ANNALES
POLONICI MATHEMATICI
98.2 (2010)

A stochastic symbiosis model with degenerate diffusion process

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Abstract. We present a model of symbiosis given by a system of stochastic differential equations. We consider a situation when the same factor influences both populations or only one population is stochastically perturbed. We analyse the long-time behaviour of the solutions and prove the asymptoptic stability of the system.

1. Introduction. The aim of this paper is to study the following system of stochastic differential equations:

$$dX(t) = ((a_1 + b_1Y(t) - c_1X(t))dt + \rho_1 dW(t))X(t), \tag{1}$$

$$dY(t) = ((a_2 + b_2X(t) - c_2Y(t))dt + \rho_2 dW(t))Y(t). \tag{2}$$

This is a stochastic version of the deterministic symbiosis model [7] (Gause and Witt, 1935)

$$x' = (a_1 + b_1y - c_1x)x, \qquad y' = (a_2 + b_2x - c_2y)y. \tag{3}$$

where the functions $x(t)$, $y(t)$ represent, respectively, the size of the first and the second population at time t. We assume that a_i, b_i, c_i $(i = 1, 2)$ are positive constants. The coefficients a_i $(i = 1, 2)$ are ideal growth rates, b_i $(i = 1, 2)$ are symbiosis rates, c_i $(i = 1, 2)$ are death rates.

In the model described by (1) and (2) the stochastic processes $X(t)$, $Y(t)$ represent, respectively, the first and the second population, $W(t)$ is a one-dimensional standard Wiener process, the constants a_i, b_i, c_i $(i = 1, 2)$, as in the deterministic model, are positive, and ρ_i $(i = 1, 2)$ are diffusion coefficients. We assume that fluctuations of the environment randomly change the reproduction rates of the populations and the random noise is proportional to the number of individuals. We consider two kinds of stochastic perturbations:

(i) strongly correlated, i.e. $\rho_1 > 0$, $\rho_2 > 0$,

2010 *Mathematics Subject Classification*: 47D07, 35K15, 60J60, 60H10, 92D25.
Key words and phrases: symbiosis model, diffusion process, Markov semigroups, asymptotic stability.

DOI: 10.4064/ap98-2-2

(ii) only one population is stochastically perturbed; by symmetry, we assume that the second population is perturbed, i.e. $\rho_1 = 0$, $\rho_2 > 0$.

Case (i) corresponds to the situation when the same factor (like weather conditions) affects both populations.

We consider system (1), (2) on the assumption that $b_1 b_2 < c_1 c_2$, because in the deterministic model (3) we observe that the sizes of both populations go to infinity if $b_1 b_2 \geq c_1 c_2$. The inequality $b_1 b_2 < c_1 c_2$ has an interesting biological interpretation, namely it means that the symbiosis coefficients are smaller than the death rates. The goal of this paper is to study the long-time behaviour of the solutions of system (1), (2).

We prove that the probability distributions of the process $(X(t), Y(t))$ are absolutely continuous with respect to Lebesgue measure. Let $U(x, y, t)$ be the density of the distribution of $(X(t), Y(t))$. We give a sufficient and a necessary condition for the asymptotic stability of system (1), (2), i.e. the convergence of $U(x, y, t)$ to an invariant density $U_*(x, y)$. We also find the support of $U_*(x, y)$. If the system is not asymptotically stable, we prove that $\lim_{t\to\infty} Y(t) = 0$ a.e. We also show that in this case either $\lim_{t\to\infty} X(t) = 0$ a.e., or the probability distribution of the process $X(t)$ converges weakly to some probability measure.

A model of symbiosis in which stochastic perturbations are weakly correlated was considered in [28]. System (1), (2) has similar properties, but methods of the proof are more complicated. In the same way as in [28] we prove the existence, uniqueness, positivity and non-extinction property of the solutions. But it is more difficult to obtain the asymptotic stability, because the Fokker–Planck equation corresponding to system (1), (2) is of a degenerate type. We use Hörmander's theorem ([2], [9], [15], [16], [19]) in order to show that a semigroup connected with the Fokker–Planck is integral and has a continuous kernel. Using support theorems ([1], [3], [13], [29]) we find a set E on which the kernel is positive. Next we prove that E is an "attractor". Then we apply some facts concerning integral Markov semigroups ([20], [21], [24] and [27]) to show that the semigroup connected with the Fokker–Planck equation satisfies the "Foguel alternative", i.e. it is either asymptotically stable or "sweeping". Finally, we find a Khasminskiĭ function which excludes "sweeping" and in this way we obtain the asymptotic stability.

A similar technique was applied to study properties of a stochastic prey-predator model [25]. It should be noted that in [28] the proof of the asymptotic stability is much easier because the semigroup connected with the Fokker–Planck equation is integral and has a continuous and strictly positive kernel. Therefore, in order to obtain the asymptotic stability it is sufficient to construct a Khasminskiĭ function. When system (1), (2) is not asymptotically stable we prove similar results to those in [28] using the same methods.

In particular, we show that in this case one of the populations becomes extinct.

2. Mathematical results and their interpretation. In this section we formulate the main results of this paper.

THEOREM 1. *If* $b_1b_2 < c_1c_2$ *then for any initial condition* $(X(0), Y(0)) \in \mathbb{R}^2_+$ *there exists a unique solution* $(X(t), Y(t))$ *of system* (1), (2) *for* $t \geq 0$ *and the solution remains in* $\mathbb{R}^2_+$ *with probability* 1, *i.e.* $(X(t), Y(t)) \in \mathbb{R}^2_+$ *for all* $t \geq 0$ *almost surely.*

The proof of this theorem is identical to the case of weakly correlated perturbations (see Theorem 2 in [28]).

The asymptotic behaviour of system (1), (2) depends on the constants b_1, b_2, c_1, c_2, ρ_1, ρ_2, $\tilde{a}_1 = a_1 - \rho_1^2/2$, $\tilde{a}_2 = a_2 - \rho_2^2/2$.

THEOREM 2. *Let* $b_1b_2 < c_1c_2$. *If* $(X(t), Y(t))$ *is a solution of system* (1), (2), *then for every* $t > 0$ *the distribution of* $(X(t), Y(t))$ *has a density* $U(t, x, y)$.

(I) *Let* $\tilde{a}_1, \tilde{a}_2 > 0$. *In case* (i), *assume that* $\tilde{a}_1 \neq \tilde{a}_2$ *or* $\rho_1 \neq \rho_2$. *Then there exists a unique invariant density* $U_*(x, y)$ *such that*

$$\lim_{t\to\infty} \iint_{\mathbb{R}^2_+} |U(x, y, t) - U_*(x, y)|\, dx\, dy = 0. \tag{4}$$

(II) *If* $\tilde{a}_1, \tilde{a}_2 < 0$ *then*

$$\lim_{t\to\infty} X(t) = 0 \quad \textit{a.e.} \quad \textit{and} \quad \lim_{t\to\infty} Y(t) = 0 \quad \textit{a.e.}$$

(III) *Let* $\tilde{a}_1 > 0$, $\tilde{a}_2 < 0$ *and* $\tilde{a}_1 b_2 + \tilde{a}_2 c_1 > 0$. *Then there exists a unique invariant density* $U_*(x, y)$ *such that*

$$\lim_{t\to\infty} \iint_{\mathbb{R}^2_+} |U(x, y, t) - U_*(x, y)|\, dx\, dy = 0. \tag{5}$$

(IV) *Let* $\tilde{a}_1 > 0$, $\tilde{a}_2 < 0$ *and* $\tilde{a}_1 b_2 + \tilde{a}_2 c_1 < 0$. *Then in case* (i) *we have* $\lim_{t\to\infty} Y(t) = 0$ *a.e. and the distribution of the process* $X(t)$ *converges weakly to the measure with density*

$$f_*(x) = Cx^{2\tilde{a}_1/\rho_1^2 - 1} \exp(-2c_1 x/{\rho_1}^2),$$

where $C = (2c_1/{\rho_1}^2)^{2\tilde{a}_1/{\rho_1}^2} / \Gamma(2\tilde{a}_1/{\rho_1}^2)$, *while in case* (ii) *we have* $\lim_{t\to\infty} Y(t) = 0$ *a.e. and* $\lim_{t\to\infty} X(t) = a_1/c_1$ *a.e.*

In cases (I) and (III) the support of the invariant density U_* depends on the coefficients ρ_1, ρ_2, $\tilde{a}_1$, $\tilde{a}_2$. By the *support* of a measurable function f we simply mean the set

$$\operatorname{supp} f = \{(x, y) \in X : f(x, y) \neq 0\}.$$

We have the following result.

