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# Addendum to the paper "Decomposition into special cubes and its application to quasi-subanalytic geometry"

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In [7], we demonstrate how to achieve the model completeness and o-minimality of the real field with restricted quasianalytic functions (a result due to Rolin–Speissegger–Wilkie [13]) by means of a technique of decomposition into special cubes; see [8–11] for other applications of this method. Therein we asked, inter alia, whether, given a polynomially bounded o-minimal expansion $\mathcal{R}$ of the real field, the structure generated by global smooth $\mathcal{R}$-definable functions is model complete. We should note that this follows immediately from Wilkie's complement theorem [14] (see also [12, 6]). In this Addendum, we also wish to indicate that Gabrielov's proof [5] of the complement theorem can be adapted to the real field with restricted smooth $\mathcal{R}$-definable functions.

Gabrielov's approach relies on certain three preliminary lemmas. Below we state their quasianalytic versions, whose proofs can be repeated mutatis mutandis. Next, we shall outline our proof of the complement theorem based on those lemmas. Denote by $Q_n$ the algebra of those $\mathcal{R}$-definable functions that are smooth in the vicinity of the closed cube $[0,1]^n$. The algebras $Q_n$ give rise to the notions of Q-analytic, Q-semianalytic and Q-subanalytic subsets of the cubes $[0,1]^n$, $n \in \mathbb{N}$.

Lemma 1. *Consider a Q-semianalytic subset $E$ of $[0,1]^n$ of the form*

$$E := \{x \in [0,1]^n : f_1(x) = \ldots = f_k(x) = 0,\, g_1(x) > 0,\, \ldots,\, g_l(x) > 0\}$$

*with $f_i, g_j \in Q_n$. Then the closure $\overline{E}$ and frontier $\partial E$ are Q-semianalytic too.*

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*Moreover, $\overline{E}$ and $\partial E$ can be described by functions which are polynomials in $x$, in the functions $f_i$, $g_j$, and in their (finitely many) partial derivatives.*

*Consequently, if $F$ is a Q-subanalytic subset of $[0,1]^m$, then so are its closure $\overline{F}$ and frontier $\partial F$.*

Remark. As an easy generalization, one can formulate a parametric version of the above lemma, in which the $\mathcal{R}$-definable functions involved in the description depend smoothly on parameters.

By a *Q-leaf* we mean a set of the form

$$L := \{x \in [0,1]^n : f_1(x) = \ldots = f_k(x) = 0,\, g_1(x) > 0,\, \ldots,\, g_l(x) > 0\},$$

where $f_i, g_j \in Q_n$ and

$$\frac{\partial(f_1, \ldots, f_k)}{\partial(x_{i_1}, \ldots, x_{i_k})}(x) \neq 0 \quad \text{for some } 1 \leq i_1 < \cdots < i_k \leq n \text{ and for all } x \in L.$$

Lemma 2. *Every Q-semianalytic subset $E$ of $[0,1]^n$ is a finite union of Q-leaves.*

The image of a Q-leaf $L \subset [0,1]^n$ under a projection $\pi : \mathbb{R}^n \to \mathbb{R}^m$, $n \geq m$, will be called an *immersed Q-leaf* if the restriction of $\pi$ to $L$ is an immersion. By combining Lemma 2 with the technique of fiber cutting (see e.g. [4, 5, 2, 3, 1, 7]), one can obtain

Lemma 3. *Every Q-subanalytic subset $F$ of $[0,1]^m$ is a finite union of immersed Q-leaves.*

By a *Q-cell* we mean a cell given by smooth functions with Q-subanalytic graphs. Now we can readily outline our proof of the following main result wherefrom the complement theorem follows immediately.

Main Theorem. *Consider Q-subanalytic subsets $F_1, \ldots, F_r$ of $[0,1]^m$. Then there exists a Q-cell decomposition $\mathcal{C}$ of $[0,1]^m$ which is compatible with the sets $F_i$, $i = 1, \ldots, r$.*

We proceed by a double induction with respect to $m$ and

$$d := \max\{\dim F_1, \ldots, \dim F_r\}.$$

The case $m = 0$ is trivial, and so take $m > 0$. Again, the case $d = 0$ is evident, and we may suppose $d > 0$. By virtue of Lemma 3, we can assume that $F_i$ are immersed Q-leaves, i.e.

$$F_i = p(E_i), \quad p : \mathbb{R}^n \to \mathbb{R}^m, \quad p(x_1, \ldots, x_m) = (x_1, \ldots, x_n),$$

for all $i = 1, \ldots, r$. Denote by $q : \mathbb{R}^n \to \mathbb{R}^{m-1}$ and $\pi : \mathbb{R}^m \to \mathbb{R}^{m-1}$ the canonical projections onto the first $m-1$ coordinates; obviously, $\pi \circ p = q$.

