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Diffeomorphisms conformal on distributions

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Abstract. Let $f : M \to N$ be a local diffeomorphism between Riemannian manifolds. We define the eigenvalues of f to be the eigenvalues of the self-adjoint, positive definite operator $df^* df : TM \to TM$, where df^* denotes the operator adjoint to df. We show that if f is conformal on a distribution D, then $\dim V_\lambda \geq 2\dim D - \dim M$, where V_λ denotes the eigenspace corresponding to the coefficient of conformality λ of f. Moreover, if f has distinct eigenvalues, then there is locally a distribution D such that f is conformal on D if and only if $2\dim D < \dim M + 1$.

1. Introduction. Let (M, g_M) and (N, g_N) be Riemannian manifolds with $n = \dim M = \dim N$. Let $f : M \to N$ be a local diffeomorphism. Fix $x \in M$. Let $(df_x)^* : T_{f(x)}N \to T_xM$ be the adjoint operator to $df_x : T_xM \to T_{f(x)}N$, i.e.

$$g_N(Y, df_x X) = g_M((df_x)^* Y, X), \qquad X \in T_xM,\ Y \in T_{f(x)}N.$$

Then the self-adjoint and positive definite operator $S_x = (df_x)^* df_x : T_xM \to T_xM$ has n real and positive eigenvalues $\lambda_1(x) \geq \cdots \geq \lambda_n(x)$. The eigenspace of S_x corresponding to the eigenvalue $\lambda_i(x)$ is denoted by $V_{\lambda_i(x)} \subset T_xM$. An *eigenvector of* f at x is an eigenvector of S_x. We call $\lambda_1, \ldots, \lambda_n$ the *eigenvalues of* f. We say that $\lambda_1, \ldots, \lambda_n$ are *distinct* if they are distinct at every point, that is, $\lambda_i(x) \neq \lambda_j(x)$ for all $i \neq j$ and $x \in M$. Note that given any n distinct functions $\lambda_1, \ldots, \lambda_n$ on M, there is locally a diffeomorphism $f : U \to N$, U open in M, such that the λ_i are the eigenvalues of f (see [2]).

Let D be a distribution on M. We say that f is *conformal on* D if

$$g_N(f_{*x}X, f_{*x}Y) = \lambda(x) g_M(X, Y), \qquad X, Y \in D_x, \qquad x \in M,$$

for some smooth and positive function $\lambda : M \to \mathbb{R}$, which is called the *coefficient of conformality* of f. Properties of such transformations for codimension one distributions on Riemannian manifolds were considered in [5] and [6]. For contact manifolds, conformality of some diffeomorphisms with respect to contact distributions was investigated in [4] and [7].

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Given a diffeomorphism $f : M \to N$ between Riemannian manifolds, one may ask if there is a distribution D on M such that f is conformal on D. Moreover, integrability of D is of interest. In [3] the author investigated the case when M and N are open subsets of Euclidean space $\mathbb{R}^3$. Then, if f has three everywhere distinct eigenvalues, there are locally exactly two smooth 2-dimensional distributions D_1, D_2 such that f is conformal on each of them. The integrability condition leads to a second order nonlinear equation. It turns out that integrability of one of these distributions does not imply integrability of the other.

In dimensions higher than three the situation is much more complicated. However, we are able to state some local criteria both for existence and nonexistence of diffeomorphisms conformal on distributions.

We have the following results. Let $f : M \to N$ be a local diffeomorphism. Assume that f has everywhere distinct eigenvalues. Then there is locally a k-dimensional distribution D such that f is conformal on D if and only if $k \leq (\dim M + 1)/2$ (Theorem 3.3). For nonexistence the assumption that f has distinct eigenvalues can be relaxed. Namely, if $\lambda_l > \lambda_{l+1}$ or $\lambda_{l+1} > \lambda_{l+2}$ for $\dim M = 2l + 1$ and $\lambda_l > \lambda_{l+1}$ for $\dim M = 2l$, then for $k > (\dim M + 1)/2$ there is no k-dimensional distribution D such that f is conformal on D (Theorem 3.2). Thus if a diffeomorphism f is conformal on a distribution D and $\dim D > (\dim M + 1)/2$, then f cannot have distinct eigenvalues. A stronger result is valid: $\dim V_\lambda \geq 2 \dim D - \dim M$, where V_λ is the eigenspace of f corresponding to the coefficient of conformality λ (Theorem 3.4).