THEOREM 3. *Let $b_1 b_2 < c_1 c_2$. If both populations are stochastically perturbed then in case* (I) *we have four subcases:*

(a_1) *If* $\rho_2 > \rho_1$ *and* $\tilde{a}_2\rho_1 > \tilde{a}_1\rho_2$ *then* $\operatorname{supp} U_* = \mathbb{R}^2_+$.

(b_1) *If* $\rho_2 \le \rho_1$ *and* $\tilde{a}_2\rho_1 > \tilde{a}_1\rho_2$ *then there exists a constant* C_1 *such that*

$$\operatorname{supp} U_* = \{(x,y) \in \mathbb{R}^2_+ : x < C_1 y^{\rho_1/\rho_2}\}. \tag{6}$$

(c_1) *If* $\rho_2 < \rho_1$ *and* $\tilde{a}_2\rho_1 < \tilde{a}_1\rho_2$ *then* $\operatorname{supp} U_* = \mathbb{R}^2_+$.

(d_1) *If* $\rho_2 \ge \rho_1$ *and* $\tilde{a}_2\rho_1 < \tilde{a}_1\rho_2$ *then there exists a constant* C_1 *such that*

$$\operatorname{supp} U_* = \{(x,y) \in \mathbb{R}^2_+ : y < C_1 x^{\rho_2/\rho_1}\}. \tag{7}$$

If both populations are stochastically perturbed then in case (III) *we have two subcases:*

(a_3) *If* $\rho_2 < \rho_1$ *then* $\operatorname{supp} U_* = \mathbb{R}^2_+$.

(b_3) *If* $\rho_2 \ge \rho_1$ *then there exists a constant* C_1 *such that*

$$\operatorname{supp} U_* = \{(x,y) \in \mathbb{R}^2_+ : y < C_1 x^{\rho_2/\rho_1}\}. \tag{8}$$

If the second population is stochastically perturbed then

$$\operatorname{supp} U_* = (a_1/c_1, \infty) \times \mathbb{R}_+. \tag{9}$$

REMARK 1. As equations (1), (2) are symmetrical we omit some cases in the statement of Theorem 2. For example, we do not take into account the case $\tilde{a}_1 < 0$, $\tilde{a}_2 > 0$, $\tilde{a}_2 b_1 + \tilde{a}_1 c_2 > 0$, because it is symmetrical to (III), and the case $\tilde{a}_1 < 0$, $\tilde{a}_2 > 0$, $\tilde{a}_2 b_1 + \tilde{a}_1 c_2 < 0$, symmetrical to (IV).

REMARK 2. In order to explain the role of the coefficients $\tilde{a}_1$, $\tilde{a}_2$ we consider the following equation of population growth:

$$dN(t) = (a\,dt + \rho\,dW(t))N(t), \tag{10}$$

where $a > 0$ is the growth rate, $\rho > 0$ is a diffusion coefficient, $W(t)$ is a one-dimensional standard Wiener process, and $N(t)$ is a real stochastic process. The solution of this equation is of the form

$$N(t) = N(0)\exp\{\tilde{a}t + \rho W(t)\},$$

where $\tilde{a} = a - \frac{1}{2}\rho^2$. The constant $\tilde{a}$ is called the new growth rate or for short the growth rate of the population. It is easy to check that if $\tilde{a} > 0$ then $\lim_{t\to\infty} N(t) = \infty$ a.e., and if $\tilde{a} < 0$ then $\lim_{t\to\infty} N(t) = 0$ a.e. In other words, the positivity of the new growth rate means that the stochastic perturbation is small and therefore the size of the population goes to infinity as in the deterministic case. The negativity of the new growth rate means that the stochastic perturbation is too large and therefore the population dies out. Even though we consider more complicated equations, we will call the coefficients $\tilde{a}_1$, $\tilde{a}_2$ the new growth rates or briefly growth rates of these populations.

REMARK 3. Theorems 2 and 3 have an interesting biological interpretation.

From Theorem 2 it follows that if the new growth rate of the population is positive then this population survives. Otherwise, if the new growth rate is negative then the population may die out. The most interesting is case (III), because we observe a positive influence of symbiosis. Namely, from (III) it follows that if the new growth rate for one population is positive and for the other it is negative, but benefits of symbiosis for the second population are large, then both populations survive.

Theorem 3 provides us with information about the support of the invariant density $U_*(x,y)$. If we have a nondegenerate diffusion process as in [28] then the support is the whole space, but in our case it may be some proper subset. Especially interesting are formulas (6)–(9). From Theorem 2 it follows that the distribution of the system $(X(t),Y(t))$ can converge to an invariant distribution with some density U_*. Let a pair of variables (X,Y) have the density distribution U_*. Then, for example, from (6) we obtain $X < C_1 Y^{\rho_1/\rho_2}$. This means that the second population controls the size of the first.

3. Markov semigroups. In this section we give some facts concerning Markov semigroups.

Let (X,Σ,m) be a σ-finite measure space. Denote by D the subset of $L^1 = L^1(X,\Sigma,m)$ which consists of all densities, i.e.

$$D = \{f \in L^1 : f \ge 0,\ \|f\| = 1\}.$$

A linear mapping $P : L^1 \to L^1$ is called a *Markov operator* if $P(D) \subset D$.

The Markov operator P is called an *integral* operator if there exists a measurable function $k : X \times X \to [0,\infty)$ such that

$$Pf(x) = \int_X k(x,y) f(y)\, m(dy) \tag{11}$$

for every density f. The function k is called the *kernel* of the operator P. One can check that from the condition $P(D) \subset D$ it follows that

$$\int_X k(x,y)\, m(dx) = 1 \tag{12}$$

for almost all $y \in X$.

A family $\{P(t)\}_{t\ge 0}$ of Markov operators which satisfies:

(a) $P(0) = \mathrm{Id}$,
(b) $P(t+s) = P(t)P(s)$ for $s,t \ge 0$,
(c) for each $f \in L^1$ the function $t \mapsto P(t)f$ is continuous with respect to the L^1 norm

is called a *Markov semigroup*. A Markov semigroup $\{P(t)\}_{t\ge 0}$ is called *integral* if for each $t>0$, the operator $P(t)$ is an integral Markov operator.