Put $d_i := \dim F_i = \dim E_i$, $d_i \le d$, $i = 1, \dots, r$, and

$$E'_i := \{x \in E_i : \operatorname{rank} q|E_i = d_i\}, \qquad E''_i := \{x \in E_i : \operatorname{rank} q|E_i = d_i - 1\}.$$

Then $E_i = E'_i \cup E''_i$. Clearly, the restriction

$$\operatorname{res} q : E'_i \setminus q^{-1}(q(\partial E'_i)) \to q(E'_i) \setminus q(\partial E'_i)$$

is proper. Now observe that the set $S$ of self-intersections of the image of $\operatorname{res} q$ is a Q-subanalytic subset of $q(E'_i)$ as $S \times \{0\} = \overline{V} \cap (q(E'_i) \times \{0\})$, where

$$\begin{aligned} V := \{&(u_1, \dots, u_{m-1}, \epsilon) \in (q(E'_i) \setminus q(\partial E'_i)) \times [0,1] : \\ &\exists v = (v_1, \dots, v_{n-m+1}), w = (w_1, \dots, w_{n-m+1}) \in [0,1]^{n-m+1} : \\ &\qquad 0 < |v - w| < \epsilon,\ (u,v), (u,w) \in E'_i \setminus q^{-1}(q(\partial E'_i))\}. \end{aligned}$$

Then $T := S \cup q(\partial E'_i)$ is a Q-subanalytic set of dimension $< d$, and the restriction

$$\operatorname{res} q : E'_i \setminus q^{-1}(T) \to q(E'_i) \setminus T$$

is a topological covering, whence so is the restriction

$$\operatorname{res} \pi : p(E'_i) \setminus \pi^{-1}(T) \to q(E'_i) \setminus T.$$

Therefore, over any simply connected subset (below we shall take a Q-cell) of $q(E'_i) \setminus T$, the set $p(E'_i)$ is a finite union of the Q-subanalytic graphs of smooth functions.

Further, notice that, for each $u \in q(E''_i)$, the fiber $(E''_i)_u := q^{-1}(u) \cap E''_i$ is a smooth Q-semianalytic arc, and the restriction of $p$ to $(E''_i)_u$ is an immersion of this fiber into $\{u\} \times \mathbb{R}_{x_m}$ whence the fiber $(F_i)_u$ is a finite union of open intervals. By virtue of the parametric version of Lemma 1, the sets

$$Z_i := \bigcup_{u \in q(E''_i)} (\{u\} \times \partial p(E''_i)_u) \subset [0,1]^m$$

are Q-subanalytic of dimension $< d$. By the induction hypothesis, there exists a Q-cell decomposition $\{C_p : p = 1, \dots, s\}$ of $[0,1]^m$ compatible with the sets $Z_i$, $i = 1, \dots, r$. Clearly, for each cell $C_p$, the sets

$$W_{i,p} := \{u \in [0,1]^{m-1} : (C_p)_u \subset (E_i)_u\} \subset [0,1]^{m-1}$$

are Q-subanalytic. Again by the induction hypothesis, one can find a Q-cell decomposition $\mathcal{C}$ compatible with the sets

$$q(E'_i), \quad q(\partial E'_i), \quad W_{i,p}, \quad p(E'_i) \cap \pi^{-1} q(\partial E'_i) \quad \text{and} \quad Z_i;$$

where the first three are subsets of $[0,1]^{m-1}$, the last two are subsets of $[0,1]^m$ of dimension $< d$. Indeed, one must construct a Q-cell decomposition compatible with the subsets of $[0,1]^m$ under study, which are of dimension

$< d$, and next refine the induced Q-cell decomposition of $[0,1]^{m-1}$ so as to be compatible with the remaining subsets of $[0,1]^{m-1}$.

What remains to be done is to modify the Q-cell decomposition $\mathcal{C}$, achieved in this fashion, as follows. As we have already seen, over each Q-cell $C$ from the induced Q-cell decomposition of $[0,1]^{m-1}$ such that $C \subset q(E_i')$ but $C \cap q(\partial E_i') = \emptyset$, $i = 1, \ldots, r$, the set $p(E_i')$ is a finite union of the Q-subanalytic graphs of smooth functions. Again, one must modify $\mathcal{C}$ by partitioning its Q-cells by means of those Q-subanalytic graphs; this is, of course, linked with a successive refinement of the cube $[0,1]^{m-1}$, which is possible due to the induction hypothesis.

It is not difficult to check that eventually we attain a Q-cell decomposition $\mathcal{C}$ of $[0,1]^m$ compatible with the sets $p(E_i')$ and $p(E_i'')$, and a fortiori with the sets $F_i := p(E_i) = p(E_i') \cup p(E_i'')$. We leave the details to the reader.

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