A note on the regularity of the degenerate complex Monge–Ampère equation

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Abstract. We prove the almost $C^{1,1}$ regularity of the degenerate complex Monge–Ampère equation in a special case.

If $u$ is a smooth plurisubharmonic function, the Monge–Ampère operator is given by $(dd^c u)^n = \det(u_{p\bar{q}})d\mathcal{L}$, where $u_{p\bar{q}} = \partial^2 u / \partial z_p \partial \bar{z}_q$ and $\mathcal{L}$ is a $2n$-dimensional Lebesgue measure. For an arbitrary continuous plurisubharmonic function $u$ one can define $(dd^c u)^n$ to be a regular Borel measure. Let $\Omega$ be a strictly pseudoconvex domain in $\mathbb{C}^n$ (throughout the note we always assume $n \geq 2$). Then for any nonnegative $f$ which is continuous in $\Omega$, and $\varphi$ continuous on $\partial \Omega$, the Dirichlet problem

$$
\begin{cases}
    u \in \mathcal{PSH}(\Omega) \cap \mathcal{C}(\Omega), \\
    (dd^c u)^n = f d\mathcal{L} \quad \text{in} \quad \Omega, \\
    u = \varphi \quad \text{on} \quad \partial \Omega,
\end{cases}
$$

has a unique continuous plurisubharmonic solution $u$ (see [B-T]).

We say that $\Omega$ is a strictly pseudoconvex domain with a $C^{2,1}$ boundary if $\Omega = \{ z \in \mathbb{C}^n : \rho(z) < 0 \}$ where $\rho$ is a strictly plurisubharmonic function of class $C^{2,1}$ on a neighbourhood of $\Omega$ such that $\nabla \rho \neq 0$ on $\partial \Omega$. We say that a function $u$ is almost $C^{1,1}$ if the function $\Delta u$ is bounded. Now we can formulate the main theorem.

**Theorem 1.** Let $\Omega$ be a strictly pseudoconvex domain with a $C^{2,1}$ boundary, and $f$ be a nonnegative function on $\Omega$ such that $f^{1/(n-1)} \in C^{1,1}(\Omega)$ and $f^{1/n} \in C^{0,1}(\Omega)$. Then the Monge–Ampère equation (1) with the boundary condition $\varphi \equiv 0$ has a unique almost $C^{1,1}$ solution.

**Remark 2.** There are similar theorems for the degenerate real Monge–Ampère equation in [G] and [G-T-W] (with nonzero $\varphi \in C^{3,1}(\Omega)$). In the
complex case Krylov proved (see [K1, K2]) that if \( \Omega \) is of class \( C^{3,1}, \varphi \in C^{3,1}(\bar{\Omega}) \), \( f^{1/n} \in C^{1,1}(\bar{\Omega}) \), then \( u \in C^{1,1}(\bar{\Omega}) \).

**Remark 3.** If \( f^{1/(n-1)} \) is nonnegative and \( C^{1,1} \) on some neighbourhood of \( \bar{\Omega} \) then \( f^{1/n} \in C^{0,1}(\bar{\Omega}) \).

**Remark 4.** It is shown in [P] that the exponent \( 1/(n - 1) \) in Theorem [1] is optimal.

**Remark 5.** If a function \( u \) is almost \( C^{1,1} \), then \( u \in C^{1,\alpha} \) for all \( \alpha < 1 \). A plurisubharmonic function is almost \( C^{1,1} \) if and only if the mixed complex derivatives \( u_{pq} \) are bounded.

If \( \Omega = B \) is the unit ball then we can omit the assumption that \( f^{1/n} \in C^{0,1}(\bar{\Omega}) \).

**Theorem 6.** Let \( \Omega = B \) and let \( f \) be a nonnegative function on \( \bar{\Omega} \) such that \( f^{1/(n-1)} \in C^{1,1}(\bar{\Omega}) \). Then the Monge–Ampère equation [1] with the boundary condition \( \varphi \equiv 0 \) has a unique almost \( C^{1,1} \) solution.

In all the lemmas below we assume that \( \Omega = \{ z \in \mathbb{C}^n : \rho(z) < 0 \} \) where \( \rho \) is a strictly plurisubharmonic function of class \( C^\infty \) on a neighbourhood of \( \bar{\Omega} \) such that \( \nabla \rho \neq 0 \) on \( \partial \Omega \) and \( f \in C^\infty(\bar{\Omega}) \) is positive. In that case the solution of [1] is of class \( C^\infty \) on \( \bar{\Omega} \).