We also need two definitions concerning the asymptotic behaviour of a Markov semigroup. A density f_* is called *invariant* if $P(t)f_* = f_*$ for each $t>0$. The Markov semigroup $\{P(t)\}_{t\ge 0}$ is called *asymptotically stable* if there is an invariant density f_* such that

$$\lim_{t\to\infty} \|P(t)f - f_*\| = 0 \quad \text{for } f\in D.$$

A Markov semigroup $\{P(t)\}_{t\ge 0}$ is called *sweeping* with respect to a set $A\in\Sigma$ if for every $f\in D$,

$$\lim_{t\to\infty} \int_A P(t)f(x)\, m(dx) = 0. \tag{13}$$

Theorem 4 ([25]). *Let X be a metric space and Σ be the σ-algebra of Borel sets. Let $\{P(t)\}_{t\ge 0}$ be an integral Markov semigroup with a continuous kernel $k(t,x,y)$ for $t>0$, which satisfies* (12) *for all $y\in X$. Assume that for every $f\in D$,*

$$\int_0^\infty P(t)f\, dt > 0 \quad \textit{a.e.} \tag{14}$$

Then this semigroup is asymptotically stable or sweeping with respect to compact sets.

The property that a Markov semigroup $\{P(t)\}_{t\ge 0}$ is asymptotically stable or sweeping from a sufficiently large family of sets (e.g. from all compact sets) is called the *Foguel alternative*.

4. Properties of trajectories. Now instead of system (1), (2) we consider a simpler system in which the diffusion coefficients are constant. We substitute $X(t)=e^{\xi(t)}$, $Y(t)=e^{\eta(t)}$. Using the Itô formula we obtain

$$d\xi(t) = (\tilde a_1 + b_1 e^{\eta(t)} - c_1 e^{\xi(t)})dt + \rho_1 dW(t), \tag{15}$$

$$d\eta(t) = (\tilde a_2 + b_2 e^{\xi(t)} - c_2 e^{\eta(t)})dt + \rho_2 dW(t). \tag{16}$$

The proof of parts (II) and (IV) of Theorem 2 is similar to the case with weakly correlated perturbations (see [28]). Some differences are in part (IV) when only the second population is stochastically perturbed and therefore we prove the following result.

Lemma 1. *Let $b_1b_2 < c_1c_2$, $\rho_1=0$, $\rho_2>0$. If $\tilde a_2<0$ and $a_1b_2+\tilde a_2c_1<0$ then*

$$\lim_{t\to\infty}\xi(t) = \log(a_1/c_1) \quad \textit{a.e.} \quad \textit{and} \quad \lim_{t\to\infty}\eta(t) = -\infty \quad \textit{a.e.}$$

Before the proof we recall an obvious theorem on differential equations.

THEOREM 5. *Let $P \subset \mathbb{R}$ be an open interval and $h : P \times [0,\infty) \to \mathbb{R}$ be a differentiable function. Suppose that for every compact interval $P_0 \subset P$, $h(x,t)$ converges uniformly to $g(x)$ in P_0 as $t \to \infty$. Moreover, assume that the following conditions are satisfied:*

(a) *there exists $\beta \in P_0$ such that $g(x) > 0$ for $x < \beta$ and $g(x) < 0$ for $x > \beta$,*
(b) *there exist $a, b \in P_0$ with $a < \beta < b$ such that $h(x,t) > 0$ for $x < a$, $t \in [0,\infty)$, and $h(x,t) < 0$ for $x > b$, $t \in [0,\infty)$.*

Then there exists a solution $x(t)$ of the equation

$$x'(t) = h(x,t)$$

for all $t > 0$, which satisfies the condition

$$\lim_{t\to\infty} x(t) = \beta.$$

Proof of Lemma 5. First we show that $\lim_{t\to\infty} \eta(t) = -\infty$ a.e. Multiplying (15) by b_2 and (16) by c_1 and adding these equations we have

$$d(b_2\xi(t) + c_1\eta(t)) = (a_1b_2 + \tilde{a}_2c_1 + (b_1b_2 - c_1c_2)e^{\eta(t)})dt + c_1\rho_2 dW(t).$$

Since $b_1b_2 < c_1c_2$, from the comparison theorem [8, Lemma 4, p. 120] we have

$$d(b_2\xi(t) + c_1\eta(t)) \le (a_1b_2 + \tilde{a}_2c_1)dt + c_1\rho_2 dW(t).$$

Consequently,

$$\lim_{t\to\infty} (b_2\xi(t) + c_1\eta(t)) = -\infty \qquad \text{a.e.}$$

Thus for arbitrarily small $\varepsilon > 0$ there exist t_0 and a set Ω_ε such that $\mathrm{Prob}(\Omega_\varepsilon) > 1 - \varepsilon$ and $\xi(t) < -(c_1/b_2)\eta(t)$ for $t \ge t_0$ and $\omega \in \Omega_\varepsilon$. It follows that

$$d\eta(t) \le (\tilde{a}_2 + b_2e^{-(c_1/b_2)\eta(t)})dt + \rho_2 dW(t). \tag{17}$$

Consider the equation

$$d\bar{\eta}(t) = (\tilde{a}_2/2 + b_2e^{-(c_1/b_2)\bar{\eta}(t)})dt + \rho_2 dW(t). \tag{18}$$

The Fokker–Planck equation corresponding to (18) has a stationary density

$$f_*(x) = C\exp\biggl(\frac{2}{\rho_2^2}\biggl(\frac{\tilde{a}_2}{2}\,x - \frac{b_2^2}{c_1}\,e^{-(c_1/b_2)x}\biggr)\biggr),$$

where C is some constant. From the ergodic theorem [8, Theorem 2, p. 141] it follows that

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t e^{-(c_1/b_2)\bar{\eta}(s)}\,ds = \int_{-\infty}^{\infty} f_*(x)e^{-(c_1/b_2)x}\,dx. \tag{19}$$

Since $f_*'(x) = (2/\rho_2^2)(\tilde{a}_2/2 + b_2 e^{-(c_1/b_2)x}) f_*(x)$ we have

$$\int_{-\infty}^{\infty} f_*(x)\, e^{-(c_1/b_2)x}\, dx = -\frac{\tilde{a}_2}{2b_2} > 0. \tag{20}$$

From (19), (20) we obtain

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t e^{-(c_1/b_2)\bar{\eta}(s)}\, ds = -\frac{\tilde{a}_2}{2b_2}. \tag{21}$$

From (17) we have

$$\eta(t) \le \bar{\eta}(t) + \frac{\tilde{a}_2}{2}\, t + C_1,$$

where $\bar{\eta}(t)$ is a solution of equation (18) and C_1 is some constant. Therefore,

$$\eta(t) \le \tilde{a}_2 t + b_2 \int_0^t e^{-(c_1/b_2)\bar{\eta}(s)}\, ds + \rho_2 W(t) + C_1.$$

From this, (21), and $\lim_{t\to\infty} W(t)/t = 0$ we obtain

$$\limsup_{t\to\infty} \frac{\eta(t)}{t} \le \frac{\tilde{a}_2}{2} < 0,$$

and consequently $\lim_{t\to\infty} \eta(t) = -\infty$ a.e. The process $\xi(t)$ satisfies the equation

$$d\xi(t) = (a_1 + b_1 e^{\eta(t)} - c_1 e^{\xi(t)})dt.$$

If $h(x,t) = a_1 + b_1 e^{\eta(t)} - c_1 e^x$, $g(x) = a_1 - c_1 e^x$ then from Theorem 5 we obtain

$$\lim_{t\to\infty} \xi(t) = \log(a_1/c_1) \quad \text{a.e.} \ \blacksquare$$

5. Asymptotic stability. Let $(\xi(t), \eta(t))$ be a solution of (15), (16) such that the distribution of $(\xi(0), \eta(0))$ is absolutely continuous with density $v(x,y)$. Then the random variable $(\xi(t), \eta(t))$ has the density $u(x,y,t)$ and u satisfies the Fokker–Planck equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\rho_1^2 \frac{\partial^2 u}{\partial x^2} + \rho_1\rho_2 \frac{\partial^2 u}{\partial x \partial y} + \frac{1}{2}\rho_2^2 \frac{\partial^2 u}{\partial y^2} - \frac{\partial(f_1 u)}{\partial x} - \frac{\partial(f_2 u)}{\partial y}, \tag{22}$$

where $f_1(x,y) = \tilde{a}_1 + b_1 e^y - c_1 e^x$, $f_2(x,y) = \tilde{a}_2 + b_2 e^x - c_2 e^y$.

