Generalized Cesàro operators on certain function spaces

by SUNANDA NAIK (Bhubaneswar)

Abstract. Motivated by some recent results by Li and Stević, in this paper we prove that a two-parameter family of Cesàro averaging operators $P_{b,c}$ is bounded on the Dirichlet spaces $D_{p,a}$. We also give a short and direct proof of boundedness of $P_{b,c}$ on the Hardy space $H^p$ for $1 < p < \infty$.

1. Introduction and preliminaries. Let $\Delta$ be the unit disc in the complex plane $\mathbb{C}$ and $dm(z) = r\,dr\,d\theta/\pi$, the normalized Lebesgue area measure on $\Delta$. Let $\mathcal{H}$ be the space of all analytic functions in $\Delta$. Recall that for $p > 0$, the Hardy space $H^p$ consists of $f \in \mathcal{H}$ such that

$$
\|f\|_p = \lim_{r \to 1} M_p(r, f) < \infty,
$$

where the integral mean $M_p(r, f)$ is defined by

$$
M_p(r, f) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \,d\theta \right\}^{1/p}, \quad 0 \leq r < 1.
$$

For $a, p > 0$, $D_{p,a}$ denotes the space of all $f \in \mathcal{H}$ such that

$$
\|f\|_{D_{p,a}}^p = |f(0)|^p + \int_{\Delta} |f'(z)|^p (1 - |z|)^a \,dm(z) < \infty.
$$

Let $\omega(r)$, $0 \leq r < 1$, be a positive weight function which is integrable on $[0, 1)$. We extend $\omega$ on $\Delta$ by setting $\omega(z) = \omega(|z|)$.

For $0 < p < \infty$, the weighted Bergman space $B^p_\omega$ consists of $f \in \mathcal{H}$ such that

$$
\|f\|_{B^p_\omega}^p = \int_{\Delta} |f(z)|^p \omega(z) \,dm(z) < \infty.
$$

Using the definition of integral mean, the above norm in $B^p_\omega$ can be written

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as 

\[ \|f\|_{L^p,\omega}^2 = 2 \int_0^1 M_p(r, f) \omega(r) r \, dr. \]

Note that \( B^p_\omega \) is a Banach space when \( 1 \leq p < \infty \) and Hilbert space for \( p=2 \).

For any complex numbers \( a, b, c \neq -n, n = 0, 1, 2, \ldots \), the Gaussian hypergeometric function \([AAR] [1]\) is defined by power series expansion

\[ _2F_1(a, b; c; z) := F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)} \frac{z^n}{n!} \quad (|z| < 1), \]

where \( (a, n) \) is the shifted factorial defined by Appel’s symbol

\[ (a, n) := a(a+1)\ldots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad n \in \mathbb{N} = \{1, 2, \ldots\}, \]

and \( (a, 0) = 1 \) for \( a \neq 0 \).

For \( f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{H} \), the Cesàro operators of type \((1, b; c)\) or simply the generalized Cesàro operators are defined as

\[ \mathcal{P}^{b,c} f(z) := \sum_{n=0}^{\infty} \left( \frac{1}{A_n^{b+1;c}} \sum_{k=0}^{n} b_{n-k} a_k \right) z^n, \]

where

\[ A_n^{a,b;c} = \frac{(a, n)(b, n)}{(c, n)(1, n)}, \quad A_n^{b,c} = \frac{(b, n)}{(c, n)} \]

and \( b_k \) is given by \( b_0 = 1 \), and for \( k \geq 1 \),

\[ b_k = \frac{1 + b - c}{c} A_k^{b+1;c+1} = \frac{1 + b - c}{b} A_k^{b,c}. \]