Throughout this paper all manifolds, maps and distributions are assumed to be smooth.

2. Geometric lemmas. In this section we prove some geometric results about ellipsoids. By an *ellipsoid* E in $\mathbb{R}^n$ we mean a set

$$E = \{x \in \mathbb{R}^n : \langle Py, y\rangle = 1\},$$

where $\langle\cdot,\cdot\rangle$ denotes the standard inner product and P is a positive definite symmetric $n \times n$ matrix. Let $\lambda_1 \geq \cdots \geq \lambda_n > 0$ be the eigenvalues of P. Put $\mu_i = 1/\sqrt{\lambda_i}$. Let V_λ denote the eigenspace corresponding to the eigenvalue λ. If P has distinct eigenvalues, then the eigenspaces are 1-dimensional and μ_i are the lengths of the semi-principal axes of E.

We have the following lemma, generalizing a result from [1, p. 194].

Lemma 2.1. *Let E be an ellipsoid in $\mathbb{R}^{2l+1}$.*

(1) *For any $k \leq l + 1$ there is a k-dimensional subspace intersecting E along a sphere.*

(2) *If* $\lambda_l > \lambda_{l+1}$ *or* $\lambda_{l+1} > \lambda_{l+2}$, *then*

   (i) *every* $(l+1)$*-dimensional subspace intersecting* E *along a sphere contains the eigenspace* $V_{\lambda_{l+1}}$,

   (ii) *there is no* k*-dimensional subspace intersecting* E *along a sphere for any* $k > l+1$.

Proof. Let P be the matrix representing the ellipsoid E and let $\alpha_1, \dots, \alpha_{2l+1}$ be an orthonormal basis in $\mathbb{R}^{2l+1}$ consisting of eigenvectors of P corresponding to the eigenvalues $\lambda_1 \geq \dots \geq \lambda_{2l+1}$ respectively. For $x \in \mathbb{R}^{2l+1}$ let $(x_1, \dots, x_{2l+1})$ denote its coordinates in this basis, $x = \sum_i x_i \alpha_i$.

If in the sequence $\lambda_1 \geq \dots \geq \lambda_{2l+1}$ there are l consecutive equalities then there is an $(l+1)$-dimensional (and hence k-dimensional for $k \leq l+1$) subspace intersecting E along a sphere. Therefore, we may assume that $\lambda_i > \lambda_{i+1}$ and $\lambda_j > \lambda_{j+1}$ for some $i, j = 1, \dots, 2l$ such that $j - i < l+1$ and $i < l+1 \leq j$.

Consider the following subsets of $\{1, \dots, 2l+1\}$:

$$A_1 = \{1, \dots, i\}, \quad A_2 = \{i+1, \dots, l\}, \quad A_3 = \{l+2, \dots, j\},$$
$$A_4 = \{j+1, \dots, l+j-i\}, \quad A_5 = \{l+j-i+1, \dots, 2l+1\}.$$

Then $\sharp A_1 = \sharp(A_3 \cup A_5) = i$ and $\sharp A_2 = \sharp A_4 = l - i$, where $\sharp A$ denotes the number of elements of A. Put

$$D_0 : \quad \begin{cases} \sqrt{\lambda_k - \lambda_{l+1}}\, x_k = \sqrt{\lambda_{l+1} - \lambda_{m_k}}\, x_{m_k}, & k = 1, \dots, i, \\ \sqrt{\lambda_{l+1} - \lambda_{p_k}}\, x_{p_k} = \sqrt{\lambda_{n_k} - \lambda_{l+1}}\, x_{n_k}, & k = 1, \dots, l-i. \end{cases}$$

where $\{m_k\}_{k=1,\dots,i} = A_3 \cup A_5$, $\{n_k\}_{k=1,\dots,l-i} = A_2$ and $\{p_k\}_{k=1,\dots,l-i} = A_4$. Since all the coefficients on the left hand sides are nonzero, D_0 is an $(l+1)$-dimensional subspace in $\mathbb{R}^{2l+1}$.

