Abstract. A homology theory of Banach manifolds of a special form, called \textit{FSQL}-
manifolds, is developed, and also a homological degree of \textit{FSQL}-mappings between \textit{FSQL}-
manifolds is introduced.

1. Introduction. In this article the results of the article [3] are generalized to Banach manifolds of a special form, namely to Fredholm Special Quasi Linear (\textit{FSQL}) manifolds. In other words, a homology theory of such manifolds is devised and also the homological degree of \textit{FSQL}-mappings between them is introduced. Every \textit{FSQL}-mapping is an \textit{FQL}-mapping [10][1] and vice versa. However, \textit{FSQL}-mappings are more convenient for the structure of \textit{FSQL}-manifolds.

It is known that the degree of a mapping is a strong tool for proving the existence of solutions of various mathematical problems. For instance, various variants of the nonlinear Hilbert problem [7], [10], etc.) have been solved with the help of the degree of \textit{FQL}-mappings. Moreover, the homological degree of mappings transforms topological problems into algebraic ones. In this case, the problem of finding the degree of a mapping will be reduced to a combinatorial problem.

2. Definition of \textit{FSQL}-manifolds and \textit{FSQL}-mappings. Let $\xi_p = (X_p, \varphi_p, V_p)$ and $\xi_{p,r} = (X_{p,r}, \varphi_{p,r}, V_{p,r})$ be affine bundles with identical total space $X_p$ and with base spaces $V_p$, $V_{p,r}$ which are $p$- and $r$-manifolds ($r \geq p$), respectively.

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(1) See [1] for the proof that every \textit{FQL}-mapping is an \textit{FSQL}-mapping.
DEFINITION 2.1. \( \xi_{p,r} \) is called an \((r-p)\)-division of \( \xi_p \) if

\[
\forall \alpha' \in V_{p,r} \exists \alpha \in V_p, \quad \varphi_{p,r}^{-1}(\alpha') \subset \varphi_p^{-1}(\alpha) \quad \text{and} \quad \text{codim}(\varphi_{p,r}^{-1}(\alpha')) = r - p \quad \text{in} \quad \varphi_p^{-1}(\alpha).
\]

Obviously, in this case \( V_{p,r} \) is an affine bundle with the base space \( V_p \) and with fibers of dimension \( r - p \).

Let \( \eta_m = (Y_m, \psi_m, B_m) \) also be an affine bundle, the base space of which is an \( m \)-manifold.

DEFINITION 2.2. A continuous mapping \( f_{p,m} : X_p \to Y_m \) is called a Fredholm Special Linear (FSL) mapping between the affine bundles \( \xi_p \) and \( \eta_m \) if for some \( r \) there exists an \((r-p)\)-division \( \xi_{p,r} = (X_p, \varphi_{p,r}, V_{p,r}) \) of \( \xi_p \) and an \((r-m)\)-division \( \eta_{m,r} = (Y_m, \psi_{m,r}, B_{m,r}) \) of \( \eta_m \) with the same dimension \( r \) of the base spaces, such that \( f_{p,m} \) induces a bimorphism between \( \xi_{p,r} \) and \( \eta_{m,r} \).

From this point on, we will denote such \( f_{p,m} \) as \( f_{p,m,r} \). We will also call the restriction of an FSL-mapping to any subset of \( X_p \) an FSL-mapping.

Obviously, if \( f_{p,m,r} \) is a bimorphism between \( \xi_{p,r} \) and \( \eta_{m,r} \), then it is also a bimorphism between some \((\nu-r)\)-divisions \( \xi_{p,\nu} = (X_p, \varphi_{p,\nu}, V_{p,\nu}) \) and \( \eta_{m,\nu} = (Y_m, \psi_{m,\nu}, B_{m,\nu}) \) of \( \xi_{p,r} \) and \( \eta_{m,r} \) for any \( \nu > r \).

For simplicity, let us assume that \( \xi_p \) and \( \eta_m \) are embedded in Banach spaces \( E_1 \) and \( E_2 \) with norms \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \), respectively. Let \( f_{p,m,r} : X_p \to Y_m \) be a bimorphism between \( \xi_p \) and \( \eta_m \), and \( \Delta_p \) be a bounded domain in \( X_p \).

Let

\[
\| f_{p,m,r} \|_{\Delta_p} = \sup \{ C \mid \| f_{p,m,r,\alpha'}(u) \|_2 \leq C(1 + \| u \|_1), \quad \| u \|_1 \leq C(1 + \| f_{p,m,r,\alpha'}(u) \|_2), \quad \forall u \in X_{p,\alpha'} \},
\]

where \( X_{p,\alpha'} \) is the fiber of \( \xi_{p,r} \) over \( \alpha' \in V_{p,r} \), \( f_{p,m,r,\alpha'} \) is the restriction of \( f_{p,m,r} \) onto \( X_{p,\alpha'} \), and the supremum is taken over all \( X_{p,\alpha'} \) for which \( X_{p,\alpha'} \cap \Delta_p \neq \emptyset \).

DEFINITION 2.3. A continuous mapping \( f_{p,m} : X_p \to Y_m \) is called an FSQL-mapping between the affine bundles \( \xi_p \) and \( \eta_m \) if it can be uniformly approximated in each bounded domain \( \Delta_p \) of \( X_p \) by FSL-mappings \( f_{p,m,r} \) so that

\[
\| f_{p,m,r} \|_{\Delta_p} \leq C(\Delta_p), \quad \forall r > r(\Delta_p),
\]

where \( C(\Delta_p) \) is independent of \( r \) for \( r > r(\Delta_p) \).

Now we shall give definitions of FSQL-manifolds and of FSQL-mappings between FSQL-manifolds. Let \( \tilde{X} \) be a Banach manifold and \( \{ \tilde{X}_p \} \), \( \tilde{X}_{p-1} \subset \tilde{X}_p, \quad p = 1, 2, \ldots, \) be a system of open sets covering \( \tilde{X} \), i.e. \( \tilde{X} = \cup \tilde{X}_p \). Let \( \xi_p = (X_p, \varphi_p, V_p) \) be an affine bundle, \( \Delta_p \) be a bounded domain in \( X_p \) and \( \tilde{\varphi}_p : \tilde{X}_p \to \Delta_p \) be a homeomorphism. In this case, \( (\tilde{\varphi}_p, \tilde{X}_p) \) is called a linear chart (L-chart) on \( \tilde{X} \). We shall say that a linear structure (L-structure)
is introduced on $\tilde{X}_p$ if the conditions above are satisfied. If an $L$-structure is defined on $\tilde{X}_{p+1}$, then obviously it is also defined on $\tilde{X}_p$ (as an induced structure). If $\tilde{\varphi}_p' : \tilde{X}_p' \rightarrow \Delta_p'$, $\tilde{\varphi}_p'' : \tilde{X}_p'' \rightarrow \Delta_p''$, $p', p'' \geq p$, are two $L$-structures on $\tilde{X}_p$, then the transition functions $\tilde{\varphi}_p'' \circ \tilde{\varphi}_p'^{-1} : \Delta_p' \rightarrow \Delta_p''$ and $\tilde{\varphi}_p' \circ \tilde{\varphi}_p''^{-1} : \Delta_p'' \rightarrow \Delta_p'$ arise. Let us suppose that they are FSQ-mappings between $\xi_p' = (X_p', \varphi_p', V_p')$ and $\xi_p'' = (X_p'', \varphi_p'', V_p'')$. In that case, we shall say that the two $L$-structures on $\tilde{X}_p$ are equivalent.

**Definition 2.4.** A class of equivalent $L$-structures on $\tilde{X}_p$ is called an FSQ-structure on $\tilde{X}_p$.

Obviously, an FSQ-structure on $\tilde{X}_{p+1}$ induces an FSQ-structure on $\tilde{X}_p$. An FSQ-structure on $\tilde{X}_p$ is said to be coordinated with an FSQ-structure on $\tilde{X}_{p+1}$ if it coincides with the induced structure.

**Definition 2.5.** A collection of FSQ-structures on $\tilde{X}_p$, $p = 1, 2, \ldots$, which are coordinated with each other is called an FSQ-structure on $\tilde{X}$. A Banach manifold $\tilde{X}$ with an FSQ-structure is called an FSQ-manifold.