Now we introduce a Markov semigroup connected with the Fokker–Planck equation (22). Let $X = \mathbb{R}^2$, Σ be the σ-algebra of Borel subsets of X, and m be the Lebesgue measure on (X, Σ). Let $P(t)v(x,y) = u(x,y,t)$ for $v \in D$. Since the operator $P(t)$ is a contraction on D, it can be extended to a contraction on $L^1(\mathbb{R}^2, \Sigma, m)$. Thus the operators $\{P(t)\}_{t\ge 0}$ form a Markov semigroup. Let $\mathcal{A}$ be the infinitesimal generator of the semigroup $\{P(t)\}_{t\ge 0}$, i.e.

$$\mathcal{A}v = \frac{1}{2}\,\rho_1^2\,\frac{\partial^2 v}{\partial x^2} + \rho_1\rho_2\,\frac{\partial^2 v}{\partial x \partial y} + \frac{1}{2}\,\rho_2^2\,\frac{\partial^2 v}{\partial y^2} - \frac{\partial(f_1 v)}{\partial x} - \frac{\partial(f_2 v)}{\partial y}.$$

The adjoint operator of $\mathcal{A}$ is of the form

$$\mathcal{A}^* v = \frac{1}{2}\,\rho_1^2\,\frac{\partial^2 v}{\partial x^2} + \rho_1\rho_2\,\frac{\partial^2 v}{\partial x \partial y} + \frac{1}{2}\,\rho_2^2\,\frac{\partial^2 v}{\partial y^2} + f_1\,\frac{\partial v}{\partial x} + f_2\,\frac{\partial v}{\partial y}.$$

Let $\mathcal{P}(t,x,y,A)$ be the transition probability function for the diffusion process $(\xi(t),\eta(t))$, i.e. $\mathcal{P}(t,x,y,A) = \mathrm{Prob}((\xi(t),\eta(t)) \in A)$ and $(\xi(t),\eta(t))$ is a solution of system (15), (16) with the initial condition $(\xi(0),\eta(0)) = (x,y)$.

The aim of this section is to prove the asymptotic stability of the semigroup $\{P(t)\}_{t\ge 0}$, because it implies the convergence in L^1 of the densities $u(x,y,t)$ of the process $(\xi(t),\eta(t))$ to the invariant density. Therefore instead of proving parts (I) and (III) of Theorem 2 we show the asymptotic stability of this semigroup.

If stochastic perturbations are weakly correlated then the semigroup connected with the Fokker–Planck equation is an integral Markov semigroup with a continuous and strictly positive kernel. In our case we use Hörmander's theorem on the existence of smooth densities of the transition probability function for degenerate diffusion processes in order to prove that this semgroup is integral and has a continuous kernel. Let us recall this theorem now.

Consider the Stratonovitch stochastic differential equation

$$dX(t) = \sigma(X(t)) \circ dW(t) + \sigma_0(X(t))dt,$$

where $W(t)$ is an m-dimensional Brownian motion, $\sigma(x) = [\sigma_j^i(x)]$ is a $d\times m$ matrix and $\sigma_0(x)$ is a vector in $\mathbb{R}^d$ with components $\sigma_0^i(x)$ for every $x\in\mathbb{R}^d$. Let $\sigma_j(x)$ $(j=0,\dots,m)$ be a vector in $\mathbb{R}^d$ with components $\sigma_j^i(x)$ for every $x\in\mathbb{R}^d$. If $a(x)$ and $b(x)$ are two vector fields on $\mathbb{R}^d$ then their Lie bracket $[a,b]$ is the vector field on $\mathbb{R}^d$ given by

$$[a,b]_j(x) = \sum_{k=1}^{d}\biggl(a_k\,\frac{\partial b_j}{\partial x_k}(x) - b_k\,\frac{\partial a_j}{\partial x_k}(x)\biggr).$$

Theorem 6 (Hörmander). *If for every $x\in\mathbb{R}^d$ the vectors*

$$\sigma_1(x),\dots,\sigma_m(x),\ [\sigma_i,\sigma_j](x)_{0\le i,j\le m},\ [\sigma_i,[\sigma_j,\sigma_k]](x)_{0\le i,j,k\le m},\dots$$

span the space $\mathbb{R}^d$ then the transition probability function $\mathcal{P}(t,x,A)$ has a density $k(t,y,x)$ and $k\in C^\infty((0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d)$.

Lemma 2. *Let $\rho_1,\rho_2>0$. If $\rho_1\ne\rho_2$ or $\tilde a_1\ne\tilde a_2$ then $\{P(t)\}_{t\ge 0}$ is an integral Markov semigroup with a continuous kernel k.*

Proof. Let $\sigma_0(\xi,\eta)=(\tilde a_1+b_1e^\eta-c_1e^\xi,\tilde a_2+b_2e^\xi-c_2e^\eta)$ and $\sigma_1(\xi,\eta)=(\rho_1,\rho_2)$. Then

$$[\sigma_0,\sigma_1](\xi,\eta)=(c_1\rho_1e^\xi-b_1\rho_2e^\eta,c_2\rho_2e^\eta-b_2\rho_1e^\xi),$$
$$[\sigma_1,[\sigma_0,\sigma_1]](\xi,\eta)=(c_1\rho_1^2e^\xi-b_1\rho_2^2e^\eta,c_2\rho_2^2e^\eta-b_2\rho_1^2e^\xi).$$

If $\rho_1\neq\rho_2$ then for every $(\xi,\eta)\in\mathbb{R}^2$ the vectors $\sigma_1(\xi,\eta)$, $[\sigma_0,\sigma_1](\xi,\eta)$, $[\sigma_1,[\sigma_0,\sigma_1]](\xi,\eta)$ span $\mathbb{R}^2$. If $\rho_1=\rho_2$ and $\tilde a_1\neq\tilde a_2$ then the vectors

$$\sigma_1(\xi,\eta)=(\rho_1,\rho_1),\qquad [\sigma_0,\sigma_1](\xi,\eta)=\rho_1(c_1e^\xi-b_1e^\eta,c_2e^\eta-b_2e^\xi),$$
$$[\sigma_0,[\sigma_0,\sigma_1]](\xi,\eta)=\rho_1(\tilde a_1c_1e^\xi-\tilde a_2b_1e^\eta,\tilde a_2c_2e^\eta-\tilde a_1b_2e^\xi)$$

span $\mathbb{R}^2$ for every $(\xi,\eta)\in\mathbb{R}^2$. From Hörmander's theorem it follows that $\mathcal{P}(t,x_0,y_0,\cdot)$ has a density $k(t,x,y;x_0,y_0)$ and $k\in C^\infty((0,\infty)\times\mathbb{R}^2\times\mathbb{R}^2)$. Thus

$$P(t)f(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}k(t,x,y;u,v)f(u,v)\,du\,dv. \tag{23}$$

This means that $\{P(t)\}_{t\ge0}$ is an integral Markov semigroup. ∎

Remark 4. Let $\rho_1,\rho_2>0$. If $\rho_1=\rho_2$ and $\tilde a_1=\tilde a_2$ then considering a solution of system (15), (16) with initial conditions $\xi(0)=x_0$, $\eta(0)=y_0$ such that

$$y_0=x_0+\log\frac{b_2+c_1}{b_1+c_2}$$

we obtain

$$\eta(t)=\xi(t)+\log\frac{b_2+c_1}{b_1+c_2}$$

and the transition density $k(t,x,y;x_0,y_0)$ does not exist.