These operators were introduced in \([AHLNP]\) and have been studied to prove the boundedness on Hardy spaces, BMOA and Bloch space. For \( b = \gamma + 1 \) and \( c = 1 \), we obtain the Cesàro operators of order \( \gamma \), or simply the \( \gamma \)-Cesàro operators \( \mathcal{P}^{1+\gamma,1} f = C^\gamma f \) (\( \text{Re} \gamma > -1 \)). In particular, for \( \gamma = 0 \), we obtain the classical Cesàro operator \( \mathcal{P}^{1,1} f = Cf \). Many authors, for example \([M],[Si1],[Si2]\), have studied the boundedness of \( \mathcal{C} \) on \( H^p, 0 < p < \infty \), and the same problem for the Bergman space has been studied by \([Si3]\). Also in \([G]\) the boundedness of \( \mathcal{C} \) on the Dirichlet space \( \mathcal{D}_{p,a} \) has been proved when \( p = 2 \) and \( a > 0 \). For any \( \gamma \in \mathbb{C} \) with \( \text{Re} \gamma > -1 \), the operators \( C^\gamma \) were introduced in \([Si]\) and proved to be bounded on Hardy space. Subsequently the boundedness of \( C^\gamma \) has been studied by Xia \([X]\) on \( H^p \) spaces, BMOA and Bloch space, and Stević \([S1]\) proved their boundedness on Dirichlet space. In this article we generalize Stević’s results by proving the boundedness of the operator \( \mathcal{P}^{b,c} \) on Dirichlet space \( \mathcal{D}_{p,a} \) for \( p > 1 \), and using this result we prove its boundedness on weighted Bergman space \( B^p_\omega \). For some extension in the case of the unit polydisk \( \Delta^n \) we refer to \([CS1],[CS2],[CLS]\) and \([S2]\).
In [AHLNP] the authors have proved that the operator $P^{b,c}$ is bounded on when $0 < p \leq 1$, and for $1 < p < \infty$ the problem remained open. Recently Li [L] gave a partial solution by proving the boundedness of $P^{b,c}$ on $H^p$ for $p > 1$ when $\text{Re}(b+1) > \text{Re}c \geq 1$. In this article we produce a different proof of that result.

Given a weight $\omega$ we define the function

$$\psi(r) = \frac{1}{\omega(r)} \int_r^1 \omega(u) \, du \quad \text{for } 0 \leq r < 1,$$

and we call it the distortion function of $\omega$. We put $\psi(z) = \psi(|z|)$ for each $z \in \Delta$.

1.1. DEFINITION. A weight $\omega$ is admissible if it satisfies the following conditions:

(i) There is a positive constant $A = A(\omega)$ such that

$$\omega(r) \geq A \frac{1}{1-r} \int_r^1 \omega(u) \, du \quad \text{for } 0 \leq r < 1.$$

(ii) There is a positive constant $B = B(\omega)$ such that

$$\omega'(r) \leq B \frac{1}{1-r} \omega(r) \quad \text{for } 0 \leq r < 1.$$

(iii) For each sufficiently small positive $\delta$ there is a positive constant $C = C(\delta, \omega)$ such that

$$\sup_{0 \leq r < 1} \frac{\omega(r)}{\omega(r + \delta \psi(r))} \leq C.$$

For details on admissible weights, see [Si4]. The following theorem was proved in [Si4].

1.2. THEOREM. Suppose $1 \leq p < \infty$ and $\omega$ is an admissible weight with distortion function $\psi$. Then

$$\int_{\Delta} |f(z)|^p \omega(z) \, dm(z) \sim |f(0)|^p \int_{\Delta} |f'(z)|^p \psi(z)^p \omega(z) \, dm(z)$$

for all $f \in \mathcal{H}$.

The notation $\sim$ means that there are finite positive constants $C$ and $C'$ independent of $f$ (but possibly dependent on $p$) such that the left and right sides $L(f)$ and $R(f)$ satisfy

$$CR(f) \leq L(f) \leq C'R(f)$$

for all analytic $f$.

1.3. EXAMPLE. A straightforward computation shows that the standard weight $\omega(r) = (1-r)^\alpha$, $\alpha > -1$, for $r \in (0, 1)$ is admissible and its distortion function satisfies $\psi(r) \sim 1 - r$. 
Henceforth $C$ denotes a positive constant whose value is different at different occurrences. The constant may depend on the parameters $a, p$ and we write in that case $C(a, p)$.

