\left(\frac{18b^q}{1-e^{q+q\beta'-c}}\right)^{1/q}(e^{q\beta'/2}+e^{-q\beta'/2})^{1/q}e^{-\beta' t}\qquad\text{for all } t\ge 0.$$

Hence, $h_{\beta'}\in L_p$ and

$$\|h_{\beta'}\|_{L_p}<\frac{b\cdot 18^{1/q}}{(p\beta')^{1/p}(1-e^{q+q\beta'-c})^{1/q}}(e^{q\beta'/2}+e^{-q\beta'/2})^{1/q}.$$

Therefore, using the conclusions in Example 5.1 we deduce that, if

$$\frac{2^{1/q}N^2 b}{(\beta' p)^{1/p}(1-e^{q-c})^{1/q}}+\frac{bN\cdot 18^{1/q}}{(p\beta')^{1/p}(1-e^{q+q\beta'-c})^{1/q}}(e^{q\beta'/2}+e^{-q\beta'/2})^{1/q}<1, \tag{5.5}$$

then there exists a center-stable manifold of L_p-class for mild solutions of (5.2).

EXAMPLE 5.3. For a fixed $n\in\mathbb{N}^*$, consider the equation

$$\begin{aligned} &w_t(x,t)=a(t)[w_{xx}(x,t)+n^2w(x,t)]+\varphi(t)\sin(w(x,t)),\\ &\qquad\qquad 0<x<\pi,\ t\ge 0,\\ &w(0,t)=w(\pi,t)=0,\ t\ge 0, \end{aligned}\tag{5.6}$$

where φ is defined as in (5.3), while $a(\cdot)\in L_{1,\mathrm{loc}}(\mathbb{R}_+)$ satisfies $\gamma_1\ge a(t)\ge\gamma_0>0$ for fixed γ_0,γ_1 and a.e. $t\ge 0$.

We put $X:=L_2[0,\pi]$, and let $A:X\supset D(A)\to X$ be defined by $A(y)=y''+n^2y$, with

$$D(A)=\{y\in X: y \text{ and } y' \text{ are absolutely continuous}, y'\in X,\ y(0)=y(\pi)=0\}.$$

Putting $A(t):=a(t)A$, we can now rewrite (5.6) as

$$\frac{du}{dt}=A(t)u+f(t,u)\qquad\text{for } u(t)=w(\cdot,t)$$

where $f:\mathbb{R}_+\times X\to X$ with $f(t,u)=\varphi(t)\sin(u)$.

As in the above examples, A is a sectorial operator and generates an analytic semigroup $(T(t))_{t\geq 0}$, and $\sigma(A)$ satisfies the conditions as in Examples 5.1 and 5.2. Therefore, $A(t)$ "generates" the evolution family $(U(t,s))_{t\geq s\geq 0}$ defined by

$$U(t,s) = T\Big(\int_s^t a(\tau)\,d\tau\Big).$$

Arguing as in Examples 5.1 and 5.2 we see that the analytic semigroup $(T(t))_{t\geq 0}$ has an exponential trichotomy with projections P_k $(k = 0,1,2)$ and trichotomy constants N, α, β where α is as small as required. Moreover:

(i) $\|T(t)|_{P_0X}\| \leq Ne^{-\beta t}$,
(ii) $\|T(-t)|_{P_1X}\| = \|(T(t)|_{P_1X})^{-1}\| \leq Ne^{-\beta t}$,
(iii) $\|T(t)|_{P_2X}\| \leq Ne^{\alpha t}$,

for all $t \geq 0$. From this, it is straightforward to check that $(U(t,s))_{t\geq s\geq 0}$ has an exponential trichotomy with projections P_k $(k = 0,1,2)$ and trichotomy constants $N, \beta\gamma_0, \alpha\gamma_1$, by the following estimates:

$$\|U(t,s)|_{P_0X}\| = \Big\|T\Big(\int_s^t a(\tau)\,d\tau\Big)\Big|_{P_0X}\Big\| \leq Ne^{-\beta\gamma_0(t-s)},$$

$$\|U(s,t)_{|}\| = \|(U(t,s)|_{P_1X})^{-1}\| = \Big\|T\Big(-\int_s^t a(\tau)\,d\tau\Big)\Big|_{P_1X}\Big\| \leq Ne^{-\beta\gamma_0(t-s)},$$

$$\|U(t,s)|_{P_2X}\| = \Big\|T\Big(\int_s^t a(\tau)\,d\tau\Big)\Big|_{P_2X}\Big\| \leq Ne^{\alpha\gamma_1(t-s)},$$

for all $t \geq s \geq 0$. By Theorem 4.2, if (5.5) holds, then there exists a center-stable manifold of L_p-class for mild solutions of (5.6).

6. Admissibly unstable manifolds for equations defined on the whole line. We now consider the case where $(U(t,s))_{t\geq s}$ and f are defined on the whole line. That is, we will consider the integral equation

$$x(t) = U(t,s)x(s) + \int_s^t U(t,\xi)f(\xi,x(\xi))\,d\xi \quad \text{for a.e. } t \geq s. \tag{6.1}$$

As in Section 1, the solutions of (6.1) are the mild solutions of the equation

$$\frac{dx}{dt} = A(t)x + f(t,x), \qquad t\in\mathbb{R},\ x\in X,$$

where $A(t)$, $t \in \mathbb{R}$, are (in general) unbounded operators in X, which are coefficients of the well-posed Cauchy problem

$$\begin{cases} \frac{du(t)}{dt} = A(t)u(t), & t \geq s, \\ u(s) = x_s \in X, \end{cases}$$

whose solutions are given by $x(t) = U(t,s)x(s)$ as mentioned in the Introduction. In this case, admissibly (local- or invariant-) stable manifolds on $\mathbb{R}$ are defined and their existence is proved in a similar way to the case of equations defined on $\mathbb{R}_+$ (see [13, Thm. 4.7]). Therefore, we will focus on admissibly unstable manifolds which are defined below.

6.1. Local-unstable manifolds of $\mathcal{E}_{\mathbb{R}}$-class. We shall prove the existence of an admissibly local-unstable manifold under the conditions that $(U(t,s))_{t\geq s}$ has an exponential dichotomy and f is local φ-Lipschitz and in the class (M,φ,ρ) for a suitable positive function $\varphi \in E'_{\mathbb{R}}$.

Definition 6.1. A set $\mathbf{U} \subset \mathbb{R} \times X$ is said to be a *local-unstable manifold of $\mathcal{E}_{\mathbb{R}}$-class* for the solutions of (6.1) if for every $t \in \mathbb{R}$ the phase space X splits into a direct sum $X = X_0(t) \oplus X_1(t)$ with positive inclination, and if there exist positive constants ρ, ρ_0, ρ_1 and a family of Lipschitz continuous mappings

$$h_t : B_{\rho_0} \cap X_1(t) \to B_{\rho_1} \cap X_0(t), \qquad t \in \mathbb{R},$$

with Lipschitz constants independent of t, such that

(i) $\mathbf{U} = \{(t, x + h_t(x)) \in \mathbb{R} \times (X_1(t) \oplus X_0(t)) : x \in B_{\rho_0} \cap X_1(t)\}$,
(ii) $\mathbf{U}_t := \{x + h_t(x) : (t, x + h_t(x)) \in \mathbf{U}\}$ is homeomorphic to $B_{\rho_0} \cap X_1(t)$ for all $t \in \mathbb{R}$,
(iii) to each $x_0 \in \mathbf{U}_{t_0}$ there corresponds one and only one solution $x(t)$ of (6.1) such that $x(t_0) = x_0$ and the function $\chi_{(-\infty,t_0]}x(\cdot)$ belongs to the ball $\mathcal{B}_\rho$ in $\mathcal{E}^\infty_{\mathbb{R}} := \mathcal{E}_{\mathbb{R}} \cap L_\infty$.

