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Generalized Cesàro operators on certain function spaces

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Abstract. Motivated by some recent results by Li and Stević, in this paper we prove that a two-parameter family of Cesàro averaging operators $\mathcal{P}^{b,c}$ is bounded on the Dirichlet spaces $\mathcal{D}_{p,a}$. We also give a short and direct proof of boundedness of $\mathcal{P}^{b,c}$ on the Hardy space H^p for $1 < p < \infty$.

1. Introduction and preliminaries. Let Δ be the unit disc in the complex plane $\mathbb{C}$ and $dm(z) = r\,dr\,d\theta/\pi$, the normalized Lebesgue area measure on Δ. Let $\mathcal{H}$ be the space of all analytic functions in Δ. Recall that for $p > 0$, the Hardy space H^p consists of $f \in \mathcal{H}$ such that

$$\|f\|_p = \lim_{r\to 1} M_p(r,f) < \infty,$$

where the integral mean $M_p(r,f)$ is defined by

$$M_p(r,f) = \left\{\frac{1}{2\pi}\int_0^{2\pi} |f(re^{\theta})|^p\,d\theta\right\}^{1/p}, \quad 0 \le r < 1.$$

For $a, p > 0$, $\mathcal{D}_{p,a}$ denotes the space of all $f \in \mathcal{H}$ such that

$$\|f\|_{\mathcal{D}_{p,a}}^p = |f(0)|^p + \int_{\Delta} |f'(z)|^p (1-|z|)^a\,dm(z) < \infty.$$

Let $\omega(r)$, $0 \le r < 1$, be a positive weight function which is integrable on $[0,1)$. We extend ω on Δ by setting $\omega(z) = \omega(|z|)$.

For $0 < p < \infty$, the weighted Bergman space $\mathcal{B}_{\omega}^p$ consists of $f \in \mathcal{H}$ such that

$$\|f\|_{\omega,p}^p = \int_{\Delta} |f(z)|^p \omega(z)\,dm(z) < \infty.$$

Using the definition of integral mean, the above norm in B_{ω}^p can be written

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as

$$\|f\|_{\omega,p}^{p} = 2\int_{0}^{1} M_p^p(r,f)\omega(r)r\,dr.$$

Note that B_ω^p is a Banach space when $1 \le p < \infty$ and Hilbert space for $p=2$.

For any complex numbers $a, b, c \neq -n$, $n = 0, 1, 2, \ldots$, the Gaussian hypergeometric function [AAR, T] is defined by power series expansion

$$ {}_2F_1(a,b;c;z) := F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)} \frac{z^n}{n!} \qquad (|z| < 1),$$

where (a,n) is the shifted factorial defined by Appel's symbol

$$(a,n) := a(a+1)\ldots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad n \in \mathbb{N} = \{1, 2, \ldots\},$$

and $(a,0) = 1$ for $a \neq 0$.

For $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{H}$, the Cesàro operators of type $(1, b; c)$ or simply the *generalized Cesàro operators* are defined as

$$\mathcal{P}^{b,c} f(z) := \sum_{n=0}^{\infty} \left(\frac{1}{A_n^{b+1;c}} \sum_{k=0}^{n} b_{n-k} a_k \right) z^n,$$

where

$$A_k^{a,b;c} = \frac{(a,k)(b,k)}{(c,k)(1,k)}, \qquad A_k^{b;c} = \frac{(b,k)}{(c,k)}$$

