ANNALES
POLONICI MATHEMATICI
95.2 (2009)

Some envelopes of holomorphy

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Abstract. We construct some envelopes of holomorphy that are not equivalent to domains in $\mathbb{C}^n$.

1. Introduction. In [14] we exhibited domains in $\mathbb{C}^n$, $n \geq 2$, whose envelopes of holomorphy are not smoothly equivalent to domains in $\mathbb{C}^n$ ([1]). The main purpose of the present note is to present an example of a domain, which lies in $\mathbb{C}^7$, whose envelope of holomorphy is real-analytically equivalent to a domain in $\mathbb{C}^7$ but is not biholomorphic to such a domain. The construction we use yields some other examples in the same spirit. The principal ingredients of the example are the known results that the seven-sphere $\mathbb{S}^7$ does not admit a totally real embedding in $\mathbb{C}^7$ but that *every* sphere $\mathbb{S}^n$ admits a totally real immersion in $\mathbb{C}^n$.

Weinstein [18, p. 26] observed that if we take

$$\mathbb{S}^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : |x|^2 = x_1^2 + \cdots + x_{n+1}^2 = 1\},$$

then the map $\varphi : \mathbb{C}^{n+1} \to \mathbb{C}^n$ given by

$$\varphi(z) = (z_1(1 + 2iz_{n+1}), \ldots, z_n(1 + 2iz_{n+1})) \tag{1}$$

restricts to $\mathbb{S}^n$ as a Lagrangian immersion of the sphere into $\mathbb{C}^n$ that is one-to-one except that the two poles $p^{\pm} = (0, \ldots, 0, \pm 1)$ are both taken to the origin. That φ is a Lagrangian immersion means that if ϑ is the $(1,1)$-form on $\mathbb{C}^n$ given by

$$\vartheta = \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j,$$

then $\varphi^*\vartheta = 0$. It follows that the image, Σ, of $\mathbb{S}^n$ under φ is an immersed

2000 *Mathematics Subject Classification*: Primary 32D10.
Key words and phrases: envelope of holomorphy.

([1]) In [14] the methods used are those of differential topology, so it is not evident from that paper whether the envelopes of holomorphy in question may be homeomorphic to domains in $\mathbb{C}^n$. By using topological intersection theory as given in [4], it can be shown that, in fact, these domains are not even topologically equivalent to domains in $\mathbb{C}^n$. Details are given in Section 3 below.

DOI: 10.4064/ap95-2-6

totally real submanifold of $\mathbb{C}^n$. At the origin of $\mathbb{C}^n$, the two local branches of Σ meet transversally.

Thus, though among the spheres $\mathbb{S}^n$ only the one-sphere and the three-sphere embed as totally real submanifolds in $\mathbb{C}^n$, each sphere $\mathbb{S}^n$ admits a very simple totally real immersion in $\mathbb{C}^n$.

2. The main construction. We construct the desired domain as follows.

Fix once and for all an integer $n \geq 2$. Let

$$\check{\mathbb{S}}^n = \{z \in \mathbb{C}^{n+1} : z_1^2 + \cdots + z_{n+1}^2 = 1\}.$$

The map φ defined in (1) above is of maximal rank on the sphere $\mathbb{S}^n$, so there is a neighborhood U^*, which we fix at the outset, on which φ is regular. That is, the pair (U^*, φ) is a Riemann domain spread over $\mathbb{C}^n$. In what follows, all our constructions are carried out inside U^*, though we shall not again refer to this restriction.

For positive r and with $|\cdot|$ the Euclidean norm on $\mathbb{C}^{n+1}$, let

$$\Delta(r) = \{z \in \check{\mathbb{S}}^n : |z - p^+| < r\},$$

which is a strictly pseudoconvex domain with smooth boundary in $\check{\mathbb{S}}^n$, provided r is small enough. Fix $r > 0$ small enough that φ carries $\overline{\Delta(r)}$ injectively into $\mathbb{C}^n$. Having fixed r, fix an $r' \in (0, r)$.

For a compact set X in $\mathbb{C}^n$, we denote by $\mathscr{C}(X)$ the space of continuous $\mathbb{C}$-valued functions on X, and by $\mathscr{P}(X)$ the closed subalgebra of $\mathscr{C}(X)$ consisting of those functions that can be approximated uniformly on X by polynomials.