**Lemma 7.** We have \( \| u \|_{L^\infty(\Omega)} \leq C \), where \( C = C(\Omega, \| f \|_{L^\infty(\bar{\Omega})}) \).

**Proof.** From the comparison principle and the maximum principle we have \( A|z|^2 - B \leq u \leq 0 \) for \( A, B \in \mathbb{R} \) large enough.

The proof of the next lemma is similar to the proof of Lemma 11 in [G] (see also the proof of Theorem 3.1 in [B1]).

**Lemma 8.** We have \( \sup_{\Omega} \Delta u \leq C(\sup_{\partial \Omega} \Delta u + 1) \), where \( C = C(\Omega, \| f^{1/(n-1)} \|_{C^{1,1}(\Omega)}) \).

**Proof.** For \( k = 1, \ldots, n \), we have

\[
(\log f)_k = u^{pq} u_{kpq},
\]

(2)

\[
(\log f)_{kk} = u^{pq} u_{kkpq} - u^{pq} u_{kij} u_{kpq},
\]

where \( (u^{pq}) \) is the inverse of the matrix \( (u_{pq}) \).

Let

\[
h = \log( \max_{i \in \{1, \ldots, n\}} \lambda_i ) + 2A|z|^2,
\]

where \( \lambda_i \) are the eigenvalues of the matrix \( (u_{pq}) \), and \( A = (2 \sup_{z \in \Omega} |z|)^{-2} \). We can assume that \( h \) attains its maximum at some point \( z_0 \in \Omega \). We can also assume (after a change of variables) that at \( z_0 \) the matrix \( (u_{pq}) \) is diagonal and \( u_{11} = \max_{i \in \{1, \ldots, n\}} \lambda_i \). Then also the function \( \tilde{h} = \log(u_{11}) + 2A|z|^2 \) attains its maximum at \( z_0 \).
From now on, all formulas are assumed to hold at \( z_0 \). We have

\[ 0 = \tilde{h}_k = \frac{u_{1\bar{1}k}}{u_{1\bar{1}}} + 2A\bar{z}_k \]

and \( \tilde{h}_{kk} \leq 0 \). Using this and (3) we can compute

\[ 0 \geq u^{kk}\tilde{h}_{kk} = 2A\sum_k \frac{1}{u_{kk}} + u^{kk}(\log(u_{1\bar{1}}))_{kk} \]

\[ = 2A\sum_k \frac{1}{u_{kk}} + u^{kk}\left(\frac{(u_{1\bar{1}})_{kk}}{u_{1\bar{1}}} - \frac{u_{1\bar{1}k}u_{1\bar{1}k}}{(u_{1\bar{1}})^2}\right) \]

\[ = 2A\sum_k \frac{1}{u_{kk}} + \frac{(\log f)_{1\bar{1}}}{u_{1\bar{1}}} + \frac{1}{u_{1\bar{1}}}u^{pp}u^{qq}u_{kp\bar{p}}u_{qp\bar{p}} - 4A^2\sum_k |z_k|^2. \]

By the definition of \( A \), we obtain

\[ 0 \geq A\sum_k \frac{1}{u_{kk}} + (\log f)_{1\bar{1}} + \frac{1}{u_{1\bar{1}}}u^{pp}u^{qq}u_{kp\bar{p}}u_{kp\bar{p}}. \]

By the inequality between the arithmetic and geometric means, we have

\[ \sum_k \frac{1}{u_{kk}} \geq \left(\frac{u_{1\bar{1}}}{f}\right)^{1/(n-1)}. \]

The inequality between the root-mean square and the geometric mean, and (2), give us

\[ \sum_k u^{pp}u^{qq}u_{kp\bar{p}}u_{kp\bar{p}} \geq \sum_{k \geq 2} |u^{pp}u_{kp\bar{p}}|^2 \geq \frac{1}{n-1} |(\log f)_{1\bar{1}} - u_{11\bar{1}} u_{1\bar{1}1}|^2. \]

Using (5), (6), (7), and (4) we infer

\[ 0 \geq A\left(\frac{u_{1\bar{1}}}{f}\right)^{1/(n-1)} + (n-1)\frac{(f^{1/(n-1)})_{1\bar{1}}}{f^{1/(n-1)}u_{1\bar{1}}} + 4A\operatorname{Re}\left((f^{1/(n-1)})_{1\bar{1}}z_1\right). \]

Multiplying both sides of the last inequality by \( f^{1/(n-1)}u_{1\bar{1}} \), we get

\[ 0 \geq A(u_{1\bar{1}})^{n/(n-1)} + (n-1)(f^{1/(n-1)})_{1\bar{1}} + 4A\operatorname{Re}\left((f^{1/(n-1)})_{1\bar{1}}z_1\right) \]

and the lemma follows.