and b_k is given by $b_0 = 1$, and for $k \geq 1$,

$$b_k = \frac{1+b-c}{c} A_{k-1}^{b+1;c+1} = \frac{1+b-c}{b} A_k^{b;c}.$$

These operators were introduced in [AHLNP] and have been studied to prove the boundedness on Hardy spaces, BMOA and Bloch space. For $b = \gamma + 1$ and $c = 1$, we obtain the Cesàro operators of order γ, or simply the γ-Cesàro operators $\mathcal{P}^{1+\gamma,1} f = \mathcal{C}^\gamma f$ ($\operatorname{Re}\gamma > -1$). In particular, for $\gamma = 0$, we obtain the classical Cesàro operator $\mathcal{P}^{1,1} f = \mathcal{C} f$. Many authors, for example [M], [Si1] and [Si2], have studied the boundedness of $\mathcal{C}$ on H^p, $0 < p < \infty$, and the same problem for the Bergman space has been studied by [Si3]. Also in [G] the boundedness of $\mathcal{C}$ on the Dirichlet space $\mathcal{D}_{p,a}$ has been proved when $p = 2$ and $a > 0$. For any $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > -1$, the operators $\mathcal{C}^\gamma$ were introduced in [St] and proved to be bounded on Hardy space. Subsequently the boundedness of $\mathcal{C}^\gamma$ has been studied by Xia [X] on H^p spaces, BMOA and Bloch space, and Stević [S1] proved their boundedness on Dirichlet space. In this article we generalize Stević's results by proving the boundedness of the operator $\mathcal{P}^{b,c}$ on Dirichlet space $\mathcal{D}_{p,a}$ for $p > 1$, and using this result we prove its boundedness on weighted Bergman space B_ω^p. For some extension in the case of the unit polydisk Δ^n we refer to [CS1], [CS2], [CLS] and [S2].

In [AHLNP] the authors have proved that the operator $\mathcal{P}^{b,c}$ is bounded on when $0 < p \le 1$, and for $1 < p < \infty$ the problem remained open. Recently Li [L] gave a partial solution by proving the boundedness of $\mathcal{P}^{b,c}$ on H^p for $p > 1$ when $\mathrm{Re}(b+1) > \mathrm{Re}\, c \ge 1$. In this article we produce a different proof of that result.

Given a weight ω we define the function

$$\psi(r) = \frac{1}{\omega(r)} \int_r^1 \omega(u)\, du \quad \text{for } 0 \le r < 1,$$

and we call it the *distortion function* of ω. We put $\psi(z) = \psi(|z|)$ for each $z \in \Delta$.

1.1. Definition. A weight ω is *admissible* if it satisfies the following conditions:

(i) There is a positive constant $A = A(\omega)$ such that

$$\omega(r) \ge \frac{A}{1-r} \int_r^1 \omega(u)\, du \quad \text{for } 0 \le r < 1.$$

(ii) There is a positive constant $B = B(\omega)$ such that

$$\omega'(r) \le \frac{B}{1-r}\, \omega(r) \quad \text{for } 0 \le r < 1.$$

(iii) For each sufficiently small positive δ there is a positive constant $C = C(\delta, \omega)$ such that

$$\sup_{0 \le r < 1} \frac{\omega(r)}{\omega(r + \delta\psi(r))} \le C.$$

For details on admissible weights, see [Si4]. The following theorem was proved in [Si4].

1.2. Theorem. *Suppose $1 \le p < \infty$ and ω is an admissible weight with distortion function ψ. Then*

$$\int_\Delta |f(z)|^p \omega(z)\, dm(z) \sim |f(0)|^p + \int_\Delta |f'(z)|^p \psi(z)^p \omega(z)\, dm(z)$$

for all $f \in \mathcal{H}$.

The notation $\sim$ means that there are finite positive constants C and C' independent of f (but possibly dependent on p) such that the left and right sides $L(f)$ and $R(f)$ satisfy

$$CR(f) \le L(f) \le C'R(f)$$

for all analytic f.

1.3. Example. A straightforward computation shows that the standard weight $\omega(r) = (1-r)^\alpha$, $\alpha > -1$, for $r \in (0,1)$ is admissible and its distortion function satisfies $\psi(r) \sim 1 - r$.

Henceforth C denotes a positive constant whose value is different at different occurrences. The constant may depend on the parameters a, p and we write in that case $C(a,p)$.

2. Boundedness of generalized Cesàro operators. In this section, we discuss the boundedness of generalized Cesàro operators on Dirichlet spaces $\mathcal{D}_{p,a}$ and Hardy spaces H^p.

$\mathcal{P}^{b,c}$ has an equivalent integral representation (see [AHLNP]) which is given in the following lemma.

2.4. Lemma. *For $b, c \in \mathbb{C}$ with $\operatorname{Re}(b+1) > \operatorname{Re} c > 0$, we have*

$$\mathcal{P}^{b,c} f(z) = \frac{z^{-b}}{B} \int_0^z \zeta^{c-1} (z-\zeta)^{b-c} \frac{f(\zeta)}{(1-\zeta)^{b+1-c}} F(\zeta)\, d\zeta, \tag{2.5}$$

where $F(\zeta) = F(c-1, c-b-1; c; \zeta)$ and $B = B(c, b+1-c)$ is the usual beta function.