We will show that D_0 intersects E along a sphere of radius μ_{l+1}. Since E is given by

$$\lambda_1 x_1^2 + \dots + \lambda_{2l+1} x_{2l+1}^2 = 1, \tag{1}$$

$x \in E$ satisfies

$$\begin{aligned} -\lambda_{l+1}(x_1^2 + \ldots + x_{2l+1}^2) = {} & (\lambda_1 - \lambda_{l+1}) x_1^2 + \dots + (\lambda_l - \lambda_{l+1}) x_l^2 \\ & + (\lambda_{l+2} - \lambda_{l+1}) x_{l+2}^2 + \dots + (\lambda_{2l+1} - \lambda_{l+1}) x_{2l+1}^2 - 1. \end{aligned}$$

Thus we only need to show that for every $x \in D_0 \cap E$,

$$\begin{aligned} & (\lambda_1 - \lambda_{l+1}) x_1^2 + \dots + (\lambda_l - \lambda_{l+1}) x_l^2 \\ & \quad + (\lambda_{l+2} - \lambda_{l+1}) x_{l+2}^2 + \dots + (\lambda_{2l+1} - \lambda_{l+1}) x_{2l+1}^2 = 0. \end{aligned} \tag{2}$$

This follows from the definition of D_0. Therefore D_0 intersects E along a sphere of radius μ_{l+1}. Moreover, for any $k < l+1$, a k-dimensional subspace of D_0 cuts E along a sphere.

Assume now that $\lambda_l > \lambda_{l+1}$ or $\lambda_{l+1} > \lambda_{l+2}$. Let D be an $(l+1)$-dimensional subspace cutting E along a sphere of radius μ_{l+1}. We will show that D contains $V_{\lambda_{l+1}}$. We have

$$D: \quad \begin{cases} c_{1,1}x_1 + \cdots + c_{1,2l+1}x_{2l+1} = 0, \\ \qquad \cdots \\ c_{l,1}x_1 + \cdots + c_{l,2l+1}x_{2l+1} = 0, \end{cases} \tag{3}$$

for some $c_{i,j}$. Since D cuts E along a sphere of radius μ_{l+1}, points in $E \cap D$ solve (1) and $\lambda_{l+1}x_1^2 + \cdots + \lambda_{l+1}x_{2l+1}^2 = 1$. In addition, if x satisfies (2), so does tx for any $t \in \mathbb{R}$. Thus we see that if $x \in D$, then x satisfies (2). Put

$$C = \begin{bmatrix} c_{1,1} & \ldots & c_{1,l} \\ \vdots & \ddots & \vdots \\ c_{l,1} & \ldots & c_{l,l} \end{bmatrix}, \quad \widetilde{C} = \begin{bmatrix} c_{1,l+2} & \ldots & c_{1,2l+1} \\ \vdots & \ddots & \vdots \\ c_{l,l+2} & \ldots & c_{l,2l+1} \end{bmatrix}.$$

Suppose $\det C = \det \widetilde{C} = 0$. Consider first the condition $\det C = 0$. Then $\operatorname{rank} C < l$, so $\dim(D \cap \operatorname{Lin}(\alpha_1, \ldots, \alpha_l)) \geq 1$. Moreover, $D \cap \operatorname{Lin}(\alpha_1, \ldots, \alpha_l)$ cuts an ellipsoid $E \cap \operatorname{Lin}(\alpha_1, \ldots, \alpha_l)$ along a sphere of radius not greater than μ_l. Since D intersects E along a sphere of radius μ_{l+1}, we get $\mu_l = \mu_{l+1}$. Thus $\lambda_l = \lambda_{l+1}$. Similarly $\det \widetilde{C} = 0$ implies $\lambda_{l+1} = \lambda_{l+2}$. Thus $\lambda_l = \lambda_{l+1} = \lambda_{l+2}$, which is impossible. Therefore, without loss of generality, we may assume that $\lambda_l > \lambda_{l+1}$ and $\det C \neq 0$. Thus we can write (3) in the form

$$D: \quad \begin{cases} x_1 = a_{1,l+1}x_{l+1} + a_{1,l+2}x_{l+2} + \cdots + a_{1,2l+1}x_{2l+1}, \\ \qquad \cdots \\ x_l = a_{l,l+1}x_{l+1} + a_{l,l+2}x_{l+2} + \cdots + a_{l,2l+1}x_{2l+1}, \end{cases} \tag{3$'$}$$

for some $a_{i,j}$. Moreover, $V_{\lambda_{l+1}} = \operatorname{Lin}(\alpha_{l+1}, \ldots, \alpha_p)$ for some $p \geq l+1$. Thus (2) becomes

$$\begin{aligned} (\lambda_1 - \lambda_{l+1})x_1^2 + \cdots + (\lambda_l - \lambda_{l+1})x_l^2 \\ + (\lambda_{p+1} - \lambda_{l+1})x_{p+1}^2 + \cdots + (\lambda_{2l+1} - \lambda_{l+1})x_{2l+1}^2 = 0. \end{aligned} \tag{2$'$}$$