Lemma 3. *If $\rho_1=0$, $\rho_2>0$ then $\{P(t)\}_{t\ge0}$ is an integral Markov semigroup with a continuous kernel k.*

Proof. Let $\sigma_0(\xi,\eta)=(a_1+b_1e^\eta-c_1e^\xi,\tilde a_2+b_2e^\xi-c_2e^\eta)$ and $\sigma_1(\xi,\eta)=(0,\rho_2)$. Since $[\sigma_0,\sigma_1](\xi,\eta)=\rho_2(-b_1e^\eta,c_2e^\eta)$, for every $(\xi,\eta)\in\mathbb{R}^2$ the vectors $\sigma_1(\xi,\eta)$ and $[\sigma_0,\sigma_1](\xi,\eta)$ span the space $\mathbb{R}^2$. Using the same arguments as in the proof of the previous lemma we conclude that $\{P(t)\}_{t\ge0}$ is an integral Markov semigroup with a continuous kernel. ∎

Now we find the support of the kernel k. If a diffusion process is nondegenerate then the support is the whole space, but in our case we use support theorems in order to find the set in which the kernel is positive. Now we describe the method based on support theorems [1, 3, 29]. Fix a point $(x_0,y_0)\in\mathbb{R}^2$ and a continuous function $\phi:[0,T]\to\mathbb{R}$. Consider the system

$$x'_\phi(t) = \rho_1\phi(t) + f_1(x_\phi(t), y_\phi(t)), \quad x_\phi(0) = x_0, \tag{24}$$

$$y'_\phi(t) = \rho_2\phi(t) + f_2(x_\phi(t), y_\phi(t), \quad y_\phi(0) = y_0, \tag{25}$$

where $f_1(x,y) = \tilde{a}_1 + b_1e^y - c_1e^x$, $f_2(x,y) = \tilde{a}_2 + b_2e^x - c_2e^y$, $\rho_1 = 0$ in case (ii). Let $D_{x_0,y_0;\phi}$ be the Fréchet derivative of the function

$$h \mapsto \mathbf{x}_{\phi+h}(T) \quad \text{with} \quad \mathbf{x}_{\phi+h} = \begin{bmatrix} x_{\phi+h} \\ y_{\phi+h} \end{bmatrix}.$$

If for some ϕ the derivative $D_{x_0,y_0;\phi}$ has rank 2 then $k(T,x,y;x_0,y_0) > 0$ for $x = x_\phi(T)$ and $y = y_\phi(T)$. The derivative $D_{x_0,y_0;\phi}$ can be found by means of the perturbation method for ordinary differential equations. Namely, let $\Lambda(t) = \mathbf{f}'(\mathbf{x}_\phi(t))$, where $\mathbf{f} = \left[\begin{smallmatrix} f_1 \\ f_2 \end{smallmatrix}\right]$ and $\mathbf{f}'$ is its Jacobian. Let $Q(t,t_0)$, for $T \ge t \ge t_0 \ge 0$, be a matrix function such that $Q(t_0,t_0) = I$, $\partial Q(t,t_0)/\partial t = \Lambda(t)Q(t,t_0)$ and $\mathbf{v} = \left[\begin{smallmatrix} \rho_1 \\ \rho_2 \end{smallmatrix}\right]$. Then

$$D_{x_0,y_0;\phi}h = \int_0^T Q(T,s)\mathbf{v}h(s)\,ds. \tag{26}$$

Lemma 4. *Assume that $\rho_1,\rho_2 > 0$, $\tilde{a}_1,\tilde{a}_2 > 0$ and $\tilde{a}_2\rho_1 > \tilde{a}_1\rho_2$. Let $M_0 = \log((b_2+c_1)/(b_1+c_2))$ if $\rho_2 = \rho_1$, and*

$$M_0 = \log\left(\frac{\rho_1(b_2\rho_1 + c_1\rho_2)}{\rho_2(b_1\rho_2 + c_2\rho_1)}\left(\frac{\rho_2(\tilde{a}_2\rho_1 - \tilde{a}_1\rho_2)}{(\rho_1-\rho_2)(b_2\rho_1 + c_1\rho_2)}\right)^{1-\rho_2/\rho_1}\right)$$

if $\rho_2 < \rho_1$. Let $E = \mathbb{R}^2$ when $\rho_2 > \rho_1$, and $E = E(M_0)$ when $\rho_2 \le \rho_1$, where

$$E(M_0) = \{(x,y) \in \mathbb{R}^2 : y > (\rho_2/\rho_1)x + M_0\}.$$

Then for each $(x_0,y_0) \in E$ and for almost every $(x,y) \in E$ there exists $T > 0$ such that $k(T,x,y;x_0,y_0) > 0$.

Proof. First, we show that there exists a constant C such that the rank of $D_{x_0,y_0,\phi}$ is 2 if $y \ne x + C$, where $x = x_\phi(T)$ and $y = y_\phi(T)$. Let $\varepsilon \in (0,T)$ and $h_\varepsilon = 1_{[T-\varepsilon,T]}$. Since $Q(T,s) = I + \Lambda(T)(T-s) + o(T-s)$, from (26) we obtain

$$D_{x_0,y_0;\phi}h_\varepsilon = \varepsilon\mathbf{v} + \frac{1}{2}\varepsilon^2\Lambda(T)\mathbf{v} + o(\varepsilon^2). \tag{27}$$

Since $\mathbf{v} = [\rho_1,\rho_2]$ and $\Lambda(T)\mathbf{v} = e^x[-c_1\rho_1 + b_1\rho_2e^{y-x}, b_2\rho_1 - c_2\rho_2e^{y-x}]$ there exists a constant

$$C = \log\frac{\rho_1(b_2\rho_1 + c_1\rho_2)}{\rho_2(b_1\rho_2 + c_2\rho_1)}$$

such that the vectors $\mathbf{v}$ and $\Lambda(T)\mathbf{v}$ are linearly independent if $y - x \ne C$. Consequently, $D_{x_0,y_0;\phi}$ has rank 2.

Next we substitute

$$z_\phi(t) = y_\phi(t) - \frac{\rho_2}{\rho_1}\, x_\phi(t)$$

and replace system (24), (25) by

$$x'_\phi = \rho_1\phi + g_1(x_\phi, z_\phi), \tag{28}$$

$$z'_\phi = g_2(x_\phi, z_\phi) \tag{29}$$

where

$$g_1(x,z) = \tilde{a}_1 - c_1e^x + b_1e^z e^{rx}, \qquad g_2(x,z) = \alpha e^x - \beta e^z e^{rx} + \gamma \tag{30}$$

with $\alpha = b_2 + rc_1 > 0$, $\beta = c_2 + rb_1 > 0$, $\gamma = \tilde{a}_2 - r\tilde{a}_1 > 0$, $r = \rho_2/\rho_1$.