Let $(U(t,s))_{t\geq s}$ have an exponential dichotomy with projections $P(t)$, $t \in \mathbb{R}$, and dichotomy constants $N, \beta > 0$. Then we can define the Green function as follows:

$$G(t,\tau) := \begin{cases} P(t)U(t,\tau) & \text{for } t \geq \tau, \\ -U(t,\tau)_|[I - P(\tau)] & \text{for } t < \tau. \end{cases} \tag{6.2}$$

Thus, we have

$$\|G(t,\tau)\| \leq N e^{-\beta|t-\tau|} \quad \text{for all } t \neq \tau. \tag{6.3}$$

We now prove the existence of a local-unstable manifold of $\mathcal{E}_{\mathbb{R}}$-class. To do that, we first find the form of the solutions of (6.1) which belong to admissible spaces on $(-\infty, t_0]$. We denote by $\|\cdot\|_\infty$ the sup-norm.

Lemma 6.2. *Let Standard Hypothesis* 2.10 *and Assumption* 2.12 *be satisfied with $\mathbb{I} = \mathbb{R}$. Suppose that $f : \mathbb{R} \times B_\rho \to X$ belongs to the class (M,φ,ρ). Let $x(t)$ be a solution of* (6.1) *such that for some fixed t_0 the* *function* $\chi_{(-\infty,t_0]}x(\cdot)$ *belongs to* $\mathcal{B}_\rho$. *Then for* $t \le t_0$,

$$x(t) = U(t,t_0)_| v_1 + \int_{-\infty}^{t_0} G(t,\tau) f(\tau, x(\tau))\, d\tau \tag{6.4}$$

for some $v_1 \in X_1(t_0) = (I - P(t_0))X$, *where* $G(t,\tau)$ *is the Green function defined as in* (6.2).

Proof. Let

$$y(t) := \int_{-\infty}^{t_0} G(t,\tau) f(\tau, x(\tau))\, d\tau \quad \text{for all } t \le t_0. \tag{6.5}$$

Then the function $y(\cdot)$ is bounded. Indeed, by estimates of the Green function G and of f we have

$$\begin{aligned} \|y(\cdot)\|_\infty &\le \int_{-\infty}^{t_0} Ne^{-\beta|t-\tau|} \|f(\tau, x(\tau))\|\, d\tau \\ &\le NM\Big[\int_{-\infty}^{t} e^{-\beta(t-\tau)}\varphi(\tau)\, d\tau + \int_{t}^{t_0} e^{\beta(t-\tau)}\varphi(\tau)\, d\tau\Big] \\ &\overset{(2.4)}{\le} NM\left[\frac{N_1\|\Lambda_1\varphi\|_\infty + N_2\|\Lambda_1 T_1^+\varphi\|_\infty}{1-e^{-\beta}}\right] < \infty. \end{aligned}$$

Next, by computing directly we verify that $y(\cdot)$ satisfies the integral equation

$$y(t_0) = U(t_0,t)y(t) + \int_{t}^{t_0} U(t_0,\tau) f(\tau, x(\tau))\, d\tau \quad \text{for all } t \le t_0. \tag{6.6}$$

Indeed, substituting y from (6.5) to the right-hand side of (6.6) we obtain

$$\begin{aligned} &U(t_0,t)y(t) + \int_{t}^{t_0} U(t_0,\tau) f(\tau, x(\tau))\, d\tau \\ &= U(t_0,t)\int_{-\infty}^{t_0} G(t,\tau) f(\tau, x(\tau))\, d\tau + \int_{t}^{t_0} U(t_0,\tau) f(\tau, x(\tau))\, d\tau \\ &= U(t_0,t)\int_{-\infty}^{t} U(t,\tau)P(\tau) f(\tau, x(\tau))\, d\tau \\ &\quad - U(t_0,t)\int_{t}^{t_0} U(t,\tau)_|(I-P(\tau)) f(\tau, x(\tau))\, d\tau + \int_{t}^{t_0} U(t_0,\tau) f(\tau, x(\tau))\, d\tau \end{aligned}$$

$$= \int_{-\infty}^{t} U(t_0,\tau)P(\tau)f(\tau,x(\tau))\,d\tau$$

$$- \int_{t}^{t_0} U(t_0,t)U(t,\tau)_|(I-P(\tau))f(\tau,x(\tau))\,d\tau + \int_{t}^{t_0} U(t_0,\tau)f(\tau,x(\tau))\,d\tau$$

$$= \int_{-\infty}^{t_0} U(t_0,\tau)P(\tau)f(\tau,x(\tau))\,d\tau = \int_{-\infty}^{t_0} G(t_0,\tau)f(\tau,x(\tau))\,d\tau = y(t_0);$$

here we use the fact that $U(t_0,t)U(t,\tau)_|(I-P(\tau)) = U(t_0,\tau)(I-P(\tau))$ for all $t \le \tau \le t_0$. Thus,

$$y(t_0) = U(t_0,t)y(t) + \int_{t}^{t_0} U(t_0,\tau)f(\tau,x(\tau))\,d\tau.$$

On the other hand,

$$x(t_0) = U(t_0,t)x(t) + \int_{t}^{t_0} U(t_0,\tau)f(\tau,x(\tau))\,d\tau.$$

Therefore $x(t_0) - y(t_0) = U(t_0,t)[x(t)-y(t)]$.

We need to prove that $x(t_0)-y(t_0) \in (I-P(t_0))X$. Applying the operator $P(t_0)$ to the expression $x(t_0)-y(t_0) = U(t_0,t)[x(t)-y(t)]$, we have

$$\begin{aligned}\|P(t_0)[x(t_0)-y(t_0)]\| &= \|U(t_0,t)P(t)[x(t)-y(t)]\| \\ &\le Ne^{-\beta(t_0-t)}\|P(t)\|\,\|x(t)-y(t)\|.\end{aligned}$$

Since $\sup_{t\in\mathbb{R}}\|P(t)\| < \infty$ and $\|x(t)-y(t)\| \le \|x(\cdot)\|_\infty + \|y(\cdot)\|_\infty < \infty$, letting $t \to -\infty$ we obtain

$$\|P(t_0)[x(t_0)-y(t_0)]\| = 0.$$

This means that $v_1 := x(t_0)-y(t_0) \in (I-P(t_0))X = X_1(t_0)$, finishing the proof. ∎

Remark 6.3. By computing directly in a similar way to Remark 3.7, we can see that the converse of Lemma 6.2 is also true. Thus all solutions of (6.4) satisfy (6.1) for $s \le t \le t_0$.

Lemma 6.4. *Under the assumptions of Lemma* 6.2, *for any positive numbers ρ and M, if f belongs to the class (M,φ,ρ) such that*

$$\frac{N}{1-e^{-\beta}}(N_1\|\Lambda_1T_1^+\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty) < \min\left\{1, \frac{\rho}{2M}\right\} \quad \text{and} \quad N\|h_\beta\|_E < 1,$$

then for $r = \rho/\max\{2N, 2NN_1\|e_\beta\|_{E_\mathbb{R}}\}$ and $t_0 \in \mathbb{R}$, there corresponds to each $v_1 \in B_r \cap X_1(t_0)$ one and only one solution $u(t)$ of (6.1) *on $(-\infty,t_0]$ such that $(I-P(t_0))u(t_0) = v_1$ and the function $\chi_{(-\infty,t_0]}u(\cdot)$ belongs to $\mathcal{B}_\rho$. Moreover, there exist positive constants μ and C_μ independent of t_0 such that* *for any two solutions $u_1(t), u_2(t)$ corresponding to different values $v_1, v_2 \in B_r \cap X_1(t_0)$ we have*

$$\|u_1(t) - u_2(t)\| \le C_\mu e^{-\mu(t_0-t)}\|v_1 - v_2\| \quad \text{for } t \le t_0. \tag{6.7}$$