**Remark 9.** Using (5) and (6) we can prove the above lemma for \( C = C(\Omega, -f^{1/(n-1)}\Delta \log f) \) (cf. Theorem 2 in [22]).

In the two next lemmas we shall prove an *a priori* estimate for first derivatives.

**Lemma 10.** We have

\[ \|u\|_{C^{0,1}(\partial\Omega)} \leq C, \]

where \( C = C(\Omega, \|f\|_{L^\infty(\Omega)}) \).
Proof. By the comparison principle $A\rho \leq u \leq 0$ for $A$ large enough. So on the boundary we have $|\nabla u| \leq |\nabla (A\rho)|$.

Lemma 11. We have

\[ \|u\|_{C^{0,1}(\Omega)} \leq C, \]

where $C = C(\Omega, \|f^{1/n}\|_{C^{0,1}}, \|f\|_{L^\infty(\Omega)})$.

Proof. Let $L = u^{pq}\partial_{pq}$. Since $u$ is strictly plurisubharmonic, the operator $L$ is elliptic. Consider the function $w_i = \pm u_{x_i} + A|z|^2$ (we use the standard identification $\mathbb{C}^n \cong \mathbb{R}^{2n}$, $z = (z_1, \ldots, z_n) = (x_1, \ldots, x_{2n})$). Then for $A$ large enough, by (2) and the inequality between the arithmetic and geometric means we have

\[ Lw_i = \pm \frac{f_{x_i}}{f} + A\sum u^{pq} \geq \frac{\pm n(f^{1/n})_{x_i} + A}{f^{1/n}} \geq 0. \]

The maximum principle and Lemma 10 give us (8). □

To obtain an a priori estimate for second order derivatives on the boundary we can fix a point $z_0 \in \partial\Omega$ and after a change of coordinates assume that

\[ z_0 = 0, \]

\[ \rho_{x_i} = \begin{cases} 0 & \text{for } i = 1, \ldots, 2n - 2, 2n, \\ -1 & \text{for } i = 2n - 1, \end{cases} \]

\[ \rho_{pq}(0) = u_{pq}(0) = 0 \quad \text{when } p \neq q, \ p, q = 1, \ldots, n - 1, \]

\[ u_{pp}(0) = b_p = -u_n(0)\rho_{pp}(0) \quad \text{and} \quad b_1 \leq \cdots \leq b_{n-1}, \]

and we can write

\[ \rho(z) = -\Re z_n + \sum c_{pq}z_pz_q + O(|z^3|). \]

Then we obtain:

Lemma 12. For $i, j = 1, \ldots, n - 1$ we have

\[ |u_{pq}(0)| \leq C, \]

where $C = C(\Omega, \|f\|_{L^\infty(\Omega)})$.

By standard methods from [C-K-N-S], we obtain

Lemma 13. For $i = 1, \ldots n - 1$ we have

\[ |u_{i\bar{n}}(0)| \leq C, \]

where $C = C(\Omega, \|f^{1/n}\|_{C^{0,1}})$.

Proof. Let $L = u^{pq}\partial_{pq}$. For $k = 1, \ldots, 2n - 2$ consider the function $w_k = \pm T_ku + (u_{x_k})^2 + (u_{x_n})^2 + A|z|^2 - Bx_{2n-1}$, where $T_k = \rho_{x_{2n-1}}\partial_{x_k} - \rho_{x_k}\partial_{x_{2n-1}}$. 
Observe that
\[ u^{pq} u_{x_{2n-1}q} = 2u^{pq} u_{nq} + iu^{pq} u_{x_{2n}q} = 2\delta_{pn} + iu^{pq} u_{x_{2n}q}, \]
\[ u^{pq} u_{x_{2n-1}p} = 2u^{pq} u_{pn} - iu^{pq} u_{x_{2n}p} = 2\delta_{qn} - iu^{pq} u_{x_{2n}p}, \]
\[ L((u_{xk})^2) = 2u_{xk}(\log f)_{xk} + u^{pq} u_{xkq} u_{xkp}, \]
\[ \rho^{pq} \rho_{xkp} u_{xkq} \leq \sqrt{\rho^{pq} \rho_{xkp} \rho_{xkp}} \sqrt{u^{pq} u_{xkq} u_{xkq}}, \]
\[ \sum u_{pq} = O\left( \sum u_{pq} \right). \]