2. Boundedness of generalized Cesàro operators. In this section, we discuss the boundedness of generalized Cesàro operators on Dirichlet spaces $D_{p,a}$ and Hardy spaces $H^p$.

$P^{b,c}$ has an equivalent integral representation (see [AHLNP]) which is given in the following lemma.

2.4. Lemma. For $b, c \in \mathbb{C}$ with $\Re(b + 1) > \Re c > 0$, we have

$$P^{b,c}f(z) = \frac{z^{-b}}{B} \int_0^1 \phi_t(z)^{c-1}(z - \phi_t(z))^{b-c} \frac{f(\phi_t(z))}{(1 - \phi_t(z))^{b+1-c}} F(\phi_t(z)) \phi'_t(z) dt,$$

where $F(\zeta) = F(c - 1, c - b - 1; c; \zeta)$ and $B = B(c, b + 1 - c)$ is the usual beta function.

For each $t \in [0, 1]$, we choose the path of integration between 0 and $z$ as

$$\Gamma(t) = \phi_t(z) = \frac{tz}{1 + (t - 1)z}.$$

Using this path in (2.5), we have

$$P^{b,c}f(z) = \frac{1}{B} \int_0^1 \frac{t^{c-1}(1 - t)^{b-c}}{(1 + (t - 1)z)^c} f(\phi_t(z)) F(\phi_t(z)) dt.$$

Define

$$T_t f(z) = \omega_t(z)f(\phi_t(z))F(\phi_t(z)) \quad \text{for} \ t \in (0, 1],$$

where $\omega_t(z) = t/(1 + (t - 1)z)$ and $F(z) = F(c - 1, c - b - 1; c; z)$. Then

$$P^{b,c}f(z) = \frac{1}{B} \int_0^1 \frac{1}{t} T_t(f(z))(1 - t)^{b-c} dt.$$

Now we recall the following result from [S1].

2.8. Lemma. Let $f \in D_{p,a}$, $p > 0$, $a > p - 1$. Then there is a constant $C = C(a, p)$ such that

$$|f(z)| \leq \frac{C}{(1 - |z|)^{(a+2)/p-1}} \|f\|_{D_{p,a}}.$$
We will make use of this lemma to prove our next result which is given in the following theorem.

2.9. Theorem. Let \( b, c \in \mathbb{C} \) be such that \( \Re(b+1) > \Re c > 0 \). Suppose \( a > p - 1 \) and \( f \in D_{p,a} \). If \( p > 1 \) and \( \Re c \geq 1 \), then there are constants \( C = C(a,p) \) and \( \beta > 0 \) such that

\[
\|T_i\|_{D_{p,a}} \leq Ct^\beta \|f\|_{D_{p,a}}.
\]

For \( 0 < \Re c < 1 \), the above inequality is true if \( p \geq 2 \).

Proof. For the sake of simplicity, assume \( c \) to be real and positive, since the proof for \( c \) complex can be easily modified. Suppose \( f \in D_{p,a} \). Then using the definition of \( T_i \) we have

\[
\|T_i\|^p_{D_{p,a}} = |T_i(f(0))|^p + \int_\Delta |(\omega_t^c(z)f(\phi_t(z))F(\phi_t(z)))'|^p (1 - |z|)^a \, dm(z)
\]

\[
\leq t^{cp} |f(0)|^p + c_p (I_1 + I_2 + I_3),
\]

where

\[
I_1 = \int_\Delta |(\omega_t^c(z))'|^p |f(\phi_t(z))F(\phi_t(z))|^p (1 - |z|)^a \, dm(z),
\]

\[
I_2 = \int_\Delta |\omega_t^c(z)|^p |f(\phi_t(z))'|^p |F(\phi_t(z))|^p (1 - |z|)^a \, dm(z),
\]

\[
I_3 = \int_\Delta |\omega_t^c(z)|^p |f(\phi_t(z))|^p |F(\phi_t(z))'|^p (1 - |z|)^a \, dm(z).
\]