Proof. For $v_1 \in B_r \cap X_1(t_0)$ we will prove that the transformation T defined by

$$(Tx)(t) = \begin{cases} U(t,t_0)_| v_1 + \int_{-\infty}^{t_0} G(t,\tau) f(\tau, x(\tau))\, d\tau & \text{for } t \le t_0, \\ 0 & \text{for } t > t_0, \end{cases}$$

acts from $\mathcal{B}_\rho$ into $\mathcal{B}_\rho$ and is a contraction. Indeed, for $x(\cdot) \in \mathcal{B}_\rho$ we have $\|f(t, x(t))\| \le M\varphi(t)$; therefore, putting

$$y(t) = \begin{cases} U(t,t_0)_| v_1 + \int_{-\infty}^{t_0} G(t,\tau) f(\tau, x(\tau))\, d\tau & \text{for } t \le t_0, \\ 0 & \text{for } t > t_0, \end{cases}$$

we obtain

$$\|y(t)\| \le Ne^{-\beta|t-t_0|}\|v_1\| + NM \int_{-\infty}^{t_0} e^{-\beta|t-\tau|}\varphi(\tau)\, d\tau. \tag{6.8}$$

It now follows from the admissibility of L_∞ that $y(\cdot) \in L_\infty(\mathbb{R}, X)$ and

$$\|y(\cdot)\|_\infty \le N\|v_1\| + \frac{NM}{1-e^{-\beta}}(N_1\|\Lambda_1 T_1^+\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty).$$

Using the fact that $\|v_1\| \le \frac{\rho}{2N}$ and

$$\frac{N}{1-e^{-\beta}}(N_1\|\Lambda_1 T_1^+\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty) < \frac{\rho}{2M},$$

we therefore obtain $\|y(\cdot)\|_\infty \le \rho$.

It follows from (6.8) and the admissibility of $E_\mathbb{R}$ that $y(\cdot) \in \mathcal{E}_\mathbb{R}$ and

$$\|y(\cdot)\|_{\mathcal{E}_\mathbb{R}} \le NN_1\|v_1\|\,\|e_\beta\|_{E_\mathbb{R}} + \frac{NM}{1-e^{-\beta}}(N_1\|\Lambda_1 T_1^+(\varphi)\|_{E_\mathbb{R}} + N_2\|\Lambda_1\varphi)\|_{E_\mathbb{R}}) \le \rho$$

Hence the transformation T acts from $\mathcal{B}_\rho$ to $\mathcal{B}_\rho$. We now estimate

$$\begin{aligned} \|Tx(t) - Ty(t)\| &\le \int_{-\infty}^{\infty} \|G(t,\tau)\|\,\|f(\tau,x(\tau)) - f(\tau,y(\tau))\|\, d\tau \\ &\le N\int_0^\infty e^{-\beta|t-\tau|}\varphi(\tau)\, d\tau\, \|x(\cdot) - y(\cdot)\|_\infty. \end{aligned}$$

Therefore,

$$\|Tx - Ty\|_\infty \le \frac{N}{1-e^{-\beta}}(N_1\|\Lambda_1 T_1^+\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty)\|x-y\|_\infty. \tag{6.9}$$

On the other hand,

$$\|Tx(t) - Ty(t)\| \leq \int_{-\infty}^{\infty} \|G(t,\tau)\| \, \|f(\tau, x(\tau)) - f(\tau, y(\tau))\| \, d\tau$$
$$\leq N \int_{0}^{\infty} e^{-\beta|t-\tau|}\varphi(\tau)\|x(\tau) - y(\tau)\| \, d\tau.$$

Since $\|x(\cdot) - y(\cdot)\| \in E_{\mathbb{R}}$ and $e^{-\beta|t-\cdot|}\varphi(\cdot) \in E'_{\mathbb{R}}$, by "Hölder's inequality" (2.6) we obtain

$$\|Tx(t) - Ty(t)\| \leq N\|e^{-\beta|t-\cdot|}\varphi(\cdot)\|_{E'_{\mathbb{R}}} \| \, \|x(\cdot) - y(\cdot)\| \|_{E_{\mathbb{R}}} = Nh_\beta(t)\|x - y\|_{\mathcal{E}_{\mathbb{R}}}.$$

By Standing Hypothesis 2.10 we then see that $h_\beta(\cdot) \in E_{\mathbb{R}}$, and hence

$$\|Tx - Ty\|_{\mathcal{E}_{\mathbb{R}}} \leq N\|h_\beta\|_{E_{\mathbb{R}}}\|x - y\|_{\mathcal{E}_{\mathbb{R}}}. \tag{6.10}$$

Putting $l = \max\big\{N\|h_\beta\|_{E_{\mathbb{R}}}, \frac{N}{1-e^{-\beta}}(N_1\|\varLambda_1 T_1^+\varphi\|_\infty + N_2\|\varLambda_1\varphi\|_\infty)\big\}$, by (6.9) and (6.10) we obtain $\|Tx - Ty\|_{\mathcal{E}^\infty_{\mathbb{R}}} \leq l\|x - y\|_{\mathcal{E}^\infty_{\mathbb{R}}}$. It follows from the assumptions that $l < 1$. Hence, $T : \mathcal{B}_\rho \to \mathcal{B}_\rho$ is a contraction.

Thus, there exists a unique $z(\cdot) \in \mathcal{B}_\rho$ such that $Tz = z$. By definition of T we know that z is a solution of (6.4) for $t \leq t_0$, and by Remark 6.3 it is a solution of (6.1) for all $s \leq t \leq t_0$. By Lemma 6.2 and Remark 6.3, $u(\cdot) := z_{(-\infty,t_0]}$ is the unique solution of (6.1) for $t \leq t_0$ such that $\chi_{(-\infty,t_0]}u(\cdot)$ belongs to $\mathcal{B}_\rho$.

The estimate (6.7) can be proved in the same way as in [13, Thm. 3.6]. ■

From Lemmata 6.2, 6.4 and using the same arguments as in [13, Thm. 3.8] we obtain the existence of an admissible local-unstable manifold:

THEOREM 6.5. *Under the assumptions of Lemma* 6.4, *for any* $\rho>0$ *and* $M > 0$, *if the function* f *belongs to the class* (M, φ, ρ) *such that*

$$\begin{aligned} k &:= \frac{N}{1 - e^{-\beta}}(N_1\|\varLambda_1 T_1^+\varphi\|_\infty + N_2\|\varLambda_1\varphi\|_\infty) < \min\left\{1, \frac{\rho}{2M}\right\}, \\ & N\|h_\beta\|_E < 1, \end{aligned} \tag{6.11}$$

then there exists a local-stable manifold $\mathbf{U}$ *of* $\mathcal{E}_{\mathbb{R}}$*-class for the solutions of* (6.1). *Moreover, any two solutions* $u_1(t), u_2(t)$ *on* $\mathbf{U}$ *attract each other exponentially in the sense that there exist positive constants* μ *and* C_μ *independent of* $t_0 \in \mathbb{R}$ *such that*

$$\begin{aligned} &\|u_1(t) - u_2(t)\| \\ &\quad \leq C_\mu e^{-\mu(t_0-t)}\|P(t_0)u_1(t_0) - P(t_0)u_2(t_0)\| \quad \textit{for } t \leq t_0. \end{aligned} \tag{6.12}$$

Proof. The proof is similar to that of [13, Thm. 3.7], replacing $\mathbb{R}_+$ by $\mathbb{R}$ and using the structure of solutions as in Lemmata 6.2 and 6.4. We just note that the family $(h_t)_{t\in\mathbb{R}}$ of Lipschitz mappings determining the local-unstable manifold of $\mathcal{E}_{\mathbb{R}}$-class is defined by

$$h_t : B_r \cap X_1(t) \to B_{\rho/2} \cap X_0(t), \qquad h_t(y) = \int_{-\infty}^{t} G(t,s)f(s,x(s))\,ds$$

for $r = \rho/\max\{2N, 2NN_1\|e_\beta\|_{E_{\mathbb{R}}}\}$ and $y \in B_r \cap X_1(t)$, where $x(\cdot)$ is the unique solution in $\mathcal{E}^\infty_{(-\infty,t]}$ of (6.1) on $(-\infty, t]$ satisfying $(I - P(t))x(t) = y$ (the existence and uniqueness of $x(\cdot)$ is obtained in Lemma 6.4). Furthermore, the Lipschitz constant of h_t is $\frac{kN}{1-k} < 1$, the same as that of g_t determining the local-stable manifold of $\mathcal{E}_{\mathbb{R}}$-class (see [13, Thm. 3.7]). ∎

From the existence of local-stable and local-unstable manifolds of $\mathcal{E}_{\mathbb{R}}$-class for (6.1) defined on the whole line we derive the following important corollary which describes the geometric picture of solutions to (6.1).