Substituting (3′) to (2′) we find that the coefficient of x_i^2, $l+1 \leq i \leq p$, equals

$$a_{1,i}^2(\lambda_1 - \lambda_{l+1}) + \cdots + a_{l,i}^2(\lambda_l - \lambda_{l+1}).$$

On the other hand, it must be equal to zero (by taking $x_{l+1} = \cdots = x_{i-1} = x_{i+1} = x_{2l+1} = 0$ and $x_i = 1$). Therefore $a_{j,i} = 0$ for $j = 1, \ldots, l$ and $i = l+1, \ldots, p$. We get

$$D: \quad \begin{cases} x_1 = a_{1,p+1}x_{p+1} + \cdots + a_{1,2l+1}x_{2l+1}, \\ \qquad \cdots \\ x_l = a_{l,p+1}x_{p+1} + \cdots + a_{l,2l+1}x_{2l+1}, \end{cases}$$

so D contains $V_{\lambda_{l+1}}$.

Suppose now that there is an $(l+1)$-dimensional subspace D' which does not contain $V_{\lambda_{l+1}}$ and intersects E along a sphere. Then, by the above, the radius of this sphere is $\mu \neq \mu_{l+1}$. Since $\dim(D \cap D') \geq 1$, we can choose a 1-dimensional subspace $l \subset D \cap D'$. It intersects E in four points, two with norm μ and two with norm μ_{l+1}. This is impossible.

We only need to show that there is no k-dimensional subspace intersecting E along a sphere if $k > l+1$ (still with the assumption $\lambda_l > \lambda_{l+1}$ or $\lambda_{l+1} > \lambda_{l+2}$). Suppose such a subspace D exists. Without loss of generality we may assume that $\lambda_l > \lambda_{l+1}$. Consider the subspaces $V_1 = \mathrm{Lin}(\alpha_1, \ldots, \alpha_{2l-1})$ and $V_2 = \mathrm{Lin}(\alpha_2, \ldots, \alpha_{2l})$. Then $\dim(D \cap V_i) \geq l$. Let W_i be an l-dimensional subspace of $D \cap V_i$. Then, by the previous part, W_1 and W_2 intersect the ellipsoids $E \cap V_1$ and $E \cap V_2$ along spheres of radii μ_l and μ_{l+1} respectively, a contradiction since $\mu_l < \mu_{l+1}$. ∎

LEMMA 2.2. *Let E be an ellipsoid in $\mathbb{R}^{2l}$.*

(1) *For any $k \leq l$ there is a k-dimensional subspace intersecting E along a sphere.*
(2) *If $\lambda_l > \lambda_{l+1}$ and $k > l$ then there is no k-dimensional subspace intersecting E along a sphere.*

Proof. Choose a basis $(\alpha_1, \ldots, \alpha_{2l})$ as in the proof of Lemma 2.1. Consider an ellipsoid $E_1 = E \cap \mathrm{Lin}(\alpha_1, \ldots, \alpha_{2l-1})$. By Lemma 2.1 there exists an l-dimensional subspace D intersecting E_1 along a sphere. Therefore D intersects E along a sphere. If we take $k < l$ then any k-dimensional subspace of D cuts E along a sphere.

Suppose there is a k-dimensional subspace D intersecting E along a sphere for $k > l$ and $\lambda_l > \lambda_{l+1}$. Let $\widetilde{D}$ be any $(l+1)$-dimensional subspace of D. Let λ be such that $\lambda_l > \lambda > \lambda_{l+1}$. Extend $E \subset \mathbb{R}^{2l}$ to an ellipsoid $\widetilde{E}$ in $\mathbb{R}^{2l+1}$ putting

$$\widetilde{E}: \quad \lambda_1 x_1^2 + \cdots + \lambda_{2l} x_{2l}^2 + \lambda x_{2l+1}^2 = 1,$$

where $x = \sum_i x_i \alpha_i + x_{2l+1} e_{2l+1}$ and e_{2l+1} is the $(2l+1)$th vector of the canonical basis in $\mathbb{R}^{2l+1}$. Then $\widetilde{D}$ intersects $\widetilde{E}$ along a sphere but $V_\lambda \not\subset \widetilde{D}$, a contradiction with Lemma 2.1. ∎