We have

\[
|(\omega_t^c(z))'| = \frac{ct^c(1-t)}{1 + (t-1)z|c+1|} \quad \text{and} \quad \frac{1}{1 - |\phi_t(z)|} \leq \frac{|1 + (t-1)z|}{1 - |z|}.
\]

The boundedness of \( F(\phi_t(z)) = F(c-1, c-b-1; c; \phi_t(z)) \) follows from \( \Re(b+1) > \Re(c-1) \), on \( |z| \leq 1 \). Since \( F(\phi_t(z)) = F(c-1, c-b-1; c; \phi_t(z)) \) is bounded, Lemma 2.8 and the above calculation shows that

\[
I_1 \leq C \|f\|^p_{D_{p,a}} \int_\Delta |(\omega_t^c(z))'|^p \frac{(1 - |z|)^a}{(1 - |\phi_t(z)|)^{a+2-p}} \, dm(z)
\]

\[
\leq C \|f\|^p_{D_{p,a}} c^p t^{cp} (1-t)^p \int_\Delta \frac{1}{|1 + (t-1)z|^{(c+1)p}} \frac{(1 - |z|)^a}{(1 - |\phi_t(z)|)^{a+2-p}} \, dm(z)
\]

\[
\leq C \|f\|^p_{D_{p,a}} t^{cp} (1-t)^p \int_\Delta \frac{|1 + (t-1)z|^{a+2-(c+2)p}}{(1 - |z|)^{2-p}} \, dm(z).
\]

For any \( b > 0 \), \( t \in [0,1] \) and \( z \in \Delta \), we have

\[
\frac{1}{|1 + (t-1)z|^b} \leq \frac{1}{(1 - |z|)^b} \quad \text{and} \quad t^b \leq |1 + (t-1)z|^b.
\]
Choose $\epsilon > 0$ such that
\[
\epsilon < \begin{cases} 
\min\{1, p - 1, (a - p + 1)/2\} & \text{if } c \geq 1, \\
\min\{1, p - 1, (a - p + 1)/2, cp\} & \text{if } 0 < c < 1.
\end{cases}
\]

By (2.11), we obtain
\[
|1 + (t - 1)z|^{a + 2 - (c + 2)p} = \frac{|1 + (t - 1)z|^{a - p + 1 - 2\epsilon}}{|1 + (t - 1)z|^{cp - \epsilon}|1 + (t - 1)z|^{p - 1 - \epsilon}} \leq \frac{1}{(1 - |z|)^{1 - \epsilon t c p - \epsilon}}.
\]

Using (2.12) in (2.10) it is easy to see that
\[
I_1 \leq C \|f\|_{D_p,a}^p (1 - t)^p t^\epsilon.
\]

Now for $I_2$ we have
\[
I_2 = \int_{\Delta} \left| \omega_c^\epsilon(z) \right|^p |f'(\phi_t(z))|^p |(\phi_t(z))'|^p |F(\phi_t(z))|^p (1 - |z|)^a \, dm(z)
\]
\[
= \int_{\Delta} \left| \omega_c^\epsilon(z) \right|^p |f'(\phi_t(z))|^p |(\phi_t(z))'|^p |F(\phi_t(z))|^p \left( \frac{1 - |z|}{1 - |\phi_t(z)|} \right)^a 
\]
\[
\times (1 - |\phi_t(z)|)^a \, dm(\phi_t(z)).
\]

Choose $\epsilon_1 > 0$ such that
\[
\epsilon_1 < \begin{cases} 
\min\{a - p + 2, 2p - 2\} & \text{if } c \geq 1, \\
\min\{a - p + 2, (c + 1)p - 2\} & \text{if } 0 < c < 1.
\end{cases}
\]

One can quickly obtain the following:
\[
|\omega_c^\epsilon(z)|^p |(\phi_t(z))'|^p |F(\phi_t(z))|^p \frac{(1 - |z|)^a}{(1 - |\phi_t(z)|)} \leq t^{(c+1)p-2} \frac{|1 + (t - 1)z|^{a-p+2-\epsilon_1}}{|1 + (t - 1)z|^{(c+1)p-2-\epsilon_1}} \leq t^{\epsilon_1} \quad \text{(using (2.11))},
\]

therefore we have
\[
I_2 \leq C \|f\|_{D_p,a}^p t^{\epsilon_1}.
\]

