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A DSM proof of surjectivity of monotone nonlinear mappings

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Abstract. A simple proof is given of a basic surjectivity result for monotone operators. The proof is based on the dynamical systems method (DSM).

1. Introduction. It is well-known that a continuous monotone function $f : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{|x|\to\infty} \frac{xf(x)}{|x|} = \infty \tag{1}$$

is surjective, i.e., the equation $f(x) = y$ is solvable for any $y \in \mathbb{R}$. Indeed, the monotonicity of f implies

$$[f(x) - f(s)](x - s) \geq 0, \qquad \forall x, s \in \mathbb{R}. \tag{2}$$

Therefore, taking $y = 0$ without loss of generality, one concludes from (1) that $f(x) \leq 0$ for $x \leq 0$ and $f(x) \geq 0$ for $x \geq 0$. Since f is continuous, it follows that there is a point x_0 such that $f(x_0) = 0$.

If $y \neq 0$ is an arbitrary real number, then the function $F(x) = f(x) - y$ satisfies inequality (2) with F in place of f, provided that (2) holds for f. Condition (1) is also satisfied for F if it holds for f:

$$\lim_{|x|\to\infty} \frac{xF(x)}{|x|} = \lim_{|x|\to\infty} \left(\frac{xf(x)}{|x|} - \frac{xy}{|x|} \right) = \infty.$$

Conditions (1) and (2) are generalized for nonlinear mappings F in a real Hilbert space H as follows:

$$\lim_{\|u\|\to\infty} \frac{(u, F(u))}{\|u\|} = \infty \tag{3}$$

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and

$$(F(u) - F(v), u - v) \geq 0 \quad \forall u, v \in H. \tag{4}$$

Here (u, v) stands for the inner product in H. Equations with monotone operators arise in many applications.

We want to prove that if F is twice Fréchet differentiable and conditions (3)–(4) hold, then F is surjective, i.e., the equation

$$F(u) = h \tag{5}$$

is solvable for every $h \in H$. This is a basic result in the theory of monotone operators (see, e.g., [1]), which can be proved without the assumption about twice Fréchet differentiability, but then its proof becomes considerably less simple. Our aim is to give a simple and short proof of this result. It is based on the dynamical systems method (DSM) developed in [2].

THEOREM 1. *Assume that $F : H \to H$ is a Fréchet differentiable mapping satisfying conditions* (3), (4). *Then equation* (5) *is solvable for any h.*

REMARK 1. If in (4) one has a strict inequality for $u \neq v$, then the solution to (5) is unique.

REMARK 2. Condition (4) and Fréchet differentiability imply that $A := F'(u) \geq 0$ for all $u \in H$.

REMARK 3. The Fréchet differentiability assumption can be weakened to semicontinuity ([3], see also [1]), but then the proof loses its elementary character.

2. Proof. Let us formulate the steps of our proof.

STEP 1. For any $a = \text{const} > 0$ the equation

$$F(u_a) + au_a = h \tag{6}$$

has a unique solution u_a.

STEP 2.

$$\sup_{0<a<1} \|u_a\| < c, \qquad c = \text{const} > 0. \tag{7}$$

By c we denote various constants independent of a.

STEP 3. Using (7), select a sequence $u_n = u_{a_n}$, $a_n \to 0$, weakly convergent in H to an element u:

$$u_n \rightharpoonup u, \qquad n \to \infty. \tag{8}$$

From (6) and (8) it follows that

$$F(u_n) \to h, \qquad n \to \infty. \tag{9}$$

From (8), (9) and (4) one concludes that u solves (5).

Let us give a *detailed proof.*

STEP 1. Consider the problem

$$\dot{v} = -A_a^{-1}[F(v) + av - h], \qquad v(0) = 0. \tag{10}$$

Here $\dot{v} := \frac{dv}{dt}$, $A_a := A + aI$, $A := F'(v)$. Problem (10) is a version of the DSM (see [2, p. 115]). We claim that:

(a) problem (10) has a unique global solution, that is, the solution defined for all $t \in [0, \infty)$,
(b) there exists $v(\infty) := \lim_{t\to\infty} v(t)$,
(c) $F(v(\infty)) + av(\infty) = h$.