Now we check that for any two points $(x_0, z_0), (x_1, z_1) \in E$ there exist a control function ϕ and $T > 0$ such that $x_\phi(0) = x_0$, $z_\phi(0) = z_0$, $x_\phi(T) = x_1$ and $z_\phi(T) = z_1$. The proof is split into the following cases.

1° Suppose $r > 1$ and $E = \mathbb{R}^2$. First we find a positive constant T and a differentiable function $z_\phi : [0,T] \to \mathbb{R}$ such that $z_\phi(0) = z_0$, $z_\phi(T) = z_1$,

$$\alpha e^{x_0} - \beta e^{z_0} e^{rx_0} = z'_\phi(0) - \gamma, \qquad \alpha e^{x_1} - \beta e^{z_1} e^{rx_1} = z'_\phi(T) - \gamma \tag{31}$$

and

$$z'_\phi(t) - \gamma < \frac{\alpha(r-1)}{r}\left(\frac{\alpha}{r\beta e^{z_\phi(t)}}\right)^{1/(r-1)} \qquad \text{for } t \in [0,T]. \tag{32}$$

In order to do it, we first determine $z'_\phi(0) =: a_0$ and $z'_\phi(T) =: a_T$ from (31). We construct the function z_ϕ separately in the three intervals $[0,\varepsilon]$, $[\varepsilon, T-\varepsilon]$ and $[T-\varepsilon, T]$, where $0 < \varepsilon < T/2$. Since $\gamma > 0$, $r > 1$ and

$$z'_\phi(0) - \gamma < \frac{\alpha(r-1)}{r}\left(\frac{\alpha}{r\beta e^{z_0}}\right)^{1/(r-1)}$$

we can construct a C^2 function $z_\phi : [0,\varepsilon] \to \mathbb{R}$ such that $z_\phi(0) = z_0$, $z'_\phi(0) = a_0$, $z'_\phi(\varepsilon) = 0$ and z_ϕ satisfies inequality (32) for $t \in [0,\varepsilon]$. In the same way we construct $z_\phi : [T-\varepsilon, T] \to \mathbb{R}$ such that $z_\phi(T) = z_1$, $z'_\phi(T) = a_T$, $z'_\phi(T-\varepsilon) = 0$ and z_ϕ satisfies (32) for $t \in [T-\varepsilon, T]$. Taking sufficiently large T we can extend $z_\phi : [0,\varepsilon] \cup [T-\varepsilon, T] \to \mathbb{R}$ to a C^2 function defined on the whole interval $[0,T]$ such that $z'_\phi(t) < \gamma$ for $t \in [\varepsilon, T-\varepsilon]$, and therefore z_ϕ satisfies (32). From (32) it follows that we can find a C^1 function x_ϕ which satisfies equation (29) and finally we can determine a continuous function ϕ from (28).

2° Assume that $r = 1$. Thus $M_0 = \log(\alpha/\beta)$. Let

$$E = E(M_0) = \{(x,z) \in \mathbb{R}^2 : z > M_0\}.$$

In this case we construct the function ϕ in the following way. First we find a positive constant T and a differentiable function $z_\phi : [0,T] \to \mathbb{R}$ such that

$z_\phi(0) = z_0$, $z_\phi(T) = z_1$,

$$e^{x_0}(\alpha - \beta e^{z_0}) = z'_\phi(0) - \gamma, \quad e^{x_1}(\alpha - \beta e^{z_1}) = z'_\phi(T) - \gamma \tag{33}$$

and

$$z'_\phi(t) - \gamma < 0, \quad \alpha - \beta e^{z(t)} < 0 \quad \text{for } t \in [0, T]. \tag{34}$$

In order to do it, we first determine $z'_\phi(0) =: a_0$ and $z'_\phi(T) =: a_T$ from (33). Next let $0 < \varepsilon < T/2$. Since $\gamma > 0$, $z_0 > M_0$ and $z'_\phi(0) - \gamma < 0$ we can construct a C^2 function $z_\phi : [0, \varepsilon] \to \mathbb{R}$ such that $z_\phi(0) = z_0$, $z'_\phi(0) = a_0$, $z'_\phi(\varepsilon) = 0$ and z_ϕ satisfies (34) for $t \in [0, \varepsilon]$. In the same way we construct $z_\phi : [T - \varepsilon, T] \to \mathbb{R}$ such that $z_\phi(T) = z_1$, $z'_\phi(T) = a_T$, $z'_\phi(T - \varepsilon) = 0$ and z_ϕ satisfies (34) for $t \in [T - \varepsilon, T]$. Taking sufficiently large T we can extend $z_\phi : [0, \varepsilon] \cup [T - \varepsilon, T] \to \mathbb{R}$ to a C^2 function defined on $[0, T]$ such that (34) holds. From (34) it follows that we can find a C^1 function x_ϕ which satisfies (29) and finally we determine a continuous function ϕ from (28).

3° Assume that $r \in (0, 1)$ and $E = E(M_0) = \{(x, z) \in \mathbb{R}^2 : z > M_0\}$, where

$$M_0 = \log\left(\frac{\alpha}{r\beta}\left(\frac{r\gamma}{\alpha(1-r)}\right)^{1-r}\right).$$

In this case we construct the function ϕ in the following way. First, we find a positive constant T and a differentiable function $z_\phi : [0, T] \to \mathbb{R}$ such that $z_\phi(0) = z_0$, $z_\phi(T) = z_1$,

$$\alpha e^{x_0} - \beta e^{z_0} e^{r x_0} = z'_\phi(0) - \gamma, \quad \alpha e^{x_1} - \beta e^{z_1} e^{r x_1} = z'_\phi(T) - \gamma, \tag{35}$$

and

$$z'_\phi(t) - \gamma < \frac{\alpha(r-1)}{r}\left(\frac{\alpha}{r\beta e^{z_\phi(t)}}\right)^{1/(r-1)} \quad \text{for } t \in [0, T], \tag{36}$$

$$\frac{\alpha(r-1)}{r}\left(\frac{\alpha}{r\beta e^{z_\phi(t)}}\right)^{1/(r-1)} + \gamma > 0 \quad \text{for } t \in [0, T]. \tag{37}$$

In order to do it, we first determine $z'_\phi(0) =: a_0$ and $z'_\phi(T) =: a_T$ from (35). Next let $0 < \varepsilon < T/2$. Since $\gamma > 0$, $z_0 > M_0$ and

$$z'_\phi(0) - \gamma < \frac{\alpha(r-1)}{r}\left(\frac{\alpha}{r\beta e^{z_0}}\right)^{1/(r-1)}$$

we can construct a C^2 function $z_\phi : [0, \varepsilon] \to \mathbb{R}$ such that $z_\phi(0) = z_0$, $z'_\phi(0) = a_0$, $z'_\phi(\varepsilon) = 0$ and z_ϕ satisfies (36) and (37) for $t \in [0, \varepsilon]$. Analogously, we construct $z_\phi : [T - \varepsilon, T] \to \mathbb{R}$ such that $z_\phi(T) = z_1$, $z'_\phi(T) = a_T$, $z'_\phi(T - \varepsilon) = 0$ and z_ϕ satisfies (36) and (37) for $t \in [T - \varepsilon, T]$. Taking sufficiently large T we can extend $z_\phi : [0, \varepsilon] \cup [T - \varepsilon, T] \to \mathbb{R}$ to a C^2 function defined on $[0, T]$ such that (36) and (37) hold. Hence we can find a C^1 function x_ϕ which satisfies (29) and finally we can determine a continuous function ϕ from (28).