For each $t \in [0,1]$, we choose the path of integration between 0 and z as

$$\Gamma(t) = \phi_t(z) = \frac{tz}{1+(t-1)z}.$$

Using this path in (2.5), we have

$$\begin{aligned}\mathcal{P}^{b,c} f(z) &\\ &= \frac{z^{-b}}{B} \int_0^1 \phi_t(z)^{c-1} (z-\phi_t(z))^{b-c} \frac{f(\phi_t(z))}{(1-\phi_t(z))^{b+1-c}} F(\phi_t(z)) \phi_t'(z)\, dt \\ &= \frac{1}{B} \int_0^1 \frac{t^{c-1}(1-t)^{b-c}}{(1+(t-1)z)^c} f(\phi_t(z)) F(\phi_t(z))\, dt.\end{aligned} \tag{2.6}$$

Define

$$T_t f(z) = \omega_t^c(z) f(\phi_t(z)) F(\phi_t(z)) \quad \text{for } t \in (0,1],$$

where $\omega_t(z) = t/(1+(t-1)z)$ and $F(z) = F(c-1, c-b-1; c; z)$. Then

$$\mathcal{P}^{b,c} f(z) = \frac{1}{B} \int_0^1 \frac{1}{t} T_t(f(z))(1-t)^{b-c}\, dt. \tag{2.7}$$

Now we recall the following result from [S1].

2.8. Lemma. *Let $f \in \mathcal{D}_{p,a}$, $p > 0$, $a > p-1$. Then there is a constant $C = C(a,p)$ such that*

$$|f(z)| \le \frac{C}{(1-|z|)^{(a+2)/p-1}} \|f\|_{\mathcal{D}_{p,a}}.$$

We will make use of this lemma to prove our next result which is given in the following theorem.

2.9. THEOREM. *Let $b, c \in \mathbb{C}$ be such that* $\operatorname{Re}(b+1) > \operatorname{Re} c > 0$. *Suppose $a > p-1$ and $f \in \mathcal{D}_{p,a}$. If $p > 1$ and $\operatorname{Re} c \ge 1$, then there are constants $C = C(a,p)$ and $\beta > 0$ such that*

$$\|T_t\|_{\mathcal{D}_{p,a}} \le Ct^{\beta}\|f\|_{\mathcal{D}_{p,a}}.$$

For $0 < \operatorname{Re} c < 1$, the above inequality is true if $p \ge 2$.

Proof. For the sake of simplicity, assume c to be real and positive, since the proof for c complex can be easily modified. Suppose $f \in \mathcal{D}_{p,a}$. Then using the definition of T_t we have

$$\begin{aligned}\|T_t\|^p_{\mathcal{D}_{p,a}} &= |T_t(f(0))|^p + \int_{\Delta} |(\omega_t^c(z) f(\phi_t(z)) F(\phi_t(z)))'|^p (1-|z|)^a \, dm(z) \\ &\le t^{cp}|f(0)|^p + c_p(I_1 + I_2 + I_3),\end{aligned}$$

where

$$\begin{aligned}I_1 &= \int_{\Delta} |(\omega_t^c(z))'|^p |f(\phi_t(z)) F(\phi_t(z))|^p (1-|z|)^a \, dm(z), \\ I_2 &= \int_{\Delta} |\omega_t^c(z)|^p |f(\phi_t(z))'|^p |F(\phi_t(z))|^p (1-|z|)^a \, dm(z), \\ I_3 &= \int_{\Delta} |\omega_t^c(z)|^p |f(\phi_t(z))|^p |F(\phi_t(z))'|^p (1-|z|)^a \, dm(z).\end{aligned}$$

We have

$$|(\omega_t^c(z))'| = \frac{ct^c(1-t)}{|1+(t-1)z|^{c+1}} \quad \text{and} \quad \frac{1}{1-|\phi_t(z)|} \le \frac{|1+(t-1)z|}{1-|z|}.$$