Let $\tilde{X}, \tilde{Y}$ be FSQ-manifolds,

$$\tilde{X} = \bigcup \tilde{X}_p, \tilde{X}_p \subset \tilde{X}_{p+1} \forall p, \quad \tilde{Y} = \bigcup \tilde{Y}_m, \tilde{Y}_m \subset \tilde{Y}_{m+1} \forall m,$$

$(\tilde{\varphi}_p, \tilde{X}_p)$, $(\tilde{\psi}_m, \tilde{Y}_m)$ be $L$-charts on $\tilde{X}, \tilde{Y}$ and $\tilde{\varphi}_p(\tilde{X}_p) = \Delta_p$, $\tilde{\psi}_m(\tilde{Y}_m) = \Omega_m$ be bounded domains in $\xi_p = (X_p, \varphi_p, V_p)$, $\eta_m = (Y_m, \psi_m, B_m)$, respectively.

**Definition 2.6.** A continuous mapping $\tilde{f} : \tilde{X} \rightarrow \tilde{Y}$ between FSQ-manifolds $\tilde{X}$ and $\tilde{Y}$ is called an FSQ-mapping if

(a) $\forall p \exists m$, $\tilde{f}(\tilde{X}_p) \subset \tilde{Y}_m$,

(b) $f_{p,m} \equiv \tilde{\psi}_m \circ \tilde{f} \circ \tilde{\varphi}_p^{-1} : \Delta_p \rightarrow \Omega_m$ is an FSQ-mapping between the domains of the affine bundles $\xi_p$ and $\eta_m$.

**3. L-homology theory of affine bundles. Singular theory.** First, note that the simplicial theory of $(n, k)$-simplexes is available in [3], it is similar to the finite-dimensional case.

Let $H$ be a real Hilbert space, $H^k$ be a linear subspace of codimension $k$ ($k \geq 0$) and $\sigma_n$ be a Euclidean $n$-simplex. We will name the Cartesian product $\sigma_n \times H^k$ a Hilbertian simplex of bi-dimension $(n, k)$ and we will denote it by $\sigma^{k}_n$, that is, $\sigma^{k}_n = \sigma_n \times H^k$. We will consider $\sigma^{k}_n$ to be oriented if $\sigma_n$ is oriented. In this case, the orientation on $\sigma_n$ is taken to be the orientation on $\sigma^{k}_n$. From this point on, we will consider $\sigma^{k}_n$ to be oriented.

**Definition 3.1.** A continuous mapping $f_n^k : \sigma^{k}_n \rightarrow X_p$ is called a singular $(n, k)$-simplex in $\xi_p$ if there exists a $k$-division ($\xi_{p'}$) of $\xi_p$ such that $f_n^k$ induces a bimorphism between $\sigma_n \times H^k$ and $\xi_{p'}$. 

It follows from this definition that each singular \((n, k)\)-simplex \(f^k_n\) induces some finite-dimensional mapping between the base spaces \(\sigma_n\) and \(V_{p'}\) \((p' = p + k)\) of these bundles.

**Definition 3.2.** A finite formal linear combination \(\tilde{c}^k_n = \sum_i g_i \cdot f^k_{n,i}\) of singular \((n, k)\)-simplexes in \(\xi_p\) with coefficients \(g_i \in \mathbb{Z}\), where \(\mathbb{Z}\) is the ring of integers, is called a singular \((n, k)\)-chain in \(\xi_p\).

We will denote by \(\tilde{C}^k_n(X_p)\) the set of all singular chains in \(\xi_p\) of bi-dimension \((n, k)\). Obviously, it is an Abelian group under addition of chains. It is a free group.

**Definition 3.3.** We define the differential \(\tilde{\partial}^k_n: \tilde{C}^k_n(X_p) \to \tilde{C}^k_{n-1}(X_p)\) \(\forall n \geq 1, \forall k \geq 0\) as follows:
\[
\tilde{\partial}^k_n f^k_n = \sum_i (-1)^i (f^k_n|_{\sigma^k_{n-1,i}})
\]
and we extend it to \(\tilde{C}^k_n(X_p)\) by additivity. Moreover,
\[
\tilde{\partial}^k_0: \tilde{C}^k_0(X_p) \to 0 \quad \forall k \geq 0.
\]

**Remark.** Here \(\sigma^k_{n-1,i}\) is the \((n-1, k)\)-boundary of the simplex \(\sigma^k_n\), which is located opposite vertex \(i\).

**Theorem 3.4.** The equality
\[
\tilde{\partial}^k_{n-1} \circ \tilde{\partial}^k_n = 0
\]
is true for each \(n \geq 1\) and \(k\).

The proof is similar to the finite-dimensional case.

Analogously to the finite-dimensional case, one can define the groups \(\text{Ker } \tilde{\partial}^k_n\), \(\text{Im } \tilde{\partial}^k_n\) and \(\tilde{H}^k_n\), i.e. the groups of \((n, k)\)-cycles, \((n, k)\)-boundaries and the \((n, k)\)-homology group (see [3]). However the theory of relative homology of \(\xi_p\), which is introduced in the following section, is more interesting.

4. **The relative \(L\)-homology of an affine bundle**

**Definition 4.1.** An \((n, k)\)-chain \(\tilde{c}^k_n \in \tilde{C}^k_n(X_p)\) is called a relative cycle of bi-dimension \((n, k)\) if \(\tilde{\partial}^k_n \tilde{c}^k_n \in \tilde{C}^k_{n-1}(X_p \setminus \Delta_p)\).

**Definition 4.2.** A relative cycle \(\tilde{c}^k_n\) is called homologous to zero if
\[
\exists \tilde{c}^k_{n+1} \in \tilde{C}^k_{n+1}(X_p), \quad \tilde{\partial}^k_{n+1} \tilde{c}^k_{n+1} = \tilde{c}^k_n \oplus \tilde{d}^k_n, \quad \tilde{d}^k_n \in \tilde{C}^k_n(X_p \setminus \Delta_p).
\]

It follows from this definition that the sum of relative \((n, k)\)-cycles homologous to zero is also homologous to zero. Therefore the set of relative \((n, k)\)-cycles homologous to zero forms a subgroup of the group of relative \((n, k)\)-cycles.
Now we define the concept of “support” of a singular simplex.

Let $f^k_n$ be a singular simplex in $X_p$. By definition, it induces a bimorphism between $\sigma_n \times H^k$ and some $k$-division $\xi_{p'} = (X_p, \varphi_{p'}, V_{p'})$ of $\xi_p$. Then $(f^k_n)^{-1}(\xi_{p'})$ induces an affine bundle $(\sigma_{n'})$, which is a $(k' - k)$-division of $\sigma_n$. Its base space $\sigma_{n'}$ is itself an affine bundle with base space $\sigma_n$ and fiber $H_{k' - k}$, $n' = n + (k' - k)$, which is the Euclidean $(k' - k)$-space. As $\sigma_n$ is convex, one can represent $\sigma_{n'}$ in the form of a Cartesian product: $\sigma_{n'} = \sigma_n \times H_{k' - k}$. Therefore the bundle $(\sigma^k_{n'})$ is also a Cartesian product, i.e. $\sigma^k_{n'} = \sigma_{n'} \times H^{k'}$, where $H^{k'}$ is a subspace of $H$ of codimension $k'$. Now we divide $\sigma_{n'}$ into $n'$-prisms $\sigma_{n',j}$, $j = 0, \pm 1, \pm 2, \ldots$, with bases $\sigma_n$. Let us choose the orientation of one $(n', k')$-prism $\sigma^k_{n',j} = \sigma_{n',j} \times H^{k'}$ arbitrarily and coordinate orientations of other $(n', k')$-prisms with it. Then any two neighboring prisms will induce opposite orientations on the common edge.

Obviously, it is possible to divide $\sigma_{n'}$ into $n'$-prisms so that the restriction of each of the mappings $f^k_n$ to a unique $\sigma_{n',j} \times H^{k'}$ contains the intersection of $f^k_n(\sigma^k_n)$ with $\Delta_p$; this is possible because of the linearity of each $f^k_n$ on $H^{k'}$, the uniform continuity of $f^k_n$ in $\alpha$, and the boundedness of $\Delta_p$. In this case all the other analogous restrictions will be outside of $\Delta_p$. Thus, we can give the following

**DEFINITION 4.3.** The restriction of a singular simplex $f^k_n$ to an $(n', k')$-prism $\sigma^k_{n'}$ is called an $(n', k')$-support of $f^k_n$ if

(a) $n' - n = k' - k$,
(b) $f^k_n(\sigma^k_{n'}) \cap \Delta_p = f^k_n(\sigma^k_n) \cap \Delta_p$.

Let us denote the $(n', k')$-support of $f^k_n$ by $f^k_{n'}$. From Definition 4.3 it follows that there can be different $(n', k')$-supports of a singular $(n, k)$-simplex. But obviously, the difference of two $(n', k')$-supports of $f^k_n$ is homologous to zero relative to $X_p \setminus \Delta_p$.