From cases 1°–3° it follows that for any $(x_0, y_0), (x, y) \in E$ there is a control function ϕ and $T > 0$ such that $x_\phi(0) = x_0$, $y_\phi(0) = y_0$, $x_\phi(T) = x$ and $y_\phi(T) = y$. From the first part of the proof we now conclude that $k(T, x, y; x_0, y_0) > 0$ if $y \neq x + C$. ■

From Lemma 4 we obtain the following results.

Lemma 5. *Assume that* $\rho_1, \rho_2 > 0$, $\tilde{a}_1, \tilde{a}_2 > 0$ *and* $\tilde{a}_2\rho_1 < \tilde{a}_1\rho_2$. *Let* $M_0 = \log((b_2 + c_1)/(b_1 + c_2))$ *if* $\rho_2 = \rho_1$, *and*

$$M_0 = \log\biggl(\frac{\rho_1(b_2\rho_1 + c_1\rho_2)}{\rho_2(b_1\rho_2 + c_2\rho_1)}\biggl(\frac{\rho_2(\tilde{a}_2\rho_1 - \tilde{a}_1\rho_2)}{(\rho_1 - \rho_2)(b_2\rho_1 + c_1\rho_2)}\biggr)^{1-\rho_2/\rho_1}\biggr)$$

if $\rho_2 > \rho_1$. *Let* $E = \mathbb{R}^2$ *when* $\rho_2 < \rho_1$ *and* $E = E(M_0)$ *when* $\rho_2 \geq \rho_1$, *where*

$$E(M_0) = \{(x, y) \in \mathbb{R}^2 : y < (\rho_2/\rho_1)x + M_0\}.$$

Then for each $(x_0, y_0) \in E$ *and for almost every* $(x, y) \in E$ *there exists* $T > 0$ *such that* $k(T, x, y; x_0, y_0) > 0$.

Lemma 6. *Assume that* $\rho_1, \rho_2 > 0$, $\tilde{a}_1 > 0$ *and* $\tilde{a}_2 < 0$. *Let* $M_0 = \log((b_2 + c_1)/(b_1 + c_2))$ *if* $\rho_2 = \rho_1$ *and*

$$M_0 = \log\biggl(\frac{\rho_1(b_2\rho_1 + c_1\rho_2)}{\rho_2(b_1\rho_2 + c_2\rho_1)}\biggl(\frac{\rho_2(\tilde{a}_2\rho_1 - \tilde{a}_1\rho_2)}{(\rho_1 - \rho_2)(b_2\rho_1 + c_1\rho_2)}\biggr)^{1-\rho_2/\rho_1}\biggr)$$

if $\rho_2 > \rho_1$. *Let* $E = \mathbb{R}^2$ *when* $\rho_2 < \rho_1$ *and* $E = E(M_0)$ *when* $\rho_2 \geq \rho_1$, *where*

$$E(M_0) = \{(x, y) \in \mathbb{R}^2 : y < (\rho_2/\rho_1)x + M_0\}.$$

Then for each $(x_0, y_0) \in E$ *and for almost every* $(x, y) \in E$ *there exists* $T > 0$ *such that* $k(T, x, y; x_0, y_0) > 0$.

In case (ii) when only the second population is stochastically perturbed we have

Lemma 7. *Assume that* $\rho_1 = 0$, $\rho_2 > 0$. *Let* $E = (\log(a_1/c_1), \infty) \times \mathbb{R}$. *Then for each* $(x_0, y_0) \in E$ *and for almost every* $(x, y) \in E$ *there exists* $T > 0$ *such that* $k(T, x, y; x_0, y_0) > 0$.

Proof. First, we show that the rank of $D_{x_0,y_0,\phi}$ is 2. Let $\varepsilon \in (0, T)$ and $h_\varepsilon = 1_{[T-\varepsilon,T]}$. Since the vectors $\mathbf{v} = [0, \rho_2]$ and $\Lambda(T)\mathbf{v} = e^y\rho_2[b_1, -c_2]$ are linearly independent, from (27) it follows that $D_{x_0,y_0;\phi}$ has rank 2.

Next we prove that for any $(x_0, y_0), (x_1, y_1) \in E$ there exist a control function ϕ and $T > 0$ such that $x_\phi(0) = x_0$, $y_\phi(0) = y_0$, $x_\phi(T) = x_1$ and $y_\phi(T) = y_1$. First we find a positive constant T and a differentiable function $x_\phi : [0, T] \to \mathbb{R}$ such that $x_\phi(0) = x_0$, $x_\phi(T) = x_1$,

$$b_1e^{y_0} = x'_\phi(0) - a_1 + c_1e^{x_0}, \qquad b_1e^{y_1} = x'_\phi(T) - a_1 + c_1e^{x_1} \tag{38}$$

and

$$x'_\phi(t) - a_1 + c_1e^{x_\phi(t)} > 0 \quad \text{and} \quad a_1 - c_1e^{x_\phi(t)} < 0 \qquad \text{for } t \in [0, T]. \tag{39}$$

In order to do it, we first determine $x'_\phi(0) =: a_0$ and $x'_\phi(T) =: a_T$ from (38). Next let $0 < \varepsilon < T/2$. Since $c_1 e^{x_0} > a_1$ and $x'_\phi(0) - a_1 + c_1 e^{x_0} > 0$ we can construct a C^2 function $x_\phi : [0,\varepsilon] \to \mathbb{R}$ such that $x_\phi(0) = x_0$, $x'_\phi(0) = a_0$, $x'_\phi(\varepsilon) = 0$ and x_ϕ satisfies (39) for $t \in [0,\varepsilon]$. In the same way we construct $x_\phi : [T-\varepsilon, T] \to \mathbb{R}$ such that $x_\phi(T) = x_1$, $x'_\phi(T) = a_T$, $x'_\phi(T-\varepsilon) = 0$ and x_ϕ satisfies (39) for $t \in [T-\varepsilon, T]$. Taking sufficiently large T we can extend $x_\phi : [0,\varepsilon] \cup [T-\varepsilon, T] \to \mathbb{R}$ to a C^2 function defined on $[0,T]$ such that (39) holds. It follows that we can find a C^1 function y_ϕ which satisfies (24) and finally we can determine a continuous function ϕ from (25). ■

LEMMA 8. *Assume that $\rho_1, \rho_2 > 0$, $\tilde{a}_1, \tilde{a}_2 > 0$, $\tilde{a}_2\rho_1 > \tilde{a}_1\rho_2$, and $\rho_2 \le \rho_1$. Let $E = E(M_0)$. Then for every density f we have*

$$\lim_{t\to\infty} \iint_E P(t)f(x)\, dx\, dy = 1. \tag{40}$$

Proof. First we substitute

$$\zeta(t) = \eta(t) - \frac{\rho_2}{\rho_1}\,\xi(t).$$

Then we replace system (15), (16) by

$$d\xi(t) = g_1(\xi(t), \zeta(t))dt + \rho_1 dW(t), \tag{41}$$

$$d\zeta(t) = g_2(\xi(t), \zeta(t))dt, \tag{42}$$

where the functions g_1, g_2 are defined by (30). Since for each $\varepsilon > 0$ we have

$$\inf\{g_2(x,z) : z \le M_0 - \varepsilon,\ x \in \mathbb{R}\} > 0 \tag{43}$$

we obtain $\liminf_{t\to\infty} \zeta_t \ge M_0$. We will prove that for almost every ω there is $t_0 = t_0(\omega)$ such that $\zeta(t,\omega) > M_0$ for $t \ge t_0$.