Further we have
\[
F'(\phi_t(z)) = \frac{(c - 1)(c - b - 1)}{c} F(c, c - b; c + 1; \phi_t(z))(\phi_t(z))',
\]
and since $F(c, c - b; c + 1; \phi_t(z))$ is bounded for $\text{Re}(b + 1) > \text{Re} c$ on $|z| \leq 1,$
we have
\[ I_3 \leq C \int_\Delta |\omega_t^c(z)|^p |f(\phi_t(z))|^p (\phi_t(z))'|^p (1 - |z|)^a \, dm(z) \]
\[ = C \int_\Delta \frac{t^{(c+1)p}}{|1 + (t - 1)z|^{(c+2)p}} |f(\phi_t(z))|^p (1 - |z|)^a \, dm(z) \]
\[ \leq C\|f\|_D^p \int_\Delta \frac{1}{|1 + (t - 1)z|^{(c+2)p}} \frac{(1 - |z|)^a}{(1 - |\phi_t(z)|)^{a+2-p}} \, dm(z) \]
\[ \leq C\|f\|_D^p \int_\Delta \frac{|1 + (t - 1)z|^{a+2-(c+3)p}}{(1 - |z|)^{2-p}} \, dm(z) \]
\[ \leq C\|f\|_D^p \int_\Delta \frac{1}{(1 - |z|)^{1-\epsilon t(c+1)p-\epsilon}} \, dm(z) \] (using (2.11));

by choosing the same \( \epsilon \) as in \( I_1 \), the last inequality shows
\[ I_3 < C\|f\|_D^p t^\epsilon. \]

Finally, combining all the above results for \( I_1, I_2 \) and \( I_3 \), we obtain
\[ \|T_t\|_{D_{p,a}}^p \leq t^{cp} |(f(0))|^p + c_p C ((t - 1)^p t^\epsilon \|f\|_D^p + t^{\epsilon_1} \|f\|_D^p + t^\epsilon \|f\|_D^p) \]
\[ \leq C t^{cp} \|f\|_D^p + c_p C \|f\|_D^p (t^\epsilon + t^{\epsilon_1} + t^\epsilon) \] (Lemma 2.8)
\[ \leq C t^\beta \|f\|_D^p \]
for \( \beta = \min\{\epsilon, \epsilon_1\} > 0 \) and \( C = \max(C, c_p C) \), completing the proof. \( \blacksquare \)

Now we are in a position to prove our next result which concerns the boundedness of generalized Cesàro operators on Dirichlet spaces.

2.13. THEOREM. Let \( b, c \in \mathbb{C} \) be such that \( \text{Re}(b+1) > \text{Re} c > 0 \). Suppose \( a > p - 1 \). Then the generalized Cesàro operator \( \mathcal{P}^{b,c} \) is bounded on \( D_{p,a} \) when \( p > 1 \) and \( \text{Re} c \geq 1 \). Further, \( \mathcal{P}^{b,c} \) is bounded on \( D_{p,a} \) for \( 0 < \text{Re} c < 1 \) if \( p \geq 2 \).

Proof. We give the proof only for \( b, c \) real with \( b + 1 > c > 0 \). For the proof of the complex case, we just need to note the following, for \( t \in (0,1) \):
\[ |t^{c-1}| = t^{\text{Re}c-1}, \quad |(1 - t)^{b-c}| = (1 - t)^{\text{Re}(b-c)} \]
and
\[ |(1 - z t)^{b+1-c}| = |1 - tz|^{\text{Re}(b+1-c)} e^{-\text{Im}(b+1-c) \arg(1-tz)}. \]

Here we choose the principal argument for \( \arg(1-tz) \) such that \( \arg(1-tz) = 0 \) at \( z = 0 \), and we note that \( |\arg(1-tz)| < \pi/2 \) for \( z \in \Delta \). Moreover, the integral \( \int_0^1 t^{c-1}(1-t)^{b-c} \, dt \) converges since by the hypotheses \( \text{Re}(b+1-c) > 0 \)
and \( \text{Re} \, c > 0 \), and therefore it suffices to assume \( b \) and \( c \) are real, and that \( b + 1 > c > 0 \) in the proof.