Analogously, we shall say that a chain $\tilde{c}^k_{n'} = \sum g_i \cdot f^k_{n',i}$ is an $(n', k')$-support of the chain $\tilde{c}^k_n = \sum g_i \cdot f^k_{n,i}$ if for each $i$ the simplex $f^k_{n',i}$ is an $(n', k')$-support of $f^k_{n,i}$.

Obviously with the help of the above construction one can construct an $(n'', k'')$-support of the chain $\tilde{c}^k_n$ for any $n'' > n'$, $k'' > k'$, where $n'' - n = k'' - k$.

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\(^{(2)}\) For example, in the case of $n = 1$ and $k' - k = 1$ the base space $\sigma_{n'}$, $n' = 2$, can be represented in the form of an infinite band. The line segment which defines the width of this band is $\sigma_n$. Having divided this band into segments, we will obtain rectangles-prisms $\sigma_{n',j}$, $j = 0, \pm 1, \pm 2, \ldots$, with bases $\sigma_n$.

One can represent each prism $\sigma_{n',j}$, $j = 0, \pm 1, \pm 2, \ldots$, in the form of $\sigma_n \times I_{k' - k}$, where $I_{k' - k}$ is a $(k' - k)$-cube.
REMARK. In view of the aforementioned construction, from this point on we will suppose that all simplexes \( f_{n',i}^{k'} \) of \( \tilde{c}_{n'}^{k'} \) are bimorphisms between \( \sigma_n \times H^k \) and \( \xi_{p''} = (X_p, \varphi_{p''}, V_{p''}) \).

Let \( \tilde{c}_n^{k} \) be a singular cycle relative to \( X_p \setminus \Delta_p \) and \( \tilde{c}_{n'}^{k'} \) be its \((n', k')\)-support. Let us orient each simplex of \( \tilde{c}_{n'}^{k'} \) so that two simplexes which have a common edge induce opposite orientations on this common edge. Then \( \tilde{c}_{n'}^{k'} \) is also a singular cycle relative to \( X_p \setminus \Delta_p \). Thus the relative cycle \( \tilde{c}_{n'}^{k'} \) is oriented (in two possible ways).

Obviously, two supports of a relative cycle \( \tilde{c}_n^{k} \) of the same bi-dimension are homologous to each other relative to \( X_p \setminus \Delta_p \).

**Lemma 4.4.** If \( \tilde{c}_n^{k} \) is a singular cycle relative to \( X_p \setminus \Delta_p \), then for every \( l > 0 \) its \((n + l, k + l)\)-support \( \tilde{c}_{n+l}^{k+l} \) is also a singular cycle relative to \( X_p \setminus \Delta_p \), and if an \((n + l, k + l)\)-support \( \tilde{c}_{n+l}^{k+l} \) of \( \tilde{c}_n^{k} \) is a singular cycle relative to \( X_p \setminus \Delta_p \) for some \( l > 0 \), then \( \tilde{c}_n^{k} \) is also a singular cycle relative to \( X_p \setminus \Delta_p \).

Indeed, as \( \tilde{c}_n^{k} \) is a singular cycle relative to \( X_p \setminus \Delta_p \), \( \tilde{d}_n^{k} \tilde{c}_n^{k} \in \tilde{C}_{n-1}^{k}(X_p \setminus \Delta_p) \). Because of the definition of a support of a chain and the construction of the prism, the boundary of the \((n + l, k + l)\)-support \( \tilde{c}_{n+l}^{k+l} \) also belongs to \( X_p \setminus \Delta_p \) for every \( l > 0 \). For the proof of the second statement of this lemma, it is enough to apply the construction from the definition of the support of a function in reverse order.

**Lemma 4.5.** If \( \tilde{c}_n^{k} \sim 0 \ (X_p, X_p \setminus \Delta_p) \), then \( \tilde{c}_{n+l}^{k+l} \sim 0 \ (X_p, X_p \setminus \Delta_p) \) for all \( l > 0 \), and if \( \tilde{c}_{n+l}^{k+l} \sim 0 \ (X_p, X_p \setminus \Delta_p) \) for some \( l > 0 \), then \( \tilde{c}_n^{k} \sim 0 \ (X_p, X_p \setminus \Delta_p) \).

Indeed, if \( \tilde{c}_n^{k} \sim 0 \ (X_p, X_p \setminus \Delta_p) \), then

\[
\exists \tilde{c}_{n+1}^{k} \in \tilde{C}_{n+1}^{k}(X_p), \quad \tilde{\partial}_{n+1}^{k} \tilde{c}_{n+1}^{k} = \tilde{c}_n^{k} \oplus \tilde{d}_n^{k}, \quad \tilde{d}_n^{k} \in \tilde{C}_{n}^{k}(X_p \setminus \Delta_p).
\]

In this case one can construct an \((n + l + 1, k + l)\)-support \( \tilde{c}_{n+l+1}^{k+l} \) of \( \tilde{c}_{n+1}^{k} \) such that

\[
\tilde{\partial}_{n+l+1}^{k+l} \tilde{c}_{n+l+1}^{k+l} = \tilde{c}_{n+l}^{k+l} \oplus \tilde{d}_{n+l}^{k+l}, \quad \tilde{d}_{n+l}^{k+l} \in \tilde{C}_{n+l}^{k+l}(X_p \setminus \Delta_p),
\]

where \( \tilde{c}_{n+l}^{k+l} \) and \( \tilde{d}_{n+l}^{k+l} \) are \((n + l, k + l)\)-supports of \( \tilde{c}_n^{k} \) and \( \tilde{d}_n^{k} \), respectively.

For the proof of the second statement of this lemma it is enough to apply the construction from the definition of support of a function in reverse order.

In view of Lemmas 4.4 and 4.5 we can give a new definition of homology to zero, which is equivalent to the previous one.

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\(^{(3)}\) Here and in the following, \( k' = k + l, n' = n + l \).

\(^{(4)}\) \( \tilde{c}_n^{k} \sim 0 \ (X_p, X_p \setminus \Delta_p) \) means that \( \tilde{c}_n^{k} \sim 0 \) relative to \( X_p \setminus \Delta_p \).
Definition 4.6 (equivalent to Definition 4.2). A relative cycle \( \tilde{c}^k_n \) is called homologous to zero if for some \( l > 0 \) its \((n + l, k + l)\)-support \( \tilde{c}^{k+l}_{n+l} \) is homologous to zero (in the sense of Definition 4.2).

5. Calculation of relative \( L \)-homology of an affine bundle. In this section we will assume that the base space \( V_{p_0} \) of the affine bundle \( \xi_{p_0} = (X_{p_0}, \varphi_{p_0}, V_{p_0}) \) does not have boundary, and the bounded domain \( \Delta_{p_0} \) is of the form \( X_{p_0} \cap B_1(R) \), where \( B_1(R) \) is the open ball in \( E_1 \) of radius \( R \) with center at zero \( (0) \).

Theorem 5.1. For any \( p_0 \) and \( k \geq 0 \),

\[
\tilde{H}^k_n(X_{p_0}, X_{p_0} \setminus \Delta_{p_0}) \simeq \begin{cases} 0, & n \neq p_0 + k, \\ \mathbb{Z}, & n = p_0 + k. \end{cases}
\]

The proof reduces to calculating \( \tilde{H}_n(V_{p_0,p_0+k}, V_{p_0,p_0+k} \setminus W_{p_0,p_0+k}) \) where \( W_{p_0,p_0+k} = \varphi_{p_0,p_0+k}(\Delta_{p_0}) \), \( \varphi_{p_0,p_0+k} \) is the projection of the \( k \)-division \( (X_{p_0}, \varphi_{p_0,p_0+k}, V_{p_0,p_0+k}) \) of \( (X_{p_0}, \varphi_{p_0}, V_{p_0}) \).

Before proving the theorem we state two relevant lemmas.

Let \( \tilde{c}^k_n = \sum g_i \cdot f^k_{n,i} \) be an \((n, k)\)-chain in \( \tilde{C}^k_n(X_{p_0}) \), \( \sigma^k_n = \sigma_n \times H^k \) be a Hilbertian \((n, k)\)-simplex and \( s : \sigma_n \to \sigma^k_n \) be a continuous section of \( \sigma_n \times H^k \). Let us consider the \( n \)-chain \( \tilde{c}_n = \sum g_i \cdot f_{n,i} \) in \( V_{p_0,p_0+k} \), where

\[
f_{n,i} = \varphi_{p_0,p_0+k} \circ f^k_{n,i} \circ s : \sigma_n \to V_{p_0,p_0+k}.
\]

In other words, \( \tilde{c}_n \) is the projection (by means of \( \varphi_{p_0,p_0+k} \)) of the chain \( \tilde{c}^k_n \) onto \( V_{p_0,p_0+k} \).