The boundedness of $F(\phi_t(z)) = F(c-1, c-b-1; c; \phi_t(z))$ follows from $\operatorname{Re}(b+1) > \operatorname{Re}(c-1)$, on $|z| \le 1$. Since $F(\phi_t(z)) = F(c-1, c-b-1; c; \phi_t(z))$ is bounded, Lemma 2.8 and the above calculation shows that

$$\begin{aligned}I_1 &\le C\|f\|^p_{\mathcal{D}_{p,a}} \int_{\Delta} |(\omega_t^c(z))'|^p \frac{(1-|z|)^a}{(1-|\phi_t(z)|)^{a+2-p}} \, dm(z) \qquad (2.10) \\ &\le C\|f\|^p_{\mathcal{D}_{p,a}} c^p t^{cp} (1-t)^p \int_{\Delta} \frac{1}{|1+(t-1)z|^{(c+1)p}} \frac{(1-|z|)^a}{(1-|\phi_t(z)|)^{a+2-p}} \, dm(z) \\ &\le C\|f\|^p_{\mathcal{D}_{p,a}} t^{cp} (1-t)^p \int_{\Delta} \frac{|1+(t-1)z|^{a+2-(c+2)p}}{(1-|z|)^{2-p}} \, dm(z).\end{aligned}$$

For any $b > 0$, $t \in [0,1]$ and $z \in \Delta$, we have

$$\frac{1}{|1+(t-1)z|^b} \le \frac{1}{(1-|z|)^b} \quad \text{and} \quad t^b \le |1+(t-1)z|^b. \qquad (2.11)$$

Choose $\epsilon > 0$ such that

$$\epsilon < \begin{cases} \min\{1, p-1, (a-p+1)/2\} & \text{if } c \geq 1, \\ \min\{1, p-1, (a-p+1)/2, cp\} & \text{if } 0 < c < 1. \end{cases}$$

By (2.11), we obtain

$$\begin{aligned} (2.12) \qquad & \frac{|1+(t-1)z|^{a+2-(c+2)p}}{(1-|z|)^{2-p}} \\ &= \frac{|1+(t-1)z|^{a-p+1-2\epsilon}}{|1+(t-1)z|^{cp-\epsilon}|1+(t-1)z|^{p-1-\epsilon}} \frac{(1-|z|)^{p-1-\epsilon}}{(1-|z|)^{1-\epsilon}} \\ &\leq \frac{1}{(1-|z|)^{1-\epsilon} t^{cp-\epsilon}}. \end{aligned}$$

Using (2.12) in (2.10) it is easy to see that

$$I_1 \leq C\|f\|^p_{\mathcal{D}_{p,a}} (1-t)^p t^{\epsilon}.$$

Now for I_2 we have

$$\begin{aligned} I_2 &= \int_{\Delta} |\omega_t^c(z)|^p |f'(\phi_t(z))|^p |(\phi_t(z))'|^p |F(\phi_t(z))|^p (1-|z|)^a \, dm(z) \\ &= \int_{\Delta} |\omega_t^c(z)|^p |f'(\phi_t(z))|^p |(\phi_t(z))'|^{p-2} |F(\phi_t(z))|^p \left(\frac{1-|z|}{1-|\phi_t(z)|}\right)^a \\ &\quad \times (1-|\phi_t(z)|)^a \, dm(\phi_t(z)). \end{aligned}$$

Choose $\epsilon_1 > 0$ such that

$$\epsilon_1 < \begin{cases} \min\{a-p+2, 2p-2\} & \text{if } c \geq 1, \\ \min\{a-p+2, (c+1)p-2\} & \text{if } 0 < c < 1. \end{cases}$$

One can quickly obtain the following:

$$\begin{aligned} |\omega_t^c(z)|^p |(\phi_t(z))'|^{p-2} \left(\frac{1-|z|}{1-|\phi_t(z)|}\right)^a &\leq t^{(c+1)p-2} \frac{|1+(t-1)z|^{a-p+2-\epsilon_1}}{|1+(t-1)z|^{(c+1)p-2-\epsilon_1}} \\ &\leq t^{\epsilon_1} \qquad \text{(using (2.11))}, \end{aligned}$$

therefore we have

$$I_2 \leq C\|f\|^p_{\mathcal{D}_{p,a}} t^{\epsilon_1}.$$

Further we have

$$F'(\phi_t(z)) = \frac{(c-1)(c-b-1)}{c} F(c, c-b; c+1; \phi_t(z))(\phi_t(z))',$$