3. Main results

LEMMA 3.1. *Let $f : M \to N$ be a local diffeomorphism between n-dimensional Riemannian manifolds (M, g_M) and (N, g_N), and D a distribution on M. Let $e_1(x), \ldots, e_n(x)$ be an orthonormal basis of T_xM consisting of eigenvectors of f at $x \in M$. Let $\lambda_1(x), \ldots, \lambda_n(x)$ be the corresponding* *eigenvalues. Then for every $x \in M$, D_x intersects the ellipsoid*

$$E_x = \Big\{ \sum_i X_i e_i(x) : \sum_i \lambda_i(x) X_i^2 = 1 \Big\} \subset T_x M$$

along a sphere if and only if f is conformal on D.

Proof. Let $X \in E_x$, $X = \sum_i X_i e_i(x)$. Then

$$|df_x X|_N^2 = g_M(S_x X, X) = \sum_{i,j} X_i X_j g_M(S_x e_i(x), e_j(x)) = \sum_i \lambda_i X_i^2 = 1,$$

where $S_x = (df_x)^* df_x$ and $|\cdot|_N$ denotes the norm in the metric g_N. Therefore $df_x(E_x) = \mathbb{S}$, where $\mathbb{S} \subset T_{f(x)}N$ is the unit sphere. If f is conformal on D, then $D_x \cap E_x$ is a sphere since its image $df_x(D_x) \cap \mathbb{S}$ is a sphere. Conversely, if $D_x \cap E_x$ is a sphere, then $df_x(D_x \cap E_x) = df_x(D_x) \cap \mathbb{S}$ is a sphere. Thus f is conformal on D. ∎

Theorem 3.2. *Let $f : M \to N$ be a local diffeomorphism between n-dimensional Riemannian manifolds, and $\lambda_1 \geq \cdots \geq \lambda_n$ the eigenvalues of f. Let $x \in M$. Assume that $\lambda_l(x) > \lambda_{l+1}(x)$ or $\lambda_{l+1}(x) > \lambda_{l+2}(x)$ if $n = 2l+1$ and $\lambda_l(x) > \lambda_{l+1}(x)$ if $n = 2l$. Then for any $k \in \mathbb{N}$ satisfying $\dim M < 2k-1$ there is no k-dimensional distribution D in any neighbourhood of x such that f is conformal on D.*

Proof. Follows from Lemmas 2.1, 2.2 and 3.1. ∎

Theorem 3.3. *Let $f : M \to N$ be a local diffeomorphism between Riemannian manifolds. Assume that f has distinct eigenvalues. Then there exists locally a k-dimensional distribution D such that f is conformal on D if and only if $k \leq (\dim M + 1)/2$.*

Proof. Nonexistence for $k > (\dim M + 1)/2$ follows from Theorem 3.2. Let $n = \dim M$ and $k \leq (n+1)/2$. Let $\lambda_1(x) > \cdots > \lambda_n(x) > 0$ be the eigenvalues of f at $x \in M$. Since f has globally distinct eigenvalues, the functions λ_i, $i = 1, \ldots, n$, are smooth. Fix $x_0 \in M$. Let $(X_1, \ldots, X_n)$ be an orthonormal basis in a neighbourhood U of x_0 such that $X_i(x)$ is an eigenvector of f at x corresponding to the eigenvalue $\lambda_i(x)$, $x \in U$, $i = 1, \ldots, n$. If $\dim M = 2l+1$ put

$$Y_i = \sqrt{\lambda_i - \lambda_{l+1}}\, X_i - \sqrt{\lambda_{l+1} - \lambda_{l+1+i}}\, X_{l+1+i}, \qquad i = 1, \ldots, l,$$

while if $\dim M = 2l$ put

$$Z_i = \sqrt{\lambda_i - \lambda_l} X_i - \sqrt{\lambda_l - \lambda_{l+i}} X_{l+i}, \qquad i = 1, \ldots, l-1.$$

Define

$$D = \begin{cases} \mathrm{Lin}(Y_1, \ldots, Y_{k-1}, X_{l+1}) & \text{if } \dim M = 2l+1, \\ \mathrm{Lin}(Z_1, \ldots, Z_{k-1}, X_l) & \text{if } \dim M = 2l. \end{cases}$$