Lemma 5.2. \( \tilde{c}^k_n \) is a cycle relative to \( X_{p_0} \setminus \Delta_{p_0} \) if and only if \( \tilde{c}_n \) is a cycle relative to \( V_{p_0,p_0+k} \setminus W_{p_0,p_0+k} \).

Indeed, if \( \tilde{c}^k_n \) is a cycle relative to \( X_{p_0} \setminus \Delta_{p_0} \), then \( \tilde{d}^k_n \tilde{c}^k_n \in \tilde{C}^k_{n-1}(X_{p_0} \setminus \Delta_{p_0}) \). As \( \tilde{c}_n \) is the projection (by means of \( \varphi_{p_0,p_0+k} \)) of \( \tilde{c}^k_n \) onto \( V_{p_0,p_0+k} \), then \( \tilde{d}_n \tilde{c}_n \in \tilde{C}^k_{n-1}(V_{p_0,p_0+k} \setminus W_{p_0,p_0+k}) \). The converse implication is self-evident.

Lemma 5.3. \( \tilde{c}^k_n \sim 0 \) \((X_{p_0}, X_{p_0} \setminus \Delta_{p_0}) \) if and only if

\( \tilde{c}_n \sim 0 \) \((V_{p_0,p_0+k}, V_{p_0,p_0+k} \setminus W_{p_0,p_0+k}) \).

Indeed, if \( \tilde{c}^k_n \sim 0 \) \((X_{p_0}, X_{p_0} \setminus \Delta_{p_0}) \), it follows that

\[
\exists \tilde{c}^k_{n+1} \in \tilde{C}^k_{n+1}(X_{p_0}), \quad \tilde{d}^k_{n+1} \tilde{c}^k_{n+1} = \tilde{c}^k_n \oplus \tilde{d}^k_n, \quad \tilde{d}^k_n \in \tilde{C}^k_n(X_{p_0} \setminus \Delta_{p_0}).
\]

\(^{(5)}\) Recall that the affine bundle \( \xi_{p_0} \) is embedded in a Banach space \( E_1 \).
Therefore
\[ \tilde{c}_{n+1} \tilde{c}_{n+1} = \tilde{c}_n \oplus \tilde{d}_n, \quad \tilde{d}_n \in \tilde{C}_n(V_{p_0,p_0+k} \setminus W_{p_0,p_0+k}), \]
where \( \tilde{c}_{n+1}, \tilde{c}_n \) and \( \tilde{d}_n \) are the projections (by means of \( \varphi_{p_0,p_0+k} \)) of the chains \( \tilde{c}_{n+1}^k, \tilde{c}_n^k \) and \( d_n^k \) onto \( V_{p_0,p_0+k}, \) respectively. The converse implication is self-evident.

**Proof of Theorem 5.1.** Let \( \tilde{c}_n^k \in [\tilde{c}_n^k] \in \tilde{H}_n^k(X_{p_0}, X_{p_0} \setminus \Delta_{p_0}), \) and \( \tilde{c}_n \) be the projection of \( \tilde{c}_n^k \) onto \( V_{p_0,p_0+k}. \) By Lemma 5.2, \( \tilde{c}_n \) is a cycle relative to \( V_{p_0,p_0+k} \setminus W_{p_0,p_0+k}. \)

1) Let \( n \neq p_0 + k. \) Then, as is known from the theory of finite-dimensional homology,
\[ \tilde{c}_n \sim 0 (V_{p_0,p_0+k}, V_{p_0,p_0+k} \setminus W_{p_0,p_0+k}), \]
i.e. the \( n \)-dimensional singular cycle \( \tilde{c}_n \) in \( V_{p_0,p_0+k} \) is homologous to zero relative to \( V_{p_0,p_0+k} \setminus W_{p_0,p_0+k}. \) By Lemma 5.3,
\[ \tilde{c}_n^k \sim 0 (X_{p_0}, X_{p_0} \setminus \Delta_{p_0}). \]
Hence,
\[ \tilde{H}_n^k(X_{p_0}, X_{p_0} \setminus \Delta_{p_0}) \cong 0 \text{ for } n \neq p_0 + k. \]

2) Let \( n = p_0 + k. \) If \( \tilde{c}_{p_0+k} \) is a cycle relative to \( V_{p_0,p_0+k} \setminus W_{p_0,p_0+k}, \) then
\[ [\tilde{c}_{p_0+k}] \in \tilde{H}_{p_0+k}(V_{p_0,p_0+k}, V_{p_0,p_0+k} \setminus W_{p_0,p_0+k}). \]
Therefore
\[ \exists d \in \mathbb{Z}, \quad [\tilde{c}_{p_0+k}] = d \cdot [\tilde{1}_{p_0+k}], \]
where \([\tilde{1}_{p_0+k}]\) is the unit element of \( \tilde{H}_{p_0+k}(V_{p_0,p_0+k}, V_{p_0,p_0+k} \setminus W_{p_0,p_0+k}). \) By Lemma 5.3,
\[ [\tilde{c}_{p_0+k}^k] = d \cdot [\tilde{1}_{p_0+k}^k], \]
where \([\tilde{1}_{p_0+k}^k]\) is the unit element of \( \tilde{H}_{p_0+k}^k(X_{p_0}, X_{p_0} \setminus \Delta_{p_0}). \) By the above-mentioned construction, the mapping
\[ [\tilde{c}_{p_0+k}^k] \mapsto d \in \mathbb{Z} \]
is an isomorphism. Thus,
\[ \tilde{H}_{p_0+k}^k(X_{p_0}, X_{p_0} \setminus \Delta_{p_0}) \cong \mathbb{Z}. \]

**Remark.** Actually we proved that
\[ \tilde{H}_n^k(X_{p_0}, X_{p_0} \setminus \Delta_{p_0}) \cong \tilde{H}_n(V_{p_0,p_0+k}, V_{p_0,p_0+k} \setminus W_{p_0,p_0+k}) \cong \begin{cases} 0, & n \neq p_0 + k, \\ \mathbb{Z}, & n = p_0 + k. \end{cases} \]
As \( \tilde{\varphi}_p(\tilde{X}_{p_0}) = \Delta_{p_0} \) and \( \Delta_{p_0} \subset X_{p_0} \), the spaces \((\tilde{X}, \tilde{X} \setminus \tilde{X}_{p_0})\) and \((X_{p_0}, X_{p_0} \setminus \Delta_{p_0})\) are homeomorphic to each other. Therefore

\[
\tilde{H}_n^k(\tilde{X}, \tilde{X} \setminus \tilde{X}_{p_0}) \cong \begin{cases} 0, & n \neq p_0 + k, \\ \mathbb{Z}, & n = p_0 + k. \end{cases}
\]

for every integer \( k \geq 0 \) \(^{(6)}\).

6. \textit{L-homological degree of an FSQL-mapping between FSQL-manifolds}. We shall consider a simpler case for the definition of \( L \)-homological degree of FSQL-mappings between FSQL-manifolds.

We will suppose that

1) The FSQL-manifolds \( \tilde{X}, \tilde{Y} \) are embedded in the Banach spaces \( E_x, E_y \) with the norms \( \| \cdot \|_x, \| \cdot \|_y \), respectively.

2) The mappings \( \tilde{\varphi}_p, \tilde{\varphi}_p^{-1}, \tilde{\psi}_m, \tilde{\psi}_m^{-1} \) are uniformly continuous.

3) \( \tilde{f}: \tilde{X} \rightarrow \tilde{Y} \) is an FSQL-mapping which satisfies an a priori estimate

\[
(6.1) \quad \| \tilde{x} \|_x \leq \Phi(\| \tilde{f}(\tilde{x}) \|_y),
\]

where \( \Phi \) is some positive monotone function.