Corollary 6.6. *Under the assumptions of Lemma* 6.4, *for any* $\rho > 0$ *and* $M > 0$, *if* f *belongs to the class* (M, φ, ρ) *such that*

$$k < \min\left\{\frac{\rho}{2M}, 1, \frac{1}{M\max\{2N, 2NN_1\|e_\beta\|_E\}}\right\} \quad \text{and} \quad N\|h_\beta\|_{E_{\mathbb{R}}} < 1,$$

where k *is defined as in* (6.11), *then there exist a local-stable manifold* $\mathbf{S}$ *and a local-unstable manifold* $\mathbf{U}$ *of* $\mathcal{E}$*-class for the solutions of* (6.1) *having the following properties:*

(a) *For each* t_0, $\mathbf{S}_{t_0} \cap \mathbf{U}_{t_0}$ *contains the unique element* v_{t_0}.
(b) *The solution* $u_0(t)$ *of* (6.1) *with initial condition* $u_0(t_0) = v_{t_0}$ *belongs to the ball* $\mathcal{B}_\rho$ *in* $\mathcal{E}^\infty_{\mathbb{R}}$.
(c) *The solutions* $u(t)$ *of* (6.1) *satisfying* $u(t_0) \in \mathbf{S}_{t_0}$ *exponentially approach* $u_0(t)$ *as* $t \to \infty$ *and exponentially recede from* $u_0(t)$ *as* $t \to -\infty$.
(d) *The solutions* $u(t)$ *of* (6.1) *satisfying* $u(t_0) \in \mathbf{U}_{t_0}$ *exponentially approach* $u_0(t)$ *as* $t \to -\infty$ *and exponentially recede from* $u_0(t)$ *as* $t \to \infty$.

Proof. (a) The condition $x \in \mathbf{S}_{t_0} \cap \mathbf{U}_{t_0}$ is equivalent to the existence of $w \in B_{\rho_0} \cap X_0(t_0)$ and $y \in B_{\rho_0} \cap X_1(t_0)$ such that $x = w + g_{t_0}w = h_{t_0}y + y$, where g_{t_0} and h_{t_0} are members of the families $(g_t)_{t\in\mathbb{R}}$ of Lipschitz continuous mappings determining $\mathbf{S}$ and $(h_t)_{t\in\mathbb{R}}$ determining $\mathbf{U}$, respectively. Then $w - h_{t_0}y = y - g_{t_0}w \in X_0(t_0) \cap X_1(t_0) = \{0\}$. It follows that $w = h_{t_0}y$ and $y = g_{t_0}w$. Therefore, $w = h_{t_0}(g_{t_0}w) = (h_{t_0} \circ g_{t_0})w$. We now estimate $g_{t_0}w$ for $w \in B_{\rho_0} \cap X_0(t_0)$ by using the formula (see [13, eq. (22)])

$$g_{t_0}(w) = \int_{t_0}^{\infty} G(t_0,s)f(s,x(s))\,ds, \tag{6.13}$$

where $w \in B_{\rho_0} \cap X_0(t_0)$ and $x(\cdot)$ is the unique solution in $\mathcal{B}_\rho$ of (6.1) on

$[t_0,\infty)$ satisfying $P(t_0)x(t_0) = w$ (the existence and uniqueness of $x(\cdot)$ is obtained in Theorem 3.3). Note that $\rho_0 = 1/\max\{2N, 2NN_1\|e_\beta\|_E\}$. By (6.13) we have

$$\begin{aligned}\|g_{t_0}(w)\| &\le \int_{t_0}^{\infty} \|G(t_0,s)\|\,\|f(s,x(s))\|\,ds \le NM\int_{t_0}^{\infty} e^{-\beta|t_0-s|}\varphi(s)\,ds\\ &\le \frac{NM}{1-e^{-\beta}}(N_1\|\Lambda_1 T_1^{+}\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty)\\ &= kM < \frac{1}{\max\{2N, 2NN_1\|e_\beta\|_E\}} = \rho_0\end{aligned}$$

(since $k < 1/(M\max\{2N, 2NN_1\|e_\beta\|_E\})$). Hence, $g_{t_0} : B_{\rho_0}\cap X_0(t_0) \to B_{\rho_0}\cap X_1(t_0)$. Similarly, $h_{t_0} : B_{\rho_0}\cap X_1(t_0) \to B_{\rho_0}\cap X_0(t_0)$. It follows that

$$h_{t_0}\circ g_{t_0} : B_{\rho_0}\cap X_0(t_0) \to B_{\rho_0}\cap X_0(t_0).$$

As g_{t_0} and h_{t_0} are both Lipschitz continuous with the same Lipschitz constant $\frac{kN}{1-k} < 1$ (see Theorem 6.5 and [13, proof of Thm. 3.8]), $h_{t_0}\circ g_{t_0}$ is a contraction. Thus, there exists a unique w_0 such that $w_0 = (h_{t_0}\circ g_{t_0})w_0$. Putting $v_{t_0} = w_0 + g_{t_0}w_0$ we find that v_{t_0} is the unique element of $\mathbf{S}_{t_0}\cap\mathbf{U}_{t_0}$.

Property (b) is a consequence of the definitions of the local-stable and local-unstable manifolds of $\mathcal{E}_{\mathbb{R}}$-class.

Properties (c) and (d) follow from (3.7) and (6.12), respectively. ∎

6.2. Invariant-unstable manifolds of $\mathcal{E}_{\mathbb{R}}$-class. In this subsection we consider the existence of an admissibly invariant-unstable manifold under the conditions that the evolution family has an exponential dichotomy, and the nonlinear term f is φ-Lipschitz continuous.

Definition 6.7. A set $\mathbf{U} \subset \mathbb{R}\times X$ is said to be an *invariant-unstable manifold of $\mathcal{E}_{\mathbb{R}}$-class* for the solutions of (6.1) if for every $t \in \mathbb{R}$ the phase space X splits into a direct sum $X = X_0(t)\oplus X_1(t)$ with positive inclination, and if there exists a family of Lipschitz continuous mappings

$$g_t : X_1(t) \to X_0(t), \quad t \in \mathbb{R},$$

with Lipschitz constants independent of t, such that

- (i) $\mathbf{U} = \{(t, x + g_t(x)) \in \mathbb{R}\times(X_1(t)\oplus X_0(t)) : x \in X_1(t)\}$,
- (ii) $\mathbf{U}_t := \{x + g_t(x) : (t, x + g_t(x)) \in \mathbf{U}\}$ is homeomorphic to $X_1(t)$ for all $t\in\mathbb{R}$,
- (iii) to each $x_0 \in \mathbf{U}_{t_0}$ there corresponds one and only one solution $x(t)$ of (6.1) such that $x(t_0) = x_0$ and the function $\chi_{(-\infty,t_0]}x(\cdot)$ belongs to $\mathcal{E}_{\mathbb{R}}$.
- (iv) $\mathbf{U}$ is invariant under (6.1) in the sense that if $x(\cdot)$ is a solution of (6.1) such that $x(t_0)\in\mathbf{U}_{t_0}$ and the function $\chi_{(-\infty,t_0]}x(\cdot)$ belongs to $\mathcal{E}_{\mathbb{R}}$, then $x(t)\in\mathbf{U}_t$ for all $t\le t_0$.

As in the previous subsection (Lemma 6.2), we can find the form of the solutions of (6.1) which belong to admissible spaces on $(-\infty, t_0]$:

LEMMA 6.8. *Let Standing Hypothesis* 2.10 *and Assumption* 2.12 *be satisfied with* $\mathbb{I} = \mathbb{R}$. *Suppose that* $f : \mathbb{R} \times X \to X$ *is* φ*-Lipschitz. Let* $x(t)$, $t \le t_0$, *be a solution of* (6.1) *such that* $\chi_{(-\infty,0]}x(\cdot)$ *belongs to* $\mathcal{E}_{\mathbb{R}}$. *Then for* $t \le t_0$,

$$x(t) = U(t,t_0)_{|}v_1 + \int_{-\infty}^{t_0} G(t,\tau)f(\tau,x(\tau))\,d\tau \tag{6.14}$$

for some $v_1 \in X_1(t_0) = (I - P(t_0))X$, *where* $G(t,\tau)$ *is the Green function* (6.2).