Claim (a) follows from *local* solvability of problem (10) and a uniform (with respect to t) bound on the norm $\|v(t)\|$. This bound is obtained below (see (13)). The local solvability follows from the standard result on local solvability of a differential equation with Lipschitz right-hand side. Our asumption about twice differentiability of F implies that the right-hand side of equation (10) is Lipschitz. For monotone F the twice differentiability assumption can be considerably weakened (see [2, 3]), but then the proof of the local solvability becomes more complicated.

Define

$$\|F(v(t)) + av(t) - h\| =: g(t), \qquad \dot{g} := \frac{dg}{dt}.$$

Using (10), one gets

$$g\dot{g} = \big((F'(v) + aI)\dot{v}, F(v(t)) + av(t) - h\big) = -g^2.$$

Thus

$$g(t) = g(0)e^{-t}. \tag{11}$$

From (11) and (10) one deduces

$$\|\dot{v}\| \le \frac{g(0)}{a}\, e^{-t}, \tag{12}$$

where the estimate $\|A_a^{-1}\| \le 1/a$ was used. This estimate holds because $A = F'(v(t)) \ge 0$ by the monotonicity of F. Integrating (12) from t to infinity yields

$$\|v(t) - v(\infty)\| \le \frac{g(0)}{a}\, e^{-t}. \tag{13}$$

Note that if $\|\dot{v}\| \le g(t)$ and $g(t) \in L^1(0, \infty)$, then $v(\infty)$ exists by the Cauchy criterion for the existence of a limit:

$$\|v(t) - v(s)\| \le \int_s^t g(\tau)\, d\tau \to 0, \qquad t, s \to \infty,\ t > s.$$

It follows from (12) that

$$\lim_{t\to\infty} \|\dot{v}\| = 0. \tag{14}$$

Therefore, passing to the limit $t \to \infty$ in (10), one gets

$$0 = -A_a^{-1}(v(\infty))[F(v(\infty)) + av(\infty) - h]. \tag{15}$$

Applying the operator $A_a(v(\infty))$ to equation (15), one sees that $v(\infty)$ solves equation (6).

Uniqueness of the solution to (6) is easy to prove: if v and w solve (6), then

$$F(v) - F(w) + a(v - w) = 0, \qquad a > 0.$$

Multiply this equation by $v - w$, use the monotonicity of F (see (4)), and conclude that $v = w$. Step 1 is completed.

STEP 2. Multiply (6) by $u_a/\|u_a\|$ to get

$$\frac{(F(u_a), u_a)}{\|u_a\|} + a\|u_a\| = \frac{(h, u_a)}{\|u_a\|}. \tag{16}$$

Since $a > 0$ and $(h, u_a)/\|u_a\| \le \|h\|$, one gets

$$\frac{(F(u_a), u_a)}{\|u_a\|} \le \|h\|. \tag{17}$$

From (17) and (3) the desired estimate (7) follows. Step 2 is completed.

STEP 3. Let us prove that (4), (8) and (9) imply (5). Let $\eta \in H$ be arbitrary, and $s > 0$ be a small number. Note that $u_n \rightharpoonup u$ and $g_n \to g$ imply $(u_n, g_n) \to (u, g)$. Using (4), one gets

$$(F(u_n) - F(u - s\eta), u_n - u + s\eta) \ge 0, \qquad \forall \eta \in H,\ s > 0. \tag{18}$$

Let $n \to \infty$ in (18). Then, using (8) and (9), one concludes that

$$\begin{aligned} (h - F(u - s\eta), s\eta) &\ge 0 \quad \forall \eta \in H,\ s > 0, \qquad \text{or} \\ (h - F(u - s\eta), \eta) &\ge 0 \quad \forall \eta \in H,\ s > 0. \end{aligned} \tag{19}$$

Let $s \to 0$ and use the continuity of F. (Here hemicontinuity of F would suffice.) Then (19) implies

$$(h - F(u), \eta) \ge 0 \quad \forall \eta \in H. \tag{20}$$

Taking $\eta = h - F(u)$ in (20), one concludes that $F(u) = h$. Step 3 is completed. Theorem 1 is proved. ■

References

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