For simplicity, suppose that \( \Phi \) is the identity mapping. Let us consider the equation

\[
(6.2) \quad \tilde{f}(x) = y_0, \quad y_0 \in \tilde{Y}.
\]

Under condition \((6.1)\), all the solutions of \((6.2)\) belong to \( \tilde{X}_{R_0} = \tilde{X} \cap B_x(R_0) \), where \( B_x(R_0) \) is the open ball in \( E_x \) of radius \( R_0 = \| y_0 \|_y \) with center at zero. According to the definition of an FSQL-manifold,

\[
\exists p_0, \forall p \geq p_0 : \tilde{X}_{R_0} \tilde{X}_p,
\]

and according to the definition of FSQL-mappings between FSQL-manifolds,

\[
\exists m_0, \forall m \geq m_0 : \tilde{f}(\tilde{X}_p) \subset \tilde{Y}_m.
\]

Let \( p \) and \( m \) be numbers for which all the above mentioned conditions are satisfied. Then to define the degree of \( \tilde{f} \) at the point \( y_0 \in \tilde{Y} \) we can consider the restriction of \( \tilde{f} \) to \( \tilde{X}_p \). As \( \tilde{\varphi}_p \) and \( \tilde{\psi}_m \) are homeomorphisms, equation \((6.2)\) holds in \( \tilde{X}_{R_0} \) if and only if the equation

\[
(6.3) \quad f_{p,m}(u) = w_0, \quad w_0 = \tilde{\psi}_m(y_0)
\]

holds in \( \tilde{\varphi}_p(\tilde{X}_{R_0}) \), where \( f_{p,m} \equiv \tilde{\psi}_m \circ \tilde{f} \circ \tilde{\varphi}_p^{-1}: \Delta_p \rightarrow \Omega_m, \tilde{\varphi}_p(\tilde{X}_{R_0}) \subset \Delta_p. \)

According to the definition of FSQL-manifolds, \( f_{p,m} \) is an FSQL-mapping between the affine bundles \( \xi_p \) and \( \eta_m \). Let \( \{f_{p,m,r}\} \) be a sequence of FSL-mappings which is uniformly convergent to \( f_{p,m} \) on \( \Delta_p \). Let us consider the

\(^{(6)}\) Recall that \( p_0 \) is the dimension of the base space \( V_{p_0} \) of the affine bundle \( \xi_{p_0} \).
equation
\begin{equation}
(6.3) \quad f_{p,m,r}(u) = w_0.
\end{equation}
We will search for its solutions in \( \tilde{\varphi}_p(\tilde{X}_{R'_0}) \), where \( \tilde{X}_{R'_0} = \tilde{X} \cap B_x(R'_0) \), \( R'_0 = \|y_0\|_y + 2\delta, \delta > 0. \)

Remark. \( \tilde{X}_{R'_0} \subset \tilde{X}_p \) for large enough \( p \), therefore \( \tilde{\varphi}_p(\tilde{X}_{R'_0}) \subset \Delta_p. \)

Obviously, \( \tilde{f}(x) \in \tilde{Y} \setminus B_y(R_0) \) at \( x \in \tilde{X} \setminus B_x(R_0) \), where \( B_y(R_0) \) is the open ball in \( E_y \) of radius \( R_0 \) with center at zero. Therefore \( \tilde{f} \) is a mapping of pairs \( (\tilde{X}, \tilde{X} \setminus B_x(R_0)) \) and \( (\tilde{Y}, \tilde{Y} \setminus B_y(R_0)) \), and \( f_{p,m} \) is a mapping of pairs \( (\Delta_p, \Delta_p \setminus \varphi_p(\tilde{X}_{R_0})), (\Omega_m, \Omega_m \setminus \psi_m(\tilde{Y}_{R_0})) \) \((7)\).

By the definition of FS\(\mathcal{L}\)-mapping,
\[
\forall u \in \Delta_p : \|f_{p,m}(u) - f_{p,m,r}(u)\|_2 < \delta_1, \quad \delta_1 > 0.
\]
for sufficiently large \( r \). As the \( L \)-charts \( \tilde{\varphi}_p, \tilde{\varphi}_p^{-1}, \tilde{\psi}_m, \tilde{\psi}_m^{-1} \) are uniformly continuous,
\[
\forall x \in \tilde{X}_p : \|\tilde{f}(x) - \tilde{\psi}_m^{-1} \circ f_{p,m,r} \circ \tilde{\varphi}_p(x)\|_y < \delta, \quad \delta > 0.
\]
for a proper choice of \( \delta_1 \). Therefore \( f_{p,m,r} \) will be a mapping of pairs \( (\Delta_p, \Delta_p \setminus \varphi_p(\tilde{X}_{R'_0})), (\Omega_m, \Omega_m \setminus \psi_m(\tilde{Y}_{R_0 - \delta})) \) for sufficiently large \( r \), where \( \tilde{Y}_{R_0 - \delta} = \tilde{Y} \cap B_y(R_0 - \delta), B_y(R_0 - \delta) \) is the open ball in \( E_y \) of radius \( R_0 - \delta \) with center at zero.

Let \([\tilde{\omega}_{r+k}^k] \in \hat{H}^k_{r+k}(\Delta_p, \Delta_p \setminus \varphi_p(\tilde{X}_{R'_0})), [\tilde{\omega}_{r+k}^k] \in [\tilde{\omega}_{r+k}^k, \tilde{\omega}_{r+k}^k = \sum g_i \cdot f_{r+k,i}^k \) and for any \( i, f_{r+k,i}^k : \sigma_{r+k} \times H^k \to \xi_{p''}, \) where \( \xi_{p''} = (x_p, \varphi_{p''}, V_{p''}) \), and \( f_{p,m,r} : \Delta_p \to \Omega_m \) is an FS\(\mathcal{L}\)-mapping which satisfies the above mentioned conditions. One can construct an affine bundle \( \xi_{p,v}, v \geq r, \) which is a common division of \( \xi_{p,r} \) and \( \xi_{p''} \). Let us take an \((r + v, v)\)-support \( \tilde{\omega}_{r+v}^v = \sum g_i \cdot f_{r+v,i}^v \) of \( \tilde{\omega}_{r+k}^k \). Then there exists a singular chain \( [\tilde{\omega}_{r+v}^v] \) is a relative cycle. As \( f_{p,m,r} \) is a mapping of the above-mentioned pairs, \( \tilde{\omega}_{r+v}^v \) is also a relative cycle, i.e. \([\tilde{\omega}_{r+v}^v] \in \hat{H}_{r+v}^v(\Omega_m, \Omega_m \setminus \psi_m(\tilde{Y}_{R_0 - \delta})).\) Obviously, the class \([\tilde{\omega}_{r+k}^k] \) corresponds to \([\tilde{\omega}_{r+v}^v] \) under the natural isomorphism \( \hat{H}_{r+v}^v(\Delta_p, \Delta_p \setminus \varphi_p(\tilde{X}_{R'_0})) \to \hat{H}^k_{r+k}(\Delta_p, \Delta_p \setminus \varphi_p(\tilde{X}_{R'_0})), \) and the class \([\tilde{\omega}_{r+k}^k] \) corresponds to \([\tilde{\omega}_{r+v}^v] \) under the natural isomorphism
\[
\hat{H}_{r+v}^v(\Omega_m, \Omega_m \setminus \psi_m(\tilde{Y}_{R_0 - \delta})) \to \hat{H}^k_{r+k}(\Omega_m, \Omega_m \setminus \psi_m(\tilde{Y}_{R_0 - \delta})).
\]
Therefore \( f_{p,m,r} \) induces a homomorphism
\[
f_{p,m,r,*} : \hat{H}^k_{r+k}(\Delta_p, \Delta_p \setminus \varphi_p(\tilde{X}_{R'_0})) \to \hat{H}^k_{r+k}(\Omega_m, \Omega_m \setminus \psi_m(\tilde{Y}_{R_0 - \delta})).
\]

\((7)\) Recall that \( \tilde{\psi}_m(\tilde{Y}_m) = \Omega_m, \) where \( (\tilde{\psi}_m, \tilde{Y}_m) \) is the \( L \)-chart on \( \tilde{Y} \).
Let $[\tilde{c}_{r+k}^k]$ be the generator of the group $\tilde{H}^{k}_{r+k}(\Delta_p, \Delta_p \setminus \tilde{\varphi}_p(\tilde{X}_{R_0}))$ and $[\tilde{c}_{r+k}^k] = f_{p,m,r,*}[\tilde{c}_{r+k}^k]$. As $\tilde{H}^{k}_{r+k}(\Omega_m, \Omega_m \setminus \tilde{\psi}_m(\tilde{Y}_{R_0-\delta})) \cong \mathbb{Z}$, some number in $\mathbb{Z}$ corresponds to the element $[\tilde{c}_{r+k}^k]$. Let us denote that number by $\deg_H(f_{p,m,r})$.