Proof. Put $y(t) := \int_{-\infty}^{t_0} \mathcal{G}(t,\tau)f(\tau,x(\tau))\,d\tau$ for $t \le t_0$ and $y(t) = 0$ for $t > t_0$. Since f is φ-Lipschitz, using (6.3) we obtain

$$\|y(t)\| \le N \int_{-\infty}^{\infty} e^{-\beta|t-\tau|}\varphi(\tau)\|z(\tau)\|\,d\tau \quad \text{for } t \le t_0.$$

Using the "Hölder inequality" (2.6) it now follows that

$$\|y(t)\| \le \|e^{-\beta|t-\cdot|}\varphi(\cdot)\|_{E'_{\mathbb{R}}}\|z\|_{E_{\mathbb{R}}}.$$

By Standing Hypothesis 2.10, the function $h_\beta(t) = \|e^{-\beta|t-\cdot|}\varphi(\cdot)\|_{E'_{\mathbb{R}}}$ belongs to $E_{\mathbb{R}}$. Therefore, by Banach lattice properties, $y(\cdot) \in \mathcal{E}_{\mathbb{R}}$ and

$$\|y(\cdot)\|_{\mathcal{E}_{\mathbb{R}}} \le \|h_\beta\|_{E_{\mathbb{R}}}\|z\|_{E_{\mathbb{R}}}.$$

By similar calculations to those in the proof of Lemma 6.2 we can see that

$$y(t_0) = U(t_0,t)y(t) + \int_{t}^{t_0} U(t,s)f(s,x(s))\,ds \quad \text{for } t \le t_0.$$

Since $x(t)$ is a solution of (6.1) we obtain

$$x(t_0) - y(t_0) = U(t_0,t)(x(t) - y(t)) \quad \text{for } t \le t_0.$$

Putting now $v_1 = x(t_0) - y(t_0)$ and applying the operator $P(t_0)$ to the above expression we have

$$\|P(t_0)[x(t_0)-y(t_0)]\| = \|U(t_0,t)P(t)[x(t)-y(t)]\| \le Ne^{-\beta(t_0-t)}\|x(t)-y(t)\|.$$

So $\|x(t) - y(t)\| \ge Ne^{\beta(t_0-t)}\|P(t_0)[x(t_0) - y(t_0)]\|$. Since $(x(\cdot) - y(\cdot))|_{(-\infty,t_0]}$ belongs to $\mathcal{E}_{(-\infty,t_0]}$ and the function $e^{\beta(t_0-t)}$, $t \le t_0$, does not belong to $E_{(-\infty,t_0]}$, the admissibility of $E_{(-\infty,t_0]}$ shows that $P(t_0)[x(t_0) - y(t_0)] = 0$. Therefore, $v_1 := x(t_0) - y(t_0) \in X_1(t_0)$. Finally, since $x(t) = U(t,t_0)_{|}v_1 + y(t)$ for $t \le t_0$, the equality (6.14) follows. ∎

REMARK 6.9. By computing directly in a similar way to Remark 3.7, we can see that the converse of Lemma 6.8 is also true. Hence all solutions of (6.14) satisfy (6.1) for $t \le t_0$.

Similarly to Lemma 6.4 we have the following lemma which describes the existence and uniqueness of certain bounded solutions.

LEMMA 6.10. *Under the assumptions of Lemma* 6.8, *let* $f : \mathbb{R} \times X \to X$ *be* φ*-Lipschitz such that* $N\|h_\beta\|_{E_\mathbb{R}} < 1$. *Then there corresponds to each* $v_1 \in X_1(t_0)$ *one and only one solution* $x(t)$ *of* (6.1) *on* $(-\infty, t_0]$ *such that* $(I - P(t_0))x(t_0) = v_1$ *and the function* $\chi_{(-\infty,0]}x(\cdot)$ *belongs to* $\mathcal{E}_\mathbb{R}$.

Proof. For each $t_0 \in \mathbb{R}$ and $v_1 \in X_1(t_0)$ we will prove that the transformation T defined by

$$(Tx)(t) = \begin{cases} U(t,t_0)_| v_1 + \int_{-\infty}^{t_0} G(t,\tau)f(\tau,x(\tau))\,d\tau & \text{for all } t \le t_0, \\ 0 & \text{for all } t > t_0, \end{cases}$$

acts from $\mathcal{E}_\mathbb{R}$ into $\mathcal{E}_\mathbb{R}$ and is a contraction.

Indeed, for $x(\cdot) \in \mathcal{E}_\mathbb{R}$ we have $\|f(t,x(t))\| \le \varphi(t)\|x(t)\|$, and therefore, putting

$$y(t) = \begin{cases} U(t,t_0)_| v_1 + \int_{t_0}^{\infty} G(t,\tau)f(\tau,x(\tau))\,d\tau & \text{for } t \le t_0, \\ 0 & \text{for } t > t_0, \end{cases}$$

we have

$$\|y(t)\| \le Ne^{-\beta|t_0-t|}\|v_1\| + NM \int_{-\infty}^{\infty} e^{-\beta|t-\tau|}\varphi(\tau)\|x(\tau)\|\,d\tau \quad \text{for all } t \in \mathbb{R}.$$

Putting $e_\beta(t) := e^{-\beta|t|}$, $t \in \mathbb{R}$, and using the "Hölder inequality" (2.6) we see that

$$\|y(t)\| \le N\|v_1\|(T_{t_0}^+ e_\beta)(t) + \|e^{-\beta|t-\cdot|}\varphi(\cdot)\|_{E'_\mathbb{R}}\|x\|_{E_\mathbb{R}} \quad \text{for all } t \in \mathbb{R}.$$

By Standing Hypothesis 2.10, the function $h_\beta(t) = \|e^{-\beta|t-\cdot|}\varphi(\cdot)\|_{E'_\mathbb{R}}$ belongs to $E_\mathbb{R}$. Therefore, by Banach lattice properties, $y(\cdot) \in \mathcal{E}_\mathbb{R}$ and

$$\|y(\cdot)\|_{\mathcal{E}_\mathbb{R}} \le NN_1\|v_1\|\,\|e_\beta\|_{E_\mathbb{R}} + \|h_\beta\|_{E_\mathbb{R}}\|x\|_{E_\mathbb{R}}.$$

Hence, the transformation T acts from $\mathcal{E}_\mathbb{R}$ into $\mathcal{E}_\mathbb{R}$.

It follows from the estimates of G and f that

$$\begin{aligned} \|T(x) - T(y)\|_\infty &\le \int_{-\infty}^{t_0} \|G(t,\tau)\|\,\|f(\tau,x(\tau)) - f(\tau,y(\tau))\|\,d\tau \\ &\le N \int_{-\infty}^{\infty} e^{-\beta|t-\tau|}\varphi(\tau)\|x(\tau) - y(\tau)\|\,d\tau. \end{aligned}$$

Since $\|x(\cdot)-y(\cdot)\| \in E_{\mathbb{R}}$ and $e^{-\beta|t-\cdot|}\varphi(\cdot) \in E'_{\mathbb{R}}$, by (2.6) we obtain

$$\|Tx(t)-Ty(t)\| \le N\|e^{-\beta|t-\cdot|}\varphi(\cdot)\|_{E'_{\mathbb{R}}}\| \, \|x(\cdot)-y(\cdot)\|\|_{E_{\mathbb{R}}} = Nh_\beta(t)\|x-y\|_{\mathcal{E}_{\mathbb{R}}}$$

for all $t \in \mathbb{R}$. Since $h_\beta \in E_{\mathbb{R}}$, we then have

$$\|Tx-Ty\|_{\mathcal{E}_{\mathbb{R}}} \le N\|h_\beta\|_{E_{\mathbb{R}}}\|x-y\|_{\mathcal{E}_{\mathbb{R}}}.$$

Hence, if $N\|h_\beta\|_{E_{\mathbb{R}}} < 1$, we conclude that $T : \mathcal{E}_{\mathbb{R}} \to \mathcal{E}_{\mathbb{R}}$ is a contraction with contraction constant $k = N\|h_\beta\|_{E_{\mathbb{R}}}$.