Then D is a k-dimensional distribution on U. By the proof of Lemma 2.1 (definition of D_0) and Lemma 3.1, f is conformal on D. ∎

THEOREM 3.4. *Let $f : M \to N$ be a local diffeomorphism between Riemannian manifolds. Let D be a distribution on M such that f is conformal on D. Let $V_{\lambda(x)}$ denote the eigenspace of f at $x \in M$ corresponding to the coefficient of conformality $\lambda(x)$. Then*

$$\dim V_{\lambda(x)} \geq 2\dim D - \dim M, \quad x \in M. \tag{4}$$

Proof. Let $n = \dim M$ and $k = \dim D$. Fix $x \in M$ and consider the notations from Lemma 3.1. Let $\mu_i(x) = 1/\sqrt{\lambda_i(x)}$. We divide the proof into two cases.

CASE 1: $n = 2l + 1$. If $k \leq l$, then

$$2\dim D - \dim M \leq 2l - (2l+1) = -1,$$

so (4) holds. If $k \geq l + 1$, then $\dim D_x \cap D_i \geq 1$, $i = 1, 2$, where

$$D_1 = \operatorname{Lin}(e_1(x), \ldots, e_{l+1}(x)), \qquad D_2 = \operatorname{Lin}(e_{l+1}(x), \ldots, e_{2l+1}(x)).$$

Let $\mu = 1/\sqrt{\lambda(x)}$ be the radius of the sphere $E_x \cap D_x$ (see Lemma 3.1). As $D_x \cap D_1$ intersects E_x along a sphere of radius $\leq \mu_{l+1}(x)$, and $D_x \cap D_2$ intersects E_x along a sphere of radius $\geq \mu_{l+1}(x)$, we get $\mu = \mu_{l+1}(x)$. Therefore $\lambda_{l+1}(x) = \lambda(x)$ and $\dim V_{\lambda(x)} \geq 1$. Moreover, for $k = l + 1$,

$$2\dim D - \dim M = 2(l+1) - (2l+1) = 1 \leq \dim V_{\lambda(x)},$$

so (4) holds.

Suppose now $k > l + 1$. Let

$$p = k - l - 1 \geq 1.$$

For $i = 0, 1, \ldots, p$ define

$$k_i = k - 2i, \qquad l_i = l - i.$$

Since $k_i - l_i = p - i + 1$,

$$k_i > l_i + 1, \qquad i = 0, 1, \ldots, p-1, \qquad k_p = l_p + 1.$$

By Lemma 2.1 ($k_0 > l_0 + 1$) there is a 2-dimensional subspace V_0 of $V_{\lambda(x)}$. Consider the ellipsoid $E_x \cap V_0^{\perp}$ and the subspace $D_x \cap V_0^{\perp}$. We have

$$\dim(T_xM \cap V_0^{\perp}) = 2l_1 + 1, \qquad \dim(D_x \cap V_0^{\perp}) \geq k_1.$$

We replace T_xM by $T_xM \cap V_0^{\perp}$ and D_x by a k_1-dimensional subspace of $D_x \cap V_0^{\perp}$ and continue the process (if $k_1 > l_1 + 1$). In the ith step ($k_i > l_i + 1$ for $i < p$) we take a 2-dimensional subspace V_i of $V_{\lambda(x)} \cap (V_0 \oplus \cdots \oplus V_{i-1})^{\perp}$.

Thus we have

$$V_0 \oplus \cdots \oplus V_{p-1} \subset V_{\lambda(x)},$$
$$\dim(T_xM \cap (V_0 \oplus \cdots \oplus V_{p-1})^{\perp}) = 2l_p + 1,$$
$$\dim(E_x \cap (V_0 \oplus \cdots \oplus V_{p-1})^{\perp}) \geq k_p.$$

By the part of the proof for $k \geq l+1$ we conclude that $\dim(V_{\lambda(x)} \cap (V_0 \oplus \cdots \oplus V_{p-1})^{\perp}) \geq 1$. Therefore

$$\begin{aligned}\dim V_{\lambda(x)} &\geq \dim(V_0 \oplus \cdots \oplus V_{p-1}) + 1 = 2p + 1\\ &= 2(k-l-1) + 1 = 2k - (2l+1)\\ &= 2\dim D - \dim M.\end{aligned}$$