and since $F(c, c-b; c+1; \phi_t(z))$ is bounded for $\operatorname{Re}(b+1) > \operatorname{Re} c$ on $|z| \leq 1$, we have

$$\begin{aligned}
I_3 &\le C \int_{\Delta} |\omega_t^c(z)|^p |f(\phi_t(z))|^p |(\phi_t(z))'|^p (1-|z|)^a \, dm(z) \\
&= C \int_{\Delta} \frac{t^{(c+1)p}}{|1+(t-1)z|^{(c+2)p}} |f(\phi_t(z))|^p (1-|z|)^a \, dm(z) \\
&\le C\|f\|^p_{\mathcal{D}_{p,a}} t^{(c+1)p} \int_{\Delta} \frac{1}{|1+(t-1)z|^{(c+2)p}} \frac{(1-|z|)^a}{(1-|\phi_t(z)|)^{a+2-p}} \, dm(z) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(Lemma 2.8)} \\
&\le C\|f\|^p_{\mathcal{D}_{p,a}} t^{(c+1)p} \int_{\Delta} \frac{|1+(t-1)z|^{a+2-(c+3)p}}{(1-|z|)^{2-p}} \, dm(z) \\
&\le C\|f\|^p_{\mathcal{D}_{p,a}} t^{(c+1)p} \int_{\Delta} \frac{1}{(1-|z|)^{1-\epsilon} t^{(c+1)p-\epsilon}} \, dm(z) \quad \text{(using (2.11))};
\end{aligned}$$

by choosing the same ϵ as in I_1, the last inequality shows

$$I_3 < C\|f\|^p_{\mathcal{D}_{p,a}} t^{\epsilon}.$$

Finally, combining all the above results for I_1, I_2 and I_3, we obtain

$$\begin{aligned}
\|T_t\|^p_{\mathcal{D}_{p,a}} &\le t^{cp}|(f(0))|^p + c_p C\big((1-t)^p t^{\epsilon}\|f\|^p_{\mathcal{D}_{p,a}} + t^{\epsilon_1}\|f\|^p_{\mathcal{D}_{p,a}} + t^{\epsilon}\|f\|^p_{\mathcal{D}_{p,a}}\big) \\
&\le Ct^{cp}\|f\|^p_{\mathcal{D}_{p,a}} + c_p C\|f\|^p_{\mathcal{D}_{p,a}}(t^{\epsilon} + t^{\epsilon_1} + t^{\epsilon}) \quad \text{(Lemma 2.8)} \\
&\le Ct^{\beta}\|f\|^p_{\mathcal{D}_{p,a}}
\end{aligned}$$

for $\beta = \min\{\epsilon, \epsilon_1\} > 0$ and $C = \max(C, c_p C)$, completing the proof. ∎

Now we are in a position to prove our next result which concerns the boundedness of generalized Cesàro operators on Dirichlet spaces.

2.13. Theorem. *Let* $b, c \in \mathbb{C}$ *be such that* $\operatorname{Re}(b+1) > \operatorname{Re} c > 0$. *Suppose* $a > p-1$. *Then the generalized Cesàro operator* $\mathcal{P}^{b,c}$ *is bounded on* $\mathcal{D}_{p,a}$ *when* $p > 1$ *and* $\operatorname{Re} c \ge 1$. *Further,* $\mathcal{P}^{b,c}$ *is bounded on* $\mathcal{D}_{p,a}$ *for* $0 < \operatorname{Re} c < 1$ *if* $p \ge 2$.

Proof. We give the proof only for b, c real with $b+1 > c > 0$. For the proof of the complex case, we just nead to note the following, for $t \in (0,1)$:

$$|t^{c-1}| = t^{\operatorname{Re} c - 1}, \qquad |(1-t)^{b-c}| = (1-t)^{\operatorname{Re}(b-c)}$$

and

$$|(1-zt)^{b+1-c}| = |1-tz|^{\operatorname{Re}(b+1-c)} e^{-\operatorname{Im}(b+1-c)\arg(1-tz)}.$$

Here we choose the principal argument for $\arg(1-tz)$ such that $\arg(1-tz) = 0$ at $z = 0$, and we note that $|\arg(1-tz)| < \pi/2$ for $z \in \Delta$. Moreover, the integral $\int_0^1 t^{c-1}(1-t)^{b-c}\, dt$ converges since by the hypotheses $\operatorname{Re}(b+1-c) > 0$

and $\operatorname{Re} c > 0$, and therefore it suffices to assume b and c are real, and that $b+1 > c > 0$ in the proof.