To prove the theorem it is sufficient to show that

\[
\| P_{b,c} f \|_{D_p,a}^p \leq C \| f \|_{D_p,a}^p
\]

for some \( C > 0 \), depending on \( b, c, p \) and \( a \). Using the integral representation given by (2.7), we have

\[
\| P_{b,c} f \|_{D_p,a}^p = \left\| \frac{1}{B} \int_0^1 t^{-1} (1-t)^{b-c} T'_t(f(z)) \, dt \right\| (1-|z|)^a dm(z)
\]

\[
\leq \left\| \frac{1}{B} \int_0^1 \left( \int_0^1 t^{-1} (1-t)^{b-c} |T'_t(f(z))| \, dt \right) \right\| (1-|z|)^a dm(z)
\]

\[
\leq |f(0)|^p + \frac{1}{B^p} \int_0^1 \left( \int_0^1 |T'_t(f(z))| (1-|z|)^a dm(z) \right)^{1/p} \left(1-t \right)^{b-c} dt
\]

(Minkowski inequality)

\[
\leq |f(0)|^p + \left( \frac{1}{B^p} \int_0^1 t^{-1} (1-t)^{b-c} \| T' \|_{D_p,a} \, dt \right)^p
\]

\[
\leq |f(0)|^p + \frac{C^p}{B^p} \| f \|_{D_p,a}^p \left( \int_0^1 t^{\beta-1} (1-t)^{b-c} dt \right)^p
\]

(Theorem 2.9)

\[
\leq C \| f \|_{D_p,a}^p + \frac{C^p}{B^p} B^p(\beta, b - c + 1) \| f \|_{D_p,a}^p
\]

(Lemma 2.8)

\[
\leq C \| f \|_{D_p,a}^p,
\]

where \( C = \max(C, (C^p/B^p)B^p(\beta, b - c + 1)) \), which completes the proof.

2.14. Theorem. Let \( b, c \in \mathbb{C} \) be such that \( \text{Re}(b+1) > \text{Re} \, c > 0 \). Then for \( \alpha > -1 \), the generalized Cesàro operator \( P_{b,c} \) is bounded on \( B_{(1-|z|^2)^\alpha}^p \) when \( p > 1 \) and \( \text{Re} \, c \geq 1 \), and also for \( 0 < \text{Re} \, c < 1 \) when \( p \geq 2 \).

Proof. Suppose \( f \in B_{(1-|z|^2)^\alpha}^p \) and \( p > 1 \). Since \( P_{b,c} f(0) = f(0) \), using the previous theorem we have

\[
\int_\Delta |(P_{b,c})' f(z)|^p (1-|z|)^a \, dm(z) \leq C \int_\Delta |f'(z)|^p (1-|z|)^a \, dm(z).
\]

Let \( \omega(r) = (1-r)^\alpha \), \( \alpha > -1 \). If we take \( a = \alpha + p \), as \( \psi(r) \sim 1-r \) is the distortion function for \( \omega(r) \), the theorem follows from the above inequality and Theorem 1.2. \( \blacksquare \)
We state the boundedness of the operators $P^{b,c}$ on Hardy spaces. We recall the following result from [GS] which we will use to prove our next theorem.

2.15. **Lemma.** For the Hardy space norms of the weighted composition operators

$$U_t f(z) = \frac{1}{1 + (t-1)z} f\left(\frac{tz}{1 + (t-1)z}\right)$$

we have:

(i) If $2 \leq p < \infty$ then

$$\|U_t\| \leq t^{1+1/p}, \quad 0 < t \leq 1.$$  

(ii) If $1 < p < 2$ then there is a constant $C_p$ depending only on $p$ such that

$$\|U_t\| \leq C_p t^{1+1/p}, \quad 0 < t \leq 1.$$  

2.16. **Theorem.** Let $b,c \in \mathbb{C}$ be such that $\text{Re}(b+1) > \text{Re}c \geq 1$. The generalized Cesàro operator $P^{b,c}$ is bounded on $H^p$ for $1 < p < \infty$.