Case 2: $n = 2l$. It is analogous to the previous one. If $k \leq l$, then

$$2\dim D - \dim M \leq 2l - 2l = 0 \leq \dim V_{\lambda(x)},$$

so (4) holds. Assume $k > l$. As before, we show that $\lambda(x) = \lambda_l(x) = \lambda_{l+1}(x)$. Put $p = k - l$ and define

$$k_i = k - 2i, \quad l_i = l - i, \quad i = 0, 1, \ldots, p.$$

Since $k_i - l_i = p - i$, we get $k_i > l_i$ for $i = 0, 1, \ldots, p-1$ and $k_p = l_p$. By Lemma 2.2 ($k_0 > l_0$) there is a 2-dimensional subspace V_0 of $V_{\lambda(x)}$. In the ith step ($k_i > l_i$) we find a 2-dimensional subspace V_i of $V_{\lambda(x)} \cap (V_0 \oplus \cdots \oplus V_{i-1})^{\perp}$. Thus

$$V_0 \oplus \cdots \oplus V_{p-1} \subset V_{\lambda(x)},$$

so

$$\dim V_{\lambda(x)} \geq 2p = 2(k-l) = 2\dim D - \dim M. \blacksquare$$

4. Examples

Example 4.1. Consider the n-dimensional torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$, $n > 3$. Let $A \in SL_n(\mathbb{Z})$. Assume each of the matrices A and $A^{\top}A$ has n distinct eigenvalues. For example, one may set

$$A = \begin{bmatrix} M_1 & & 0 \\ & \ddots & \\ 0 & & M_l \end{bmatrix} \quad \text{if } n = 2l$$

and

$$A = \begin{bmatrix} 1 & & & 0 \\ & M_1 & & \\ & & \ddots & \\ 0 & & & M_l \end{bmatrix} \quad \text{if } n = 2l+1,$$

where

$$M_i = \begin{bmatrix} 1 & 1 \\ a_i - 1 & a_i \end{bmatrix}, \qquad a_i \notin \{-3,-2,-1,0,1\}, \qquad a_i \neq a_j \text{ for } i \neq j.$$

Since $A(\mathbb{Z}^n) = \mathbb{Z}^n$, A induces a diffeomorphism $A' : T^n \to T^n$. Let $\pi : \mathbb{R}^n \to T^n$ be the natural projection. Then $A' \circ \pi = \pi \circ A$. Therefore $A'_* \circ \pi_* = \pi_* \circ A$ and $(A'_*)^* \circ \pi_* = \pi_* \circ A^\top$. Thus $S = (A'_*)^* A'_*$ satisfies

$$S \circ \pi_* = \pi_* \circ (A^\top A),$$

so S has n distinct eigenvalues. If $k > (n+1)/2$, Theorem 3.3 implies A' is not conformal on D for any k-dimensional distribution D on T^n.

Example 4.2. Consider Euclidean space $\mathbb{R}^3$ and the diffeomorphism

$$f(x_1, x_2, x_3) = (x_1, 2x_2, 3x_3), \qquad x = (x_1, x_2, x_3) \in \mathbb{R}^3.$$

Then $S_x = df(x)^\top df(x) = \mathrm{diag}(1,4,9)$ has different eigenvalues. Let $\mathcal{F}$ be the family of planes perpendicular to $N = (\sqrt{3}, 0, -\sqrt{5})$. Thus $\mathcal{F} = \{L_t\}_t$ is a 2-dimensional foliation on $\mathbb{R}^3$, where

$$L_t = \mathrm{Lin}((\sqrt{5}, 0, \sqrt{3}), (0,1,0)) + tN.$$

Moreover,

$$|df(x)(\sqrt{5}\alpha, \beta, \sqrt{3}\alpha)|^2 = 4(8\alpha^2 + \beta^2) = 4|(\sqrt{5}\alpha, \beta, \sqrt{3}\alpha)|^2.$$

Thus for every t, the map $f : L_t \to f(L_t)$ is conformal. We see that $\mathcal{F}$ is determined by the distribution D_0 from the proof of Lemma 2.1. Therefore, using D_0 and the arguments in the proof of Theorem 3.3, we can easily generalize this example to $\mathbb{R}^n$ and foliations by k-planes, provided that $k \leq (n+1)/2$.

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