To prove the theorem it is sufficient to show that

$$\|\mathcal{P}^{b,c} f\|_{\mathcal{D}_{p,a}}^p \le C \|f\|_{\mathcal{D}_{p,a}}^p$$

for some $C > 0$, depending on b, c, p and a. Using the integral representation given by (2.7), we have

$$\begin{aligned}
&\|\mathcal{P}^{b,c} f\|_{\mathcal{D}_{p,a}}^p \\
&\quad = |\mathcal{P}^{b,c} f(0)|^p + \frac{1}{B} \int_{\Delta} \Big| \int_0^1 t^{-1}(1-t)^{b-c} T_t'(f(z))\, dt \Big|^p (1-|z|)^a \, dm(z) \\
&\quad \le |\mathcal{P}^{b,c} f(0)|^p + \frac{1}{B} \int_{\Delta} \Big(\int_0^1 t^{-1}(1-t)^{b-c} |T_t'(f(z))|\, dt \Big)^p (1-|z|)^a \, dm(z) \\
&\quad \le |f(0)|^p + \frac{1}{B^p} \int_0^1 \Big(\frac{1}{t} \Big(\int_{\Delta} |T_t'(f(z))|^p (1-|z|)^a \, dm(z) \Big)^{1/p} (1-t)^{b-c}\, dt \Big)^p \\
&\qquad\qquad \text{(Minkowski inequality)} \\
&\quad \le |f(0)|^p + \Big(\frac{1}{B^p} \int_0^1 t^{-1}(1-t)^{b-c} \|T_t\|_{\mathcal{D}_{p,a}} \, dt \Big)^p \\
&\quad \le |f(0)|^p + \frac{C^p}{B^p} \|f\|_{\mathcal{D}_{p,a}}^p \Big(\int_0^1 t^{\beta-1}(1-t)^{b-c}\, dt \Big)^p \quad \text{(Theorem 2.9)} \\
&\quad \le C\|f\|_{\mathcal{D}_{p,a}}^p + \frac{C^p}{B^p} B^p(\beta, b-c+1) \|f\|_{\mathcal{D}_{p,a}}^p \quad \text{(Lemma 2.8)} \\
&\quad \le C\|f\|_{\mathcal{D}_{p,a}}^p,
\end{aligned}$$

where $C = \max(C, (C^p/B^p)B^p(\beta, b-c+1))$, which completes the proof. ∎

2.14. THEOREM. *Let $b, c \in \mathbb{C}$ be such that* $\operatorname{Re}(b+1) > \operatorname{Re} c > 0$. *Then for $\alpha > -1$, the generalized Cesàro operator $\mathcal{P}^{b,c}$ is bounded on $\mathcal{B}^p_{(1-|z|^2)^\alpha}$ when $p > 1$ and* $\operatorname{Re} c \ge 1$, *and also for* $0 < \operatorname{Re} c < 1$ *when $p \ge 2$.*

Proof. Suppose $f \in \mathcal{B}^p_{(1-|z|^2)^\alpha}$ and $p > 1$. Since $\mathcal{P}^{b,c} f(0) = f(0)$, using the previous theorem we have

$$\int_{\Delta} |(\mathcal{P}^{b,c})' f(z)|^p (1-|z|)^a \, dm(z) \le C \int_{\Delta} |f'(z)|^p (1-|z|)^a \, dm(z).$$

Let $\omega(r) = (1-r)^\alpha$, $\alpha > -1$. If we take $a = \alpha + p$, as $\psi(r) \sim 1-r$ is the distortion function for $\omega(r)$, the theorem follows from the above inequality and Theorem 1.2. ∎

We state the boundedness of the operators $\mathcal{P}^{b,c}$ on Hardy spaces. We recall the following result from [GS] which we will use to prove our next theorem.