Then (using again the inequality between the arithmetic and geometric means) for \( A \) enough large, we obtain
\[ Lw_k \geq 0. \]

We may choose a neighbourhood \( U \) of the origin such that in \( S_\varepsilon = \{ z \in U \cap \Omega : x_{2n-1} < \varepsilon \} \) we have \( x_{2n-1} \geq D|z|^2 \) for some constant \( D \). Using Lemma 10, we obtain \( (u_{xk})^2 + (u_{xn})^2 \leq D'|z|^2 \) for some other constant \( D' \). Then for \( B \) large enough \( w_i \leq 0 \) on \( \partial S_\varepsilon \) and by the maximum principle we get \( w_i \leq 0 \) on \( S_\varepsilon \). Hence, \( u_{x_i x_{2n}}(0) = -\rho_{x_i x_{2n}}(0) u_{n}(0) \) is bounded and we obtain 9.  

We can prove a similar lemma for a ball with a constant \( C \) not depending on \( \| f^{1/n} \|_{C^{0,1}} \).

**Lemma 14.** Let \( z_0 = (0, \ldots, 0, 1) \). If \( \Omega = B = \{ z \in \mathbb{C}^n : |z_0 - z|^2 < 1 \} \) then for \( i = 1, \ldots, n - 1 \) we have
\[ |u_{i\bar{n}}(0)| \leq C, \]
where \( C = C(\| f^{1/(n-1)} \|_{C^{1,1}}) \).

**Proof.** Let \( T_k = z_k \frac{\partial}{\partial z_k} - (z_n - 1) \frac{\partial}{\partial z_k} \) for \( k = 1, \ldots, n - 1 \), and let \( L \) be as above. Consider the function \( w = \pm \text{Re} T_k u + A(|z - z_0|^2 - 1) \). Since \( T_k \) on the boundary is tangential, we obtain \( |T_k f^{1/(n-1)}| < \tilde{C} \sqrt{f^{1/(n-1)}} \) where \( \tilde{C} \) depends only on \( \| f^{1/(n-1)} \|_{C^{1,1}} \). Thus for \( A \) large enough we have
\[ L(w) = \pm \frac{\text{Re} T_k f^{1/(n-1)}}{(n-1)f^{1/(n-1)}} + A \sum u^{pq} \geq \tilde{C} f^{-1/2(n-1)} + \frac{A}{2} f^{-1/n} \geq 0. \]

Using the maximum principle we obtain \( w \leq 0 \). So we have
\[ |u_{x_{2k-1}x_n}(0)| \leq C. \]
Similarly (taking \( w = \pm \text{Im} T_k (u - \varphi) + A(|z - z_0|^2 - 1) \)), we obtain
\[ |u_{x_{2k}x_n}(0)| \leq C \quad \text{for } k < n. \]

The proof of the next lemma is also the same as in \([C-K-N-S]\).
Lemma 15. If \( \| f \|_{L^\infty(\Omega)} > 0 \), then
\[
|u_{n\bar{n}}(0)| \leq C,
\]
where \( C = C(\Omega, \max_{(p,q) \neq (n,n)} |u_{p\bar{q}}(0)|, \| f \|_{L^\infty(\Omega)}, 1/\| f \|_{L^\infty(\Omega)}). \)

Proof. There exists \( R > 0 \) such that \( f \geq \| f \|_{L^\infty(\Omega)}/2 \) on some ball \( B \subset \Omega \) with radius \( R \). Then we have \( \Delta u \geq (\| f/2 \|_{L^\infty(\Omega)})^{1/n} \). Hence from the Hopf lemma we obtain \( -u_{x_2n-1}(0) \geq D \) for some constant \( D \). Thus the numbers \( 1/b_i = -u_n(0)\rho_{i\bar{i}} \) are bounded. As in [G-T-W] we can write
\[
f(0) = u_{n\bar{n}}(0) \left( \prod_{i=1}^{n-1} b_i \right) - \sum_{j=1}^{n-1} \frac{|u_{j\bar{n}}|^2}{b_j} \left( \prod_{i=1}^{n-1} b_i \right).
\]
Then
\[
0 \leq u_{n\bar{n}}(0) = \sum_{j=1}^{n-1} \frac{|u_{j\bar{n}}|^2}{b_j} + \frac{f(0)}{\prod_{i=1}^{n-1} b_i} < C. \]

After a standard regularisation argument, from the above lemmas we obtain Theorems 1 and 6.

Remark 16. From the proofs we can see that the exponent \( \frac{1}{n-1} \) in Theorems 1 and 6 can be replaced by any smaller positive number.

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