By the Banach contraction mapping theorem, the lemma follows. ∎

From Lemmata 6.8 and 6.10, using the same arguments as in [13, Thm. 4.6] we now obtain the existence of an invariant-unstable manifold of $\mathcal{E}_{\mathbb{R}}$-class:

Theorem 6.11. *Under the assumptions of Lemma* 6.8, *suppose that* $f : \mathbb{R}\times X \to X$ *is* φ*-Lipschitz such that* $NN_1\|e_\beta\|_{E_{\mathbb{R}}}\|\varphi\|_{E'_{\mathbb{R}}} + N\|h_\beta\|_{E_{\mathbb{R}}} < 1$. *Then there exists an invariant-unstable manifold* $\mathbf{U}$ *of* $\mathcal{E}_{\mathbb{R}}$*-class for the solutions of* (6.1). *Moreover, for any two solutions* $x_1(\cdot)$ *and* $x_2(\cdot)$ *belonging to* $\mathbf{U}$, *we have*

$$\|x_1(t)-x_2(t)\| \le Ce^{-\mu(t_0-t)}\|(I-P(t_0))x_1(t_0)-(I-P(t_0))x_2(t_0)\|$$

for all $t \le t_0$, *where* C, μ *are positive constants independent of* t_0, $x_1(\cdot)$ *and* $x_2(\cdot)$.

Proof. The proof is similar to that of [13, Thm. 4.6], replacing $\mathbb{R}_+$ by $\mathbb{R}$ and using the structure of bounded solutions as in Lemmata 6.8, 6.10. We just note that the family $(h_t)_{t\in\mathbb{R}}$ of Lipschitz mappings determining the unstable manifold is defined by

$$h_t : X_1(t) \to X_0(t), \qquad h_t(y) = \int_{-\infty}^{t} G(t,s)f(s,x(s))\,ds$$

for $y \in X_1(t)$, where $x(\cdot)$ is the unique solution in $\mathcal{E}_{(-\infty,t]}$ of (6.1) on $(-\infty,t]$ satisfying $(I-P(t))x(t) = y$ (the existence and uniqueness of $x(\cdot)$ is obtained in Lemma 6.10). ∎

Using similar arguments to those for Corollary 6.6, we easily obtain the following corollary, which describes the relations of solutions of (6.1) with initial values lying on the invariant-stable or invariant-unstable manifolds and the solution lying on the intersection of the two manifolds.

Corollary 6.12. *Under the assumptions of Lemma* 6.8, *let* f *be* φ*-Lipschitz such that* $NN_1\|e_\beta\|_{E_{\mathbb{R}}}\|\varphi\|_{E'_{\mathbb{R}}} + N\|h_\beta\|_{E_{\mathbb{R}}} < 1$. *Then there exist an invariant-stable manifold* $\mathbf{S}$ *and an invariant-unstable manifold* $\mathbf{U}$ *of* $\mathcal{E}_{\mathbb{R}}$*-class for the solutions of* (6.1) *having the following properties:*

(a) *For each* t_0, $\mathbf{S}_{t_0} \cap \mathbf{U}_{t_0}$ *contains the unique element* v_{t_0}.
(b) *The solution* $u_0(t)$ *of* (6.1) *with initial condition* $u_0(t_0) = v_{t_0}$ *belongs to* $\mathcal{E}_{\mathbb{R}}$.

(c) *The solutions $u(t)$ of* (6.1) *satisfying $u(t_0) \in \mathbf{S}_{t_0}$ exponentially approach $u_0(t)$ as $t \to \infty$ and exponentially recede from $u_0(t)$ as $t \to -\infty$.*

(d) *The solutions $u(t)$ of* (6.1) *satisfying $u(t_0) \in \mathbf{U}_{t_0}$ exponentially approach $u_0(t)$ as $t \to -\infty$ and exponentially recede from $u_0(t)$ as $t \to \infty$.*

We illustrate our results on the existence of invariant-stable and invariant-unstable manifolds by the following example.

EXAMPLE 6.13. We consider the problem

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = \sum_{k,l=1}^{n} D_k(a_{kl}(t,x)D_l u(t,x)) \\ \qquad\qquad + \delta u(t,x) + \varphi(t)\sin(u(t,x)), \quad t \geq s,\ x \in \Omega, \\ \sum_{k,l=1}^{n} n_k(x)a_{kl}(t,x)D_l u(t,x) = 0, \quad t \geq s,\ x \in \partial\Omega, \\ u(s,x) = f(x), \quad x \in \Omega. \end{cases} \tag{6.15}$$

Here $D_k := \partial/\partial x_k$ and Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ oriented by outer unit normal vectors $n(x)$. The coefficients $a_{k,l}(t,x) \in C_b^{\mu}(\mathbb{R}, L_\infty(\Omega))$, $\mu > 1/2$, are supposed to be real, symmetric, and uniformly elliptic in the sense that

$$\sum_{k,l=1}^{n} a_{kl}(t,x)v_k v_l \geq \eta|v|^2 \quad \text{for all } t \in \mathbb{R},\ \text{a.e. } x \in \Omega$$

and some constant $\eta > 0$. Also, the constant δ is defined by $\delta := -\frac{1}{2}\eta\lambda$, where $\lambda < 0$ denotes the largest negative eigenvalue of the Neumann Laplacian Δ_N on Ω.

Finally, the step function $\varphi(t)$ is defined for fixed constants $c > 1$ and $b > 0$ by

$$\varphi(t) = \begin{cases} b|m| & \text{if } t \in \left[\frac{2m+1}{2} - \frac{1}{e^{cm}}, \frac{2m+1}{2} + \frac{1}{e^{cm}}\right] \text{ for } m = \pm 1, \pm 2, \ldots, \\ 0 & \text{otherwise.} \end{cases} \tag{6.16}$$

Here, we note that the values of φ can be very large, however, by computing as in Example 5.2, we still have $\varphi \in L_q$ (for $1 \leq q < c$) and

$$\|\varphi(\tau)\|_{L_q} \leq \frac{4^{1/q}b}{(1 - e^{q-c})^{1/q}}.$$

We now choose the Hilbert space $X = L_2(\Omega)$ and define the operators $C(t)$ via the standard scalar product in X as

$$\langle C(t)f, g\rangle = -\sum_{k,l=1}^{n} \int_{\Omega} a_{kl} D_k f(x)\overline{D_l g(x)}\, dx$$

with $D(C(t)) = \{f \in W^{2,2}(\Omega) : \sum_{k,l=1}^{n} n_k(x)a_{kl}(t,x)D_l f(x) = 0,\ x \in \partial\Omega\}$.

We then write (6.15) as an abstract Cauchy problem

$$\begin{cases} \frac{d}{dt}u(t,\cdot) = A(t)u(t,\cdot) + F(t,u(t,\cdot)), & t \geq s \geq 0, \\ u(s,\cdot) = f \in X, \end{cases}$$

where $A(t) := C(t) + \delta$ and $F : \mathbb{R}_+ \times X \to X$ defined by $F(t,f)(x) := \varphi(t)\sin(f(x))$ for $(t,f) \in \mathbb{R}_+ \times X$, with φ defined as in (6.16).

By Schnaubelt [28, Chapt. 2, Theorem 2.8, Example 2.3], the operators $A(t)$ generate an evolution family having an exponential dichotomy with dichotomy constants N and β, provided that the Hölder constants of $a_{k,l}$ are sufficiently small. Also, the dichotomy projections $P(t)$, $t \in \mathbb{R}$, satisfy $\sup_{t\in\mathbb{R}} \|P(t)\| \leq N$.