**Definition 6.1.** The number $\deg_H(f_{p,m,r})$ is called an $L$-homological degree of the FSL-mapping $f_{p,m,r}$.

The sign of $\deg_H(f_{p,m,r})$ depends on the choice of the generators of the groups $\tilde{H}^{k}_{r+k}(\Delta_p, \Delta_p \setminus \tilde{\varphi}_p(\tilde{X}_{R_0}))$ and $\tilde{H}^{k}_{r+k}(\Omega_m, \Omega_m \setminus \tilde{\psi}_m(\tilde{Y}_{R_0-\delta}))$, but its absolute value is invariable. The latter fact is not important for the proof of the existence of a solution of equation (6.2) (see Theorem 6.6). One can prove that the degree of $f_{p,m,r}$ is well defined by Definition 6.1.

One can prove that $\{|\deg_H(f_{p,m,r})|\}$ stabilizes for sufficiently large $r$. Therefore we can give the following

**Definition 6.2.** $\deg_H(f_{p,m}) = \lim_{r \to \infty} |\deg_H(f_{p,m,r})|$.

**Definition 6.3.** $\deg_H(\tilde{f}) = \deg_H(f_{p,m})$.

As $f_{p,m} \equiv \tilde{\psi}_m \circ \tilde{f} \circ \tilde{\varphi}_p^{-1} \tilde{\psi}_m$, and $\tilde{\varphi}_p$ are homeomorphisms, the degree of $\tilde{f}$ is well defined by Definition 6.3.

**Lemma 6.4.** Let $\deg_H(f_{p,m,r}) \neq 0$. Then the equation (6.3) has a solution in $\tilde{\varphi}_p(\tilde{X}_{R_0})$.

**Proof.** As $f_{p,m,r}$ is a bimorphism, it induces some finite-dimensional continuous mapping $g_{p,m,r} : V_{p,r} \to B_{m,r}$. The commutativity of the diagram

$$
\begin{array}{ccc}
(\Delta_p, \Delta_p \setminus \tilde{\varphi}_p(\tilde{X}_{R_0})) & \xrightarrow{f_{p,m,r}} & (\Omega_m, \Omega_m \setminus \tilde{\psi}_m(\tilde{Y}_{R_0-\delta})) \\
\varphi_{p,r} \downarrow & & \downarrow \psi_{m,r} \\
(V_{p,r}, V_{p,r} \setminus \varphi_{p,r}(\tilde{X}_{R_0})) & \xrightarrow{g_{p,m,r}} & (B_{m,r}, B_{m,r} \setminus \psi_{m,r}(\tilde{Y}_{R_0-\delta}))
\end{array}
$$

yields the commutativity of

$$
\begin{array}{ccc}
\tilde{H}^0(\Delta_p, \Delta_p \setminus \tilde{\varphi}_p(\tilde{X}_{R_0})) & \xrightarrow{f_{p,m,r,*}} & \tilde{H}^0(\Omega_m, \Omega_m \setminus \tilde{\psi}_m(\tilde{Y}_{R_0-\delta})) \\
\varphi_{p,r,*} \downarrow & & \downarrow \psi_{m,r,*} \\
\tilde{H}_r(V_{p,r}, V_{p,r} \setminus \varphi_{p,r}(\tilde{X}_{R_0})) & \xrightarrow{g_{p,m,r,*}} & \tilde{H}_r(B_{m,r}, B_{m,r} \setminus \psi_{m,r}(\tilde{Y}_{R_0-\delta}))
\end{array}
$$

As $\varphi_{p,r,*}$ and $\psi_{m,r,*}$ are isomorphisms (see Theorem 5.1),

$$
\deg_H(f_{p,m,r}) = \deg_H(g_{p,m,r,*}).
$$

(8) Because of its length, the proof of this statement is given in the appendix.
Here \( \deg_H(g_{p,m,r}) \) is the homological degree of \( g_{p,m,r} \). Thus, \( \deg_H(g_{p,m,r}) \neq 0 \) as \( \deg_H(f_{p,m,r}) \neq 0 \). Then, as is known from finite-dimensional analysis,

\[ \exists \alpha_0' \in V_{p,r}, \quad g_{p,m,r}(\alpha_0') = \beta_0', \quad \beta_0' = \psi_{m,r}(w_0). \]

As \( f_{p,m,r,\alpha_0'} \) is an isomorphism between the fibers \( X_{p,\alpha_0'}(X_{p,\alpha_0'} = \varphi_{p,r}^{-1}(\alpha_0')) \) and \( Y_{m,\beta_0'}(Y_{m,\beta_0'} = \psi_{m,r}^{-1}(\beta_0')) \) of the affine bundles \( \xi_{p,r} \) and \( \eta_{m,r} \), there exists a unique point \( u_0 \in \varphi_{p,r}^{-1}(\alpha_0') \) such that

\[ f_{p,m,r}(u_0) = w_0. \]

However, in this case, it could happen that \( u_0 \notin \tilde{\varphi}_p(\tilde{X}_{R_0'}). \) Let us show that this is not the case. Obviously,

\[ \forall u \in \Delta_p : \quad \| f_{p,m}(u) - f_{p,m,r}(u) \|_2 < \delta_1, \quad \delta_1 > 0, \]

for sufficiently large \( r \). As the \( L \)-charts \( \tilde{\varphi}_p, \tilde{\varphi}_p^{-1}, \tilde{\psi}_m, \tilde{\psi}_m^{-1} \) are uniformly continuous,

\[ \forall x \in \tilde{X}_p : \quad \| \tilde{f}(x) - \tilde{\psi}_m^{-1} \circ f_{p,m,r} \circ \tilde{\varphi}_p(x) \|_y < \delta, \quad \delta > 0. \]

If \( u_0 \notin \tilde{\varphi}_p(\tilde{X}_{R_0'}) \), then \( x_0 = \tilde{\varphi}_p^{-1}(u_0) \notin X_{R_0'} \), i.e. \( \| x_0 \|_x > R_0' \). Then it follows from the estimate (6.1) that \( \| \tilde{f}(x_0) \|_y > R_0' \). As \( R_0' = R_0 + 2\delta, \quad R_0 = \| y_0 \|_y \), we have

\[ \| \tilde{\psi}_m^{-1} \circ f_{p,m,r} \circ \tilde{\varphi}_p(x_0) \|_y \geq \| \tilde{f}(x_0) \|_y - \| \tilde{f}(x_0) - \tilde{\psi}_m^{-1} \circ f_{p,m,r} \circ \tilde{\varphi}_p(x_0) \|_y \]

\[ \geq (\| y_0 \|_y + 2\delta) - \delta > \| y_0 \|_y, \]

i.e. \( \tilde{\psi}_m^{-1} \circ f_{p,m,r} \circ \tilde{\varphi}_p(x_0) \neq y_0 \), hence \( f_{p,m,r}(u_0) \neq w_0 \), which contradicts the equality (6.4). Thus \( u_0 \in \tilde{\varphi}_p(\tilde{X}_{R_0'}). \]

Using the local stability of \( |\deg_H(f_{p,m,r})| \) it is not difficult to prove the following:

**Theorem 6.5.** Let \( \{ \tilde{f}_t \} \) be a family of FSQL-mappings between \( \tilde{X} \) and \( \tilde{Y} \), which continuously depends on \( t \in [0,1] \) (uniformly in each ball) and for each \( t \in [0,1] \) an a priori estimate (6.1) is satisfied, where the function \( \Phi \) does not depend on \( t \). Then

\[ \deg_H(\tilde{f}_1) = \deg_H(\tilde{f}_0). \]

**Theorem 6.6** (9). Let \( \tilde{f} : \tilde{X} \to \tilde{Y} \) be an FSQL-mapping which satisfies an a priori estimate (6.1) and \( \deg_H(\tilde{f}) \neq 0 \). Then equation (6.2) has a solution for each \( y_0 \in \tilde{Y} \).