The case $\rho_2 = \rho_1$ is obvious, because $g_2(x, M_0) = \gamma > 0$ for all $x \in \mathbb{R}$. Consider the case $\rho_2 < \rho_1$. Thus there exists $C_0 \in \mathbb{R}$ such that $g_2(C_0, M_0) = 0$. Fix $\kappa > 0$ and $\tau > 0$. Consider the solution of system (41), (42) with initial conditions $\xi(0) = C_0 + 2\kappa$, $\zeta(0) = M_0 - \tau$. Let

$$A_{\kappa,\tau} = [C_0, C_0+\kappa] \times [M_0 - \tau, M_0], \qquad B_{\kappa,\tau} = [C_0, C_0 + 2\kappa] \times [M_0 - \tau, M_0].$$

Then from the continuity of g_1, g_2 it follows that there exist $\varepsilon, L > 0$ such that $g_2(x,z) > \varepsilon$ for $x \ge C_0 + \kappa$, $z \in [M_0 - \tau, M_0]$ and $|g_1(x,z)| \le L$ for $(x,z) \in B_{\kappa,\tau}$. Let $\bar{\xi}(t)$ be a solution of the equation $d\bar{\xi}(t) = -Ldt + \rho_1 dW(t)$ such that $\bar{\xi}(0) = C_0 + 2\kappa$. From the comparison theorem we obtain $\bar{\xi}(t) \le \xi(t)$ and $\zeta(t) > M_0 - \tau + \varepsilon t$ as long as $(\xi(t), \zeta(t)) \in B_{\kappa,\tau} \setminus A_{\kappa,\tau}$. Let $t = \tau/\varepsilon$ and $\Omega_\tau = \{\omega : \bar{\xi}(s,\omega) \ge C_0 + \kappa \text{ for } s \le t\}$. Thus $\lim_{\tau\to 0} \mathrm{Prob}(\Omega_\tau) = 1$ and $\zeta(t,\omega) > M_0$ for $\omega \in \Omega_\tau$. Now let $(\xi(t), \zeta(t))$ be any solution of system (41), (42). Then from what has already been proved and from the Markov property it follows that if $\sup_{t>0} \zeta(t,\omega) \le M_0$ then $\limsup_{t\to\infty} \xi(t,\omega) \le C_0$. In the same way we check that if $\sup_{t>0} \zeta(t,\omega) \le M_0$ then $\liminf_{t\to\infty} \xi(t,\omega)$

$\geq C_0$. Thus $\sup_{t>0}\zeta(t,\omega) \leq M_0$ implies $\lim_{t\to\infty}\xi(t,\omega) = C_0$. Assume that $\lim_{t\to\infty}\xi(t,\omega) = C_0$ with probability $> p_0 > 0$. Let $\gamma_1 = g_1(C_0, M_0)$. Then for every $\varepsilon > 0$ there exist $t_0 > 0$ and a set Ω' such that $\mathrm{Prob}(\Omega') > p_0$, $|\xi(t,\omega) - C_0| < \varepsilon$ and

$$\rho_1 dW(t) + (\gamma_1 - \varepsilon)dt \leq d\xi(t) \leq \rho_1 dW(t) + (\gamma_1 + \varepsilon)dt \tag{44}$$

for $\omega \in \Omega'$ and $t \geq t_0$. Then $\mathrm{Prob}(\{\omega \in \Omega' : |\xi(t_0+1) - C_0| < \varepsilon\}) \leq O(\varepsilon)$, which contradicts the assumption that $p_0 > 0$. Consequently, for almost every ω there exists $t_0 = t_0(\omega)$ such that $\zeta(t,\omega) > M_0$ for $t \geq t_0$ and (40) holds. ∎

From Lemma 8 we obtain the following results.

Lemma 9. *Assume that* $\rho_1, \rho_2 > 0$, $\tilde{a}_1, \tilde{a}_2 > 0$, $\tilde{a}_2\rho_1 < \tilde{a}_1\rho_2$, *and* $\rho_2 \geq \rho_1$. *Let* $E = E(M_0)$. *Then for every density* f *we have*

$$\lim_{t\to\infty} \iint_E P(t)f(x)\,dx\,dy = 1.$$

Lemma 10. *Assume that* $\rho_1, \rho_2 > 0$, $\tilde{a}_1 > 0$, $\tilde{a}_2 < 0$, *and* $\rho_2 \geq \rho_1$. *Let* $E = E(M_0)$. *Then for every density* f *we have*

$$\lim_{t\to\infty} \iint_E P(t)f(x)\,dx\,dy = 1.$$

Lemma 11. *Consider case* (ii) *when only the second population is stochastically perturbed. Assume that either* $\tilde{a}_2 > 0$, *or* $\tilde{a}_2 < 0$ *and* $a_1b_2 + \tilde{a}_2c_1 > 0$. *Let* $E = (\log(a_1/c_1), \infty) \times \mathbb{R}$. *Then for every density* f *we have*

$$\lim_{t\to\infty} \iint_E P(t)f(x)\,dx\,dy = 1.$$

Proof. The process $\xi(t)$ satisfies the differential equation

$$d\xi(t) = (a_1 + b_1e^{\eta(t)} - c_1e^{\xi(t)})dt.$$

It follows that for every ω there are two cases:

(a) there is $t_0 = t_0(\omega)$ such that $\xi(t,\omega) > \log(a_1/c_1)$ for $t > t_0$,
(b) $\lim_{t\to\infty}\xi(t,\omega) = \log(a_1/c_1)$ and $\lim_{t\to\infty}\eta(t,\omega) = -\infty$.

Case (b) is impossible because from the assumption $a_1b_2 + \tilde{a}_2c_1 > 0$ and the equation

$$d\eta(t) = (\tilde{a}_2 + b_2e^{\xi(t)} - c_2e^{\eta(t)})dt + \rho_2\,dW(t)$$

it follows that there exists $\varepsilon > 0$ such that

$$d\eta(t) \geq \varepsilon dt + \rho_2 dW(t)$$

for sufficiently large t, which contradicts the fact that $\lim_{t\to\infty}\eta(t,\omega) = -\infty$. ∎

THEOREM 7. *Let* $b_1b_2 < c_1c_2$. *In case* (i) *assume that* $\rho_1 \neq \rho_2$ *or* $\tilde{a}_1 \neq \tilde{a}_2$. *If either* $\tilde{a}_1, \tilde{a}_2 > 0$, *or* $\tilde{a}_1 > 0$, $\tilde{a}_2 < 0$ *and* $\tilde{a}_1 b_2 + \tilde{a}_2 c_1 > 0$, *then the semigroup* $\{P(t)\}_{t\ge 0}$ *is asymptotically stable.*

Proof. From Lemmas 8–11 it follows that it is sufficient to consider the restriction of the semigroup $\{P(t)\}_{t\ge 0}$ on $L^1(\mathbb{R}^2)$ to the space $L^1(E)$. From Lemmas 2, 3 we can see that $\{P(t)\}_{t\ge 0}$ is an integral Markov semigroup with a continuous kernel. According to Lemmas 4–7 we have

$$\int_0^\infty P(t)f\,dt > 0 \quad \text{a.e. on } E$$

for every $f \in D$. From Theorem 4 we conclude that $\{P(t)\}_{t\ge 0}$ satisfies the Foguel alternative. In order to exclude sweeping we construct a Khasminskiĭ function exactly as in [28]. Thus the semigroup $\{P(t)\}_{t\ge 0}$ is asymptotically stable. ∎

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Received 28.5.2009
and in final form 25.9.2009 (2024)