**Proof.** Let $f \in H^p$, $1 < p < \infty$. Our aim is to show that

$$M_p(r, P^{b,c} f) \leq C\|f\|_p$$

for some $C > 0$. Suppose $b,c \in \mathbb{C}$ with $\text{Re}(b+1) > \text{Re}c > 0$. It is sufficient to prove the assertion for real $b$ and $c$, because of the same reasoning as in the previous theorem. For each $t \in (0,1]$, the function $\phi_t$ given by

$$\phi_t(z) = \frac{tz}{1 + (t-1)z}$$

maps the disc into itself. Then (see [D] p. 29]) $f(\phi_t(z)) \in H^p$ for $z \in \Delta$. Since $F(c-1, c-b-1, c; z)$ is bounded for $b + 1 > c - 1$, we see that $F(\phi_t(z)) = F(c-1, c-b-1, c; \phi_t(z))$ is bounded on $|z| \leq 1$ and therefore $F(\phi_t(z)) f(\phi_t(z))$ is bounded on $H^p$ for $1 < p < \infty$.

We can easily obtain (see for example [GS] p. 4]) that for $c > 0$, the weight functions $\varpi_t(z) = 1/(1+(t-1)z)^c$ are bounded on $\Delta$ for each $t \in (0,1]$. Thus, for each $t \in (0,1]$ the weighted composition operators $S_t$ defined by

$$S_t f(z) = \varpi_t(z) f(\phi_t(z)) F(\phi_t(z))$$

are bounded on $H^p$. Since $c \geq 1$, $F(c-1, c-b-1, c; z)$ is bounded for
\[ b + 1 > c - 1 \] and using the second inequality of (2.11), we find

\[
\| S_t(f) \|_p = \lim_{r \to 1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |S_t(f(re^{i\theta}))|^p \, d\theta \right\}^{1/p}
\]

\[
= \lim_{r \to 1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| \left| \phi_t(re^{i\theta}) \right| \right|_{p} |U_t(f(re^{i\theta}))|^p \, d\theta \right\}^{1/p}
\]

\[
\leq C t^{1-c} \| U_t(f) \|_p.
\]

Using Lemma 2.15 in the above inequality, we have

\[
\| S_t \|_p \leq \begin{cases} 
  t^{-c+1/p} & \text{if } 2 \leq p < \infty, \\
  CP_t^{-c+1/p} & \text{if } 1 < p < 2.
\end{cases}
\]

Now \( \mathcal{P}^{b,c} \) defined in (2.6) can be written in the following form:

\[
\mathcal{P}^{b,c} f(z) = \frac{1}{B} \int_0^1 S_t f(z) t^{c-1} (1-t)^{b-c} \, dt.
\]

Using Minkowski’s inequality and the boundedness of \( S_t \) on \( H^p \), we have

\[
M_p(r, \mathcal{P}^{b,c} f) \leq \frac{1}{B} \left( \frac{1}{2\pi} \int_0^{2\pi} \left( \int_0^1 |S_t(f(re^{i\theta}))| t^{c-1} (1-t)^{b-c} \, dt \right)^p \, d\theta \right)^{1/p}
\]

\[
\leq \frac{1}{B} \left( \frac{1}{2\pi} \int_0^{2\pi} |S_t(f(re^{i\theta}))|^{p} \, d\theta \right)^{1/p} t^{c-1} (1-t)^{b-c} \, dt
\]

\[
\leq \frac{C}{B} \| f \|_p \int_0^1 t^{1/p-1} (1-t)^{b-c} \, dt = C \| f \|_p,
\]

which completes the proof. \( \nabla \)

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**References**


Sunanda Naik
Institute of Mathematics and Applications
Andharua
Bhubaneswar 751 003, India
E-mail: spn20@yahoo.com

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