**Proof.** Because of Definition 6.3

\[ \deg_H(f_{p,m}) \neq 0, \]

(9) A similar theorem, for a simple case, is proved in [10].
and because of Definition [6.2]

\[ \text{deg}_H(f_{p,m,r}) \neq 0 \]

for sufficiently large \( r \). By Lemma [6.4] in this case equation (6.3) has a solution in \( \tilde{\varphi}_p(\tilde{X}_{R_0}) \). Let

\[ N_r = \{ u \in \tilde{\varphi}_p(\tilde{X}_{R_0}) \mid f_{p,m,r}(u) = w_0 \}, \quad N = \bigcup_{r \geq r_0} N_r. \]

Let us prove that \( N \) is compact. First, we shall prove that \( N_r \) is compact. For this purpose we will construct its finite \( \varepsilon \)-covering. Let \( u_0 \in N_r \) and \( B_1(u_0, \varepsilon) \) the ball in \( E_1 \) of radius \( \varepsilon \) with center at \( u_0 \). Let us consider the function

\[ P_{u_0}(\alpha') = \inf_u \{ \| f_{p,m,r,\alpha'}(u) - w_0 \|_2 \mid u \in X_{p,\alpha'} \setminus B_1(u_0, \varepsilon) \}, \]

where \( X_{p,\alpha'} \) is the fiber of the subbundle \( \xi_{p,r} = (X_p, \varphi_{p,r}, V_{p,r}) \) above \( \alpha' \in V_{p,r} \) and \( f_{p,m,r,\alpha'} \) is the restriction of \( f_{p,m,r} \) to \( X_{p,\alpha'} \). It is continuous in \( \varphi_{p,r}(\tilde{\varphi}_p(\tilde{X}_{R_0})) \). Let \( C \) be the constant from Definition [2.3] Then for \( u \in X_{p,\alpha'_0} \setminus B_1(u_0, \varepsilon) \),

\[ (6.5) \quad \| f_{p,m,r,\alpha'_0}(u) - w_0 \|_2 = \| f_{p,m,r,\alpha'_0}(u) - f_{p,m,r,\alpha'_0}(u_0) \|_2 \]

\[ \geq \frac{1}{C} \cdot \| u - u_0 \|_1 > \frac{\varepsilon}{C}. \]

As \( \| u - u_0 \|_1 > \varepsilon \) we have \( P_{u_0}(\alpha'_0) > \varepsilon/C \). Then there exists a neighborhood \( U(\alpha'_0) \) in which

\[ P_{u_0}(\alpha') > \frac{\varepsilon}{2C}. \]

Let \( u \in X_{p,\alpha'} \setminus B_1(u_0, \varepsilon) \). Then

\[ \| f_{p,m,r,\alpha'}(u) - w_0 \|_2 \]

\[ = \| f_{p,m,r,\alpha'}(u) - f_{p,m,r,\alpha'_0}(u_0) \|_2 \]

\[ = \| (f_{p,m,r,\alpha'}(u) - f_{p,m,r,\alpha'(u_0)}) + (f_{p,m,r,\alpha'}(u_0) - f_{p,m,r,\alpha'_0}(u_0)) \|_2 \]

\[ \geq \| f_{p,m,r,\alpha'}(u) - f_{p,m,r,\alpha'}(u_0) \|_2 - \| f_{p,m,r,\alpha'}(u_0) - f_{p,m,r,\alpha'_0}(u_0) \|_2 \]

\[ \geq \| f_{p,m,r,\alpha'}(u) - f_{p,m,r,\alpha'_0}(u_0) \|_2 - \| f_{p,m,r,\alpha'_0}(u_0) \|_2 \]

\[ \geq \frac{\varepsilon}{C} - \| f_{p,m,r,\alpha'} - f_{p,m,r,\alpha'_0} \| \cdot \| u_0 \|_1. \]

Let us denote the last difference by \( A \). As the family \( \{ f_{p,m,r,\alpha'} \} \) of affine mappings is uniformly continuous in \( \alpha' \),

\[ \exists \lambda > 0, \quad \| f_{p,m,r,\alpha'} - f_{p,m,r,\alpha'_0} \| < \frac{\varepsilon}{2C \cdot \max\{ \| u_\tau \|_1 \}} \quad \text{if} \quad \rho_r(\alpha', \alpha'_0) < \lambda, \]
where \( u_{\tau} \in N_r \), and \( \rho_r(\alpha', \alpha'_0) \) is a metric on \( V_{p,\tau} \). Then
\[
A > \frac{\varepsilon}{C} - \frac{\varepsilon}{2C \cdot \max\{\|u_{\tau}\|_1\}} \cdot \max\{\|u_{\tau}\|_1\} = \frac{\varepsilon}{2C} \tag{10}
\]
So, the neighborhood \( U(\alpha'_0) \) contains a ball \( W(\alpha'_0) = \{\alpha' \mid \rho_r(\alpha', \alpha'_0) < \lambda\} \) of some radius \( \lambda \), where \( \lambda \) depends only on \( \varepsilon \). Therefore there exists a finite covering of the bounded finite-dimensional set \( \varphi_{p,r}(N_r) \) by balls \( W(\alpha'_i) \): \( \varphi_{p,r}(N_r) \subset \bigcup W(\alpha'_i) \). Then the balls \( B_1(u_l, \varepsilon) \) form an \( \varepsilon \)-covering of the set \( N_r \), as for \( u \notin \bigcup B_1(u_l, \varepsilon) \), \( u \notin N_r \) because of (6.5).

Now we will prove that \( N \) is compact. Let
\[
N^\varepsilon_r = \{u \in \tilde{\varphi}_p(\tilde{X}^\alpha_{r_0}) \mid \|f_{p,m,r}(u) - w_0\|_2 < \varepsilon\}.
\]
By the definition of \( \text{FSQL}\)-mapping for each \( \varepsilon > 0 \) there exists \( \mu \) such that
\[
\|f_{p,m,r}(u) - f_{p,m}(u)\|_2 < \frac{\varepsilon}{8C} \quad \text{for} \quad r \geq \mu \quad \text{and} \quad u \in \tilde{\varphi}_p(\tilde{X}^\alpha_{r_0}).
\]
Let \( u \in N_r \), i.e. \( f_{p,m,r}(u) = w_0 \), and \( r \geq \mu \). Then taking into account (6.6) we have
\[
\|f_{p,m,\mu}(u) - w_0\|_2 \leq \|f_{p,m,\mu}(u) - f_{p,m}(u)\|_2 + \|f_{p,m}(u) - f_{p,m,r}(u)\|_2 \\
+ \|f_{p,m,r}(u) - w_0\|_2 \leq \frac{\varepsilon}{4C}.
\]
Hence \( N_r \subset N^\varepsilon_{r/4C} \) at \( r \geq \mu \). Therefore \( N \subset N_{r_0} \cup \cdots \cup N_{\mu-1} \cup N^\varepsilon_{r/4C} \). Now we shall construct a finite \( \varepsilon \)-covering for \( N \). It is already constructed for each \( N_{r_0}, \ldots, N_{\mu-1} \); therefore it is sufficient to construct a finite covering only for \( N^\varepsilon_{r/4C} \). Let \( \varphi_{p,\mu} \) be the projection of \( \xi_{p,\mu} = (X_p, \varphi_{p,\mu}, V_{p,\mu}) \), on which \( f_{p,m,\mu} \) is defined. Let us consider a ball \( B_1(u_0, \varepsilon) \), where \( u_0 \in N^\varepsilon_{r/4C} \).

The intersection of \( N^\varepsilon_{r/4C} \) with the plane \( X_{p,\alpha''_0} \), where \( \alpha''_0 = \varphi_{p,\mu}(u_0) \), is contained in \( B_1(u_0, \varepsilon/2) \). Indeed, if \( u \notin B_1(u_0, \varepsilon/2) \), then \( \|u - u_0\|_2 > \varepsilon/2 \), hence
\[
\|f_{p,m,\mu,\alpha''_0}(u) - w_0\|_2 \\
\geq \|f_{p,m,\mu,\alpha''_0}(u) - f_{p,m,\mu,\alpha''_0}(u_0)\|_2 - \|f_{p,m,\mu,\alpha''_0}(u_0) - w_0\|_2 \\
\geq \|f_{p,m,\mu,\alpha''_0}(u) - f_{p,m,\mu,\alpha''_0}(u_0)\|_2 - \|f_{p,m,\mu,\alpha''_0}(u_0) - w_0\|_2 \\
\geq \frac{1}{C} \cdot \|u - u_0\|_1 - \frac{\varepsilon}{4C} \geq \frac{1}{C} \cdot \frac{\varepsilon}{2} - \frac{\varepsilon}{4C} = \frac{\varepsilon}{4C},
\]
i.e. \( u \notin N^\varepsilon_{r/4C} \). This contradicts the assumption. From this it follows that for the continuous function
\[
P^{u_0}_{\alpha''}(\alpha'') = \inf_u\{\|f_{p,m,\mu,\alpha''}(u) - w_0\|_2 \mid u \in X_{p,\alpha''} \setminus B_1(u_0, \varepsilon)\},
\]
(10) The set \( N_r \) is bounded, therefore \( \max\{\|u_{\tau}\|_1\} < \infty \).
we have
\[ P'_{u_0}(a''_0) > \varepsilon/4C. \]
Hence, as above, from the covering \( N_u^{\varepsilon/4C} \) by balls \( B_1(u, \varepsilon) \), one can select a finite subcovering. As \( \varepsilon \) is arbitrary, it is proved that \( N \) is compact.