We now easily see that F is φ-Lipschitz for $\varphi \in L_q$ as above. In this space, the constants N_1 and N_2 in Definition 2.3 are both 1. Note that we may choose β to be small, so that our estimates on $h_\beta(t)$ are valid.

Similarly to Example 5.2, we can estimate

$$h_\beta(t) = \Big(\int_{-\infty}^{\infty} e^{-q\beta|t-\tau|}\varphi^q(\tau)\,d\tau\Big)^{1/q}$$

by

$$h_\beta(t) \leq \frac{b \cdot 36^{1/q} e^{\beta/2} e^{-\beta|t|}}{(1 - e^{q-c+q\beta})^{1/q}} \quad \text{for all } t \in \mathbb{R}.$$

Hence, $h_\beta \in L_p$ and

$$\|h_\beta\|_{L_p} \leq \frac{b \cdot 36^{1/q} e^{\beta/2}}{(p\beta)^{1/p}(1 - e^{q-c+q\beta})^{1/q}}.$$

Here, $1/p + 1/q = 1$, and $p = \infty$ if $q = 1$.

Therefore, by Theorems 6.11 and 3.9 (precisely, its counterpart on $\mathbb{R}$), if

$$\text{(6.17)} \qquad \frac{2^{(1+q)/q} N^2 b}{(\beta p)^{1/p}(1 - e^{q-c})^{1/q}} + \frac{bN \cdot 36^{1/q} e^{\beta/2}}{(p\beta)^{1/p}(1 - e^{q-c+q\beta})^{1/q}} < 1,$$

then there exist an invariant-unstable manifold $\mathbf{U}$ and an invariant-stable manifold $\mathbf{S}$ of L_p-class for the mild solutions of (6.15).

6.3. Invariant-center-unstable manifolds of $\mathcal{E}_\mathbb{R}$-class. Using Theorem 6.11 and rescaling procedures similar to Theorem 4.2 to transform the trichotomy case to the dichotomy case, we can easily obtain the existence of an invariant-center-unstable manifold of $\mathcal{E}_\mathbb{R}$-class:

THEOREM 6.14. *Let Assumption* 4.1 *and Standing Hypothesis* 2.10 *be satisfied with* $\mathbb{I} = \mathbb{R}$ *and suppose that* $f : \mathbb{R}_+ \times X \to X$ *is* φ*-Lipschitz with*

$$N^2 N_1 \|e_{\beta'}\|_{E_\mathbb{R}} \|\varphi\|_{E'_\mathbb{R}} + N\|h_{\beta'}\|_{E_\mathbb{R}} < 1.$$

Then there exists an invariant-center-unstable manifold $\mathbf{C^u} = \{(t, \mathbf{C^u_t}) : t \in \mathbb{R}_+ \text{ and } \mathbf{C^u_t} \subset X\}$ *of* $\mathcal{E}_\mathbb{R}$*-class for the solutions of* (6.1), *with the family*

$(\mathbf{C}^{\mathbf{u}}_t)_{t\in\mathbb{R}}$ *being the graphs of the family of Lipschitz continuous mappings* $(h_t)_{t\in\mathbb{R}}$ (*i.e.*, $\mathbf{C}^{\mathbf{u}}_t := \mathrm{graph}(h_t) = \{x + h_t x : x \in \mathrm{Im}(P_1(t) + P_2(t))\}$ *for each* $t \in \mathbb{R}$) *where* $h_t : \mathrm{Im}(P_1(t) + P_2(t)) \to \mathrm{Im}\, P_0(t)$ *has Lipschitz constant*

$$l = \frac{N^2 N_1 \|e_{\beta'}\|_{E_{\mathbb{R}}} \|\varphi\|_{E'_{\mathbb{R}}}}{1 - N\|h_{\beta'}\|_{E_{\mathbb{R}}}}$$

independent of t, *such that:*

(i) *to each* $x_0 \in \mathbf{C}^{\mathbf{u}}_{t_0}$ *there corresponds one and only one solution* $u(t)$ *of* (6.1) *on* $(-\infty, t_0]$ *such that* $u(t_0) = x_0$ *and the function* $\chi_{(-\infty,t_0]} e^{\gamma\cdot} u(\cdot)$ *belongs to* $\mathcal{E}_{\mathbb{R}}$, *where* $\gamma := (\alpha + \beta)/2$.

(ii) $\mathbf{C}^{\mathbf{u}}_t$ *is homeomorphic to* $X_1(t) \oplus X_2(t)$ *for all* $t \in \mathbb{R}$, *where* $X_1(t) = \mathrm{Im}\, P_1(t)$ *and* $X_2(t) = \mathrm{Im}\, P_2(t)$,

(iii) $\mathbf{C}^{\mathbf{u}}$ *is invariant under* (6.1) *in the sense that if* $u(t)$ *is the solution of* (6.1) *such that* $u(t_0) = x_0 \in \mathbf{C}^{\mathbf{u}}_{t_0}$ *and the function* $\chi_{(-\infty,t_0]} e^{\gamma\cdot} u(\cdot)$ *belongs to* $\mathcal{E}_{\mathbb{R}}$, *then* $u(s) \in \mathbf{C}^{\mathbf{u}}_s$ *for all* $s \le t_0$,

(iv) *for any two solutions* $u_1(t), u_2(t)$ *on the center-unstable manifold* $\mathbf{C}^{\mathbf{u}}$ *there exist positive constants* μ *and* C_μ *independent of* $t_0 \in \mathbb{R}$ *such that*

$$\begin{aligned}\|x(t) - y(t)\| \le{}& C_\mu e^{(\gamma-\mu)(t_0-t)} \\ &\times \|(P_1(t_0) + P_2(t_0))x(t_0) - (P_1(t_0) + P_2(t_0))y(t_0)\|,\end{aligned} \tag{6.18}$$

for all $t \le t_0$.

Note that an invariant-center-stable manifold of $\mathcal{E}_{\mathbb{R}}$-class on the whole line is defined and its existence is proved in the same way as in the case of $\mathbb{R}_+$ (see Theorem 4.2).

From the existence of invariant-center-stable and center-unstable manifolds of $\mathcal{E}_{\mathbb{R}}$-class for (6.1) defined on the whole line we have the following important corollary describing the behavior of solutions to (6.1).

COROLLARY 6.15. *Under the assumptions of Theorem* 6.14, *suppose that* f *is* φ*-Lipschitz with*

$$N^2 N_1 \|e_{\beta'}\|_{E_{\mathbb{R}}} \|\varphi\|_{E'_{\mathbb{R}}} + (\sqrt{2} - 1) N \|h_{\beta'}\|_{E_{\mathbb{R}}} < \sqrt{2} - 1. \tag{6.19}$$

Then there exist an invariant-center-stable manifold $\mathbf{C}$ *and an invariant-center-unstable manifold* $\mathbf{C}^{\mathbf{u}}$ *of* $\mathcal{E}_{\mathbb{R}}$*-class for the solutions of equation* (6.1) *having the following properties:*

(a) *For each* $t_0 \in \mathbb{R}$, $\mathbf{C}_{t_0} \cap \mathbf{C}^{\mathbf{u}}_{t_0}$ *is homeomorphic to* $X_2(t_0) = P_2(t_0)X$.

(b) *The solution* $u_0(t)$ *of* (6.1) *with initial condition* $u_0(t_0) \in \mathbf{C}_{t_0} \cap \mathbf{C}^{\mathbf{u}}_{t_0}$ *satisfies* $e^{-\gamma|\cdot|} u_0(\cdot) \in \mathcal{E}_{\mathbb{R}}$, *where* $\gamma := (\alpha + \beta)/2$.

(c) *For the solution* $u(t)$ *of* (6.1) *satisfying* $u(t_0) \in \mathbf{C}_{t_0}$ *the function* $e^{-\gamma t} u(t)$ *exponentially approaches* $e^{-\gamma t} u_0(t)$ *as* $t \to \infty$, *and* $e^{\gamma t} u(t)$ *exponentially recedes from* $e^{\gamma t} u_0(t)$ *as* $t \to -\infty$.