2.15. LEMMA. *For the Hardy space norms of the weighted composition operators*

$$U_t f(z) = \frac{1}{1+(t-1)z}\, f\left(\frac{tz}{1+(t-1)z}\right)$$

we have:

(i) *If* $2 \le p < \infty$ *then*

$$\|U_t\| \le t^{-1+1/p}, \qquad 0 < t \le 1.$$

(ii) *If* $1 < p < 2$ *then there is a constant* C_p *depending only on* p *such that*

$$\|U_t\| \le C_p t^{-1+1/p}, \qquad 0 < t \le 1.$$

2.16. THEOREM. *Let* $b, c \in \mathbb{C}$ *be such that* $\operatorname{Re}(b+1) > \operatorname{Re} c \ge 1$. *The generalized Cesàro operator* $\mathcal{P}^{b,c}$ *is bounded on* H^p *for* $1 < p < \infty$.

Proof. Let $f \in H^p$, $1 < p < \infty$. Our aim is to show that

$$M_p(r, \mathcal{P}^{b,c} f) \le C\|f\|_p$$

for some $C > 0$. Suppose $b, c \in \mathbb{C}$ with $\operatorname{Re}(b+1) > \operatorname{Re} c > 0$. It is sufficient to prove the assertion for real b and c, because of the same reasoning as in the previous theorem. For each $t \in (0,1]$, the function ϕ_t given by

$$\phi_t(z) = \frac{tz}{1+(t-1)z}$$

maps the disc into itself. Then (see [D, p. 29]) $f(\phi_t(z)) \in H^p$ for $z \in \Delta$. Since $F(c-1, c-b-1, c; z)$ is bounded for $b+1 > c-1$, we see that $F(\phi_t(z)) = F(c-1, c-b-1, c; \phi_t(z))$ is bounded on $|z| \le 1$ and therefore $F(\phi_t(z)) f(\phi_t(z))$ is bounded on H^p for $1 < p < \infty$.

We can easily obtain (see for example [GS, p. 4]) that for $c > 0$, the weight functions $\varpi_t(z) = 1/(1+(t-1)z)^c$ are bounded on Δ for each $t \in (0,1]$. Thus, for each $t \in (0,1]$ the weighted composition operators S_t defined by

$$S_t f(z) = \varpi_t(z) f(\phi_t(z)) F(\phi_t(z))$$

are bounded on H^p. Since $c \ge 1$, $F(c-1, c-b-1, c; z)$ is bounded for

$b+1>c-1$ and using the second inequality of (2.11), we find

$$\begin{aligned}\|S_t(f)\|_p &= \lim_{r\to 1}\bigg\{\frac{1}{2\pi}\int_0^{2\pi}|S_t(f(re^{i\theta}))|^p\,d\theta\bigg\}^{1/p}\\ &= \lim_{r\to 1}\bigg\{\frac{1}{2\pi}\int_0^{2\pi}\bigg|\frac{1}{1+(t-1)z}\bigg|^{(c-1)p}|F(\phi_t(re^{i\theta}))|^p|U_tf(re^{i\theta})|^p\,d\theta\bigg\}^{1/p}\\ &\le Ct^{1-c}\|U_t(f)\|_p.\end{aligned}$$

Using Lemma 2.15 in the above inequality, we have

$$\|S_t\|_p \le \begin{cases} t^{-c+1/p} & \text{if } 2\le p<\infty,\\ CC_pt^{-c+1/p} & \text{if } 1<p<2.\end{cases}$$

Now $\mathcal{P}^{b,c}$ defined in (2.6) can be written in the following form:

$$\mathcal{P}^{b,c}f(z) = \frac{1}{B}\int_0^1 S_tf(z)t^{c-1}(1-t)^{b-c}\,dt.$$

Using Minkowski's inequality and the boundedness of S_t on H^p, we have

$$\begin{aligned}M_p(r,\mathcal{P}^{b,c}f) &\le \frac{1}{B}\bigg(\frac{1}{2\pi}\int_0^{2\pi}\bigg(\int_0^1|S_t(f(re^{i\theta}))|t^{c-1}(1-t)^{b-c}\,dt\bigg)^p\,d\theta\bigg)^{1/p}\\ &\le \frac{1}{B}\int_0^1\bigg(\frac{1}{2\pi}\int_0^{2\pi}|S_t(f(re^{i\theta}))|^p\,d\theta\bigg)^{1/p}t^{c-1}(1-t)^{b-c}\,dt\\ &\le \frac{C}{B}\|f\|_p\int_0^1 t^{1/p-1}(1-t)^{b-c}\,dt = C\|f\|_p,\end{aligned}$$

which completes the proof. ■

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