Now let \( \{ u_r \} \subset \tilde{\varphi}_p(\tilde{X}_R_0^p) \) be some sequence of solutions of \( (6.3) \). As \( \{ u_r \} \subset N \), there exists a subsequence converging to some \( u_0 \in N \). As \( \{ f_{p,m,r} \} \) uniformly converges to \( f_{p,m} \) in \( \tilde{\varphi}_p(\tilde{X}_R_0^p) \), \( f_{p,m}(u_0) = u_0 \). Therefore,
\[ \tilde{f}(x_0) = y_0, \text{ where } x_0 = \tilde{\varphi}_p^{-1}(u_0), \text{ i.e. } x_0 \text{ is a solution of equation } (6.2). \]

7. Appendix. The proof of stabilization of \( \{ \deg_H(f_{p,m,r}) \} \). First we recall that \( \eta_m \) is embedded in the Banach space \( E_2 \). Let \( f_{p,m,r'} : \xi_{p,r'} \to \eta_{m,r'} \) and \( f_{p,m,r''} : \xi_{p,r''} \to \eta_{m,r''} \) be two FSL-mappings which are close enough to each other in \( \Delta_p \subset X_p \). Without restriction of generality one can suppose that \( f_{p,m,r'} : \xi_{p,\nu} \to \eta_{m,\nu,1} \) and \( f_{p,m,r''} : \xi_{p,\nu} \to \eta_{m,\nu,2} \) are isomorphisms between the aforesaid bundles, where \( \xi_{p,\nu} = (X_p, \varphi_{p,\nu}, V_{p,\nu}) \), \( \nu \geq r', r'' \), is a common division of \( \xi_{p,r'}, \xi_{p,r''} \) and \( \eta_{m,\nu,1}, \eta_{m,\nu,2} \) are divisions of \( \eta_m \) of the same codimension \( \nu \). Let us introduce the following notations:

We denote the mappings \( f_{p,m,r'} \) and \( f_{p,m,r''} \) by \( f_{p,m,1} \) and \( f_{p,m,2} \), respectively. We denote the fibers of \( \xi_{p,\nu} = (X_p, \varphi_{p,\nu}, V_{p,\nu}) \) by \( X_{p,\alpha} \):
\[ X_{p,\alpha} = \varphi_{p,\nu}^{-1}(\alpha), \quad \alpha \in V_{p,\nu}. \]
We denote the fibers of \( \eta_m = (Y_m, \psi_m, B_m) \) by \( Y_{m,\beta} \):
\[ Y_{m,\beta} = \psi_m^{-1}(\beta), \quad \beta \in B_m. \]
We denote the fibers of \( \eta_{m,\nu,1} = (Y_m, \psi_{m,\nu,1}, B_{m,\nu,1}) \) by \( Y_{m,\nu,1}\beta_1 \):
\[ Y_{m,\nu,1}\beta_1 = \psi_{m,\nu,1}^{-1}(\beta_1), \quad \beta_1 \in B_{m,\nu,1}. \]
We denote the fibers of \( \eta_{m,\nu,2} = (Y_m, \psi_{m,\nu,2}, B_{m,\nu,2}) \) by \( Y_{m,\nu,2}\beta_2 \):
\[ Y_{m,\nu,2}\beta_2 = \psi_{m,\nu,2}^{-1}(\beta_2), \quad \beta_2 \in B_{m,\nu,2}. \]
Finally
\[ f_{p,m,1}(X_{p,\alpha}) = Y_{m,\nu,1}\beta_1, \quad f_{p,m,2}(X_{p,\alpha}) = Y_{m,\nu,2}\beta_2. \]
As \( f_{p,m,1} \) and \( f_{p,m,2} \) are close to each other in \( \Delta_p \), the fibers \( Y_{m,\nu,1}\beta_1 \) and \( Y_{m,\nu,2}\beta_2 \) are also close to each other for any \( \alpha \in V_{p,\nu} \), i.e.
\[ \text{dist}(Y_{m,\nu,1}\beta_1, Y_{m,\nu,2}\beta_2) = \sup\{ \rho(w, Y'_{m,\nu,1}\beta_1, Y'_{m,\nu,2}\beta_2) \mid w \in Y'_{m,\nu,2}\beta_2 \cap B_2(1) \} < \varepsilon, \quad \varepsilon > 0 \text{ (11)} \]
(11) Here \( Y'_{m,\nu,1}\beta_1, Y'_{m,\nu,2}\beta_2 \) are the subspaces of \( E_2 \) which are parallel translates of \( Y_{m,\nu,1}\beta_1, Y_{m,\nu,2}\beta_2 \) respectively through the origin of \( E_2 \), \( B_2(1) \) is the ball of radius one in \( E_2 \) with center at zero, and \( \rho(w, Y'_{m,\nu,1}\beta_1) \) is the distance between \( w \) and \( Y'_{m,\nu,1}\beta_1 \).
Therefore $Y_{m,\nu,\beta_2(\alpha),2}$ is close to $Y_{m,\beta(\alpha)}$, which contains $Y_{m,\nu,\beta_1(\alpha),1}$. Then it is possible to take the orthogonal projection of each fiber $Y_{m,\nu,\beta_2(\alpha),2}$ onto $Y_{m,\beta(\alpha)}$. Let us denote this projection by $\pi_{\beta(\alpha)}$, $\alpha \in V_{p,\nu}$. By construction:

1) $\pi_{\beta(\alpha)}$ is an affine isomorphism between $Y_{m,\nu,\beta_2(\alpha),2}$ and its image.
2) $\pi = \{\pi_{\beta(\alpha)} \mid \alpha \in V_{p,\nu}\}$ is an isomorphism between $\{Y_{m,\nu,\beta_2(\alpha),2}\}$ and its image.
3) $f_{p,m,\nu,3} = \pi \circ f_{p,m,\nu,2}$ is an FSL-mapping.
4) The mappings $f_{p,m,\nu,3}$ and $f_{p,m,\nu,2}$ are close to each other in $\Delta_p$, hence $f_{p,m,\nu,3}$ is close to $f_{p,m,\nu,1}$ in $\Delta_p$.

**Remark.** The difference between the mappings $f_{p,m,\nu,3}$ and $f_{p,m,\nu,2}$ is that $f_{p,m,\nu,1}(X_{p,\alpha})$ and $f_{p,m,\nu,3}(X_{p,\alpha})$ are contained in the same $Y_{m,\beta(\alpha)}$ for each $\alpha \in V_{p,\nu}$.

As all the mappings obeying $f_{p,m,\nu,3} = \pi \circ f_{p,m,\nu,2}$ are FSL-mappings and $\pi$ is an isomorphism,

$$\deg_H f_{p,m,\nu,3} = \deg_H (\pi \circ f_{p,m,\nu,2}) = (\deg_H \pi) \cdot (\deg_H f_{p,m,\nu,2}) = \deg_H f_{p,m,\nu,2}.$$

Now let us prove that

$$\deg_H f_{p,m,\nu,3} = \deg_H f_{p,m,\nu,1}.$$

For this purpose we take an $(\nu + 1, k)$-prism $\sigma_{\nu+1} = \sigma_{\nu} \times I_1$, where $I_1$ is a 1-cube, that is, a line segment. We will consider the singular $(\nu + 1, k)$-chain

$$\tilde{\sigma}_{\nu+1}^k = \sum g_i \cdot [t \cdot f_{p,m,\nu,1} \circ f_{\nu,i}^k(u) + (1 - t) \circ f_{p,m,\nu,3} \circ f_{\nu,i}^k(u)].$$

One can show that $\tilde{\sigma}_{\nu+1}^k = \tilde{\sigma}_{\nu}^k$, i.e. the relative cycles $\sigma_{\nu,1}^k = \sum g_i \cdot f_{p,m,\nu,1} \circ f_{\nu,i}^k$ and $\tilde{\sigma}_{\nu,3}^k = \sum g_i \cdot f_{p,m,\nu,3} \circ f_{\nu,i}^k$ are homologous to each other relative to $X_p \setminus \Delta_p$. Hence

$$\deg_H f_{p,m,\nu,3} = \deg_H f_{p,m,\nu,1}.$$

Therefore

$$\deg_H f_{p,m,\nu,2} = \deg_H f_{p,m,\nu,1}.$$

Thus,

$$\deg_H f_{p,m,r'} = \deg_H f_{p,m,r''}$$

for sufficiently large $r'$ and $r''$.

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$^{(12)}$ Because of the note mentioned above, $\tilde{\sigma}_{\nu+1}^k$ is a chain in $X_p$. 
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