(d) *For the solution $u(t)$ of (6.1) satisfying $u(t_0) \in \mathbf{C}^{\mathbf{u}}_{t_0}$ the function $e^{\gamma t}u(t)$ exponentially approaches $e^{\gamma t}u_0(t)$ as $t \to -\infty$, and $e^{-\gamma t}u(t)$ exponentially recedes from $e^{-\gamma t}u_0(t)$ as $t \to \infty$.*

Proof. (a) Let us first prove that for each $z \in X_2(t)$ there exists a unique $w \in X_0(t) \oplus X_2(t)$ such that $w = h_t(z + g_t(w)) + z$, where g_t and h_t are the members of the Lipschitz mapping families $(g_t)_{t\in\mathbb{R}}$ and $(h_t)_{t\in\mathbb{R}}$ determining the invariant-center-stable and invariant-center-unstable manifolds, respectively. Indeed, the mapping

$$\mathbf{L} : X_0(t) \oplus X_2(t) \to X_0(t) \oplus X_2(t), \qquad y \mapsto h_t(z + g_t(y)) + z,$$

satisfies

$$\begin{aligned} \|\mathbf{L}y_1 - \mathbf{L}y_2\| &= \|h_t(z + g_t(y_1)) - h_t(z + g_t(y_2))\| \\ &\le \frac{N^2 N_1 \|e_\beta\|_{E_\mathbb{R}} \|\varphi\|_{E'_\mathbb{R}}}{1 - N\|h_\beta\|_{E_\mathbb{R}}} \|g_t(y_1) - g_t(y_2)\| \le l^2 \|y_1 - y_2\|, \end{aligned}$$

where $l = \frac{N^2 N_1 \|e_\beta\|_{E_\mathbb{R}} \|\varphi\|_{E'_\mathbb{R}}}{1 - N\|h_\beta\|_{E_\mathbb{R}}}$ is the Lipschitz constant of g_t and h_t.

Since $l < \sqrt{2} - 1 < 1$ we see that $\mathbf{L}$ is a contraction. Let w be its unique fixed point. Then w is the unique element in $X_0(t) \oplus X_2(t)$ such that $w = h_t(z + g_t(w)) + z$.

Define now $\mathbf{D} : X_2(t) \to \mathbf{C}_t \cap \mathbf{C}^{\mathbf{u}}_t$ by $\mathbf{D}(z) = w + g_t(w)$, where w is the unique element in $X_0(t) \oplus X_2(t)$ such that $w = h_t(z + g_t(w)) + z$. Then we have $w + g_t(w) = z + g_t(w) + h_t(z + g_t(w)) \in \mathbf{C}_t \cap \mathbf{C}^{\mathbf{u}}_t$. The uniqueness of w implies that $\mathbf{D}$ is a well-defined mapping.

We next prove the surjectivity of $\mathbf{D}$. For $x \in \mathbf{C}_t \cap \mathbf{C}^{\mathbf{u}}_t$ there are $u \in X_0(t) \oplus X_2(t)$ and $v \in X_1(t) \oplus X_2(t)$ such that $x = u + g_t(u) = v + h_t(v)$. Then $u - h_t(v) = v - g_t(u) \in (X_0(t) \oplus X_2(t)) \cap (X_1(t) \oplus X_2(t)) = X_2(t)$. Therefore, there is a $z \in X_2(t)$ such that $u - h_t(v) = v - g_t(u) = z$. It follows that $u - h_t(z + g_t(u)) = z$. As shown above, this relation means that $\mathbf{D}z = u + g_t(u) = x$. Consequently, $\mathbf{D}$ is surjective.

We now prove that $\mathbf{D}$ is a Lipschitz mapping. Indeed, by the definition of $\mathbf{D}$ we have $\mathbf{D}(z_1) = w_1 + g_t(w_1)$ and $\mathbf{D}(z_2) = w_2 + g_t(w_2)$ for w_1 and w_2 being the unique solutions in $X_0(t) \oplus X_2(t)$ of $w_1 = h_t(z_1 + g_t(w_1)) + z_1$ and $w_2 = h_t(z_2 + g_t(w_2)) + z_2$, respectively. Then

$$\begin{aligned} (1 - l)\|w_1 - w_2\| &\le \|\mathbf{D}(z_1) - \mathbf{D}(z_2)\| \\ &= \big\| z_1 + h_t(z_1 + g_t(w_1)) + g_t(w_1) \\ &\qquad - \big(z_2 + h_t(z_2 + g_t(w_2)) + g_t(w_2)\big)\big\| \\ &\le \|z_1 - z_2\| + l\|z_1 - z_2\| + l\|g_t(w_1) - g_t(w_2)\| \\ &\qquad + \|g_t(w_2) - g_t(w_2)\| \\ &\le (1 + l)\|z_1 - z_2\| + l(l + 1)\|w_2 - w_2\|. \end{aligned}$$

Therefore, $\|\mathbf{D}(z_1) - \mathbf{D}(z_2)\| \le (1+l)\|z_1 - z_2\| + \frac{l(l+1)}{1-l}\|\mathbf{D}(z_1) - \mathbf{D}(z_2)\|$. Thus, $\|\mathbf{D}(z_1) - \mathbf{D}(z_2)\| \le \frac{1-l^2}{2-(1+l)^2}\|z_1 - z_2\|$; note that $2-(1+l)^2 > 0$ since $l < \sqrt{2}-1$. Hence, $\mathbf{D}$ is a Lipschitz mapping with Lipschitz constant $\frac{1-l^2}{2-(1+l)^2}$. It follows that $\mathbf{D}$ is continuous and injective. As shown above, $\mathbf{D}$ is surjective, therefore it is bijective. The inverse $\mathbf{D}^{-1}$ of $\mathbf{D}$ is defined as $\mathbf{D}^{-1} : \mathbf{C}_t \cap \mathbf{C}_t^{\mathbf{u}} \to X_2(t)$ with $\mathbf{D}^{-1}(w + g_t(w)) = z$ if $z = w - h_t(z + g_t(w))$.

We next prove that $\mathbf{D}^{-1}$ is also a Lipschitz mapping. Indeed, for $x_1 = w_1 + g_t(w_1)$ and $x_2 = w_2 + g_t(w_2)$ belonging to $\mathbf{C}_t \cap \mathbf{C}_t^{\mathbf{u}}$ we have

$$\begin{aligned}
\|\mathbf{D}^{-1}x_1 - \mathbf{D}^{-1}x_2\| &= \|z_1 - z_2\| \\
&\le \big\|w_1 - h_t(z_1 + g_t(w_1)) - \big(w_2 - h_t(z_2 + g_t(w_2))\big)\big\| \\
&\le \|w_1 - w_2\| + l\|z_1 - z_2\| + l^2\|w_1 - w_2\| \\
&= (1+l^2)\|w_1 - w_2\| + l\|\mathbf{D}^{-1}x_1 - \mathbf{D}^{-1}x_2\| \\
&\le \frac{1+l^2}{1-l}\|w_1 + g_t(w_1) - w_2 - g_t(w_2)\| \\
&\quad + l\|\mathbf{D}^{-1}x_1 - \mathbf{D}^{-1}x_2\| \\
&= \frac{1+l^2}{1-l}\|x_1 - x_2\| + l\|\mathbf{D}^{-1}x_1 - \mathbf{D}^{-1}x_2\|.
\end{aligned}$$

Therefore, $\|\mathbf{D}^{-1}x_1 - \mathbf{D}^{-1}x_2\| \le \frac{1+l^2}{(1-l)^2}\|x_1 - x_2\|$. Hence, $\mathbf{D}^{-1}$ is also a Lipschitz mapping. It follows that $\mathbf{D}$ is a homeomorphism, and so $\mathbf{C}_t \cap \mathbf{C}_t^{\mathbf{u}}$ is homeomorphic to $X_2(t)$ for all $t \in \mathbb{R}$.

Property (b) follows from the definitions of the invariant-center-stable and invariant-center-unstable manifolds.

Properties (c) and (d) are consequences of the inequalities (4.2) and (6.18), respectively. ■

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