For any $s \geq 0$, the set $\mathbb{S}^n \cup \overline{\Delta(s)}$ is polynomially convex and satisfies

$$\mathscr{P}(\mathbb{S}^n \cup \overline{\Delta(s)}) = \{f \in \mathscr{C}(\mathbb{S}^n \cup \overline{\Delta(s)}) : f|\Delta(s) \text{ is holomorphic}\},$$

because it is the union of a compact subset, $\mathbb{S}^n$, of $\mathbb{R}^{n+1}$ and a compact, polynomially convex subset, $\overline{\Delta(s)}$, of $\mathbb{C}^{n+1}$ that is invariant under the conjugation $x + iy \mapsto x - iy$ on $\mathbb{C}^{n+1} = \mathbb{R}^{n+1} + i\mathbb{R}^{n+1}$, so that a theorem of Smirnov and Chirka [11] shows the set to be polynomially convex. The polynomial convexity assertion and the equality of the two algebras are given in [15, Th. 8.1.26, p. 392]. (In order for this result to apply in the present situation, we need to have the approximation result that

$$\mathscr{P}(\overline{\Delta(r)}) = \{f \in \mathscr{C}(\overline{\Delta(r)}) : f|\Delta(r) \text{ is holomorphic}\}. \tag{2}$$

This equality is correct: $\Delta(r)$ is a strictly pseudoconvex domain with smooth boundary in the Stein manifold $\check{\mathbb{S}}^n$, so if f is continuous on $\overline{\Delta(r)}$ and holomorphic on $\Delta(r)$, then it can be approximated uniformly on $\overline{\Delta(r)}$ by functions g holomorphic on a neighborhood in $\check{\mathbb{S}}^n$ of $\overline{\Delta(r)}$. Moreover, the domain $\Delta(r)$ is defined by a strictly plurisubharmonic exhaustion function for $\check{\mathbb{S}}^n$, so the set $\overline{\Delta(r)}$ is convex with respect to the algebra $\mathscr{O}(\check{\mathbb{S}}^n)$, whence the approximating functions g can be approximated on $\overline{\Delta(r)}$ by functions h holomorphic on the whole of $\check{\mathbb{S}}^n$. These functions h can be extended to functions holomorphic on the whole ambient $\mathbb{C}^n$ and so can be approximated on the polynomially convex set $\overline{\Delta(r)}$ by polynomials. The desired equality (2) follows.)

Consider now the set $E = (\varphi^{-1}(\varphi(\mathbb{S}^n)) \cap b\Delta(r)) \setminus \mathbb{S}^n$, a certain compact subset of $b\Delta(r)$.

Lemma 1. *If r is small, the set E is polynomially convex.*

Proof. When s is small, each branch of the set $\varphi^{-1}(\varphi(\mathbb{S}^n)) \cap \Delta(s)$ is a totally real smooth manifold that is nearly a disc, whence each compact subset of it is polynomially convex and admits approximation of continuous functions by polynomials. Thus, E is polynomially convex as desired.

Let U_0 be an open subset of $\check{\mathbb{S}}^n$ that contains E and satisfies $\widehat{\overline{U}}_0 = \overline{U}_0$ if $\widehat{\overline{U}}_0$ denotes the polynomially convex hull of $\overline{U}_0$ and that is so small that $\widehat{\overline{U}}_0$ is disjoint from $\overline{\Delta(r')}$. Let U_1 be a second neighborhood in $\check{\mathbb{S}}^n$ of E with the property that the polynomially convex hull $\widehat{\overline{U}}_1$ is contained in U_0.

Lemma 2. *There is a bounded holomorphic function g on $\Delta(r)$ with $|g| < 1$ on $\Delta(r) \setminus \overline{U}_0$ and with the nonempty level set $\Sigma_\alpha = \{z \in \Delta(r) : |g(z)| = \alpha\}$ contained in U_0 for certain $\alpha > 1$.*

Proof. By the embedding theorem of Fornæss and Henkin [5], there exist a strictly convex domain W in $\mathbb{C}^N$ for some sufficiently large N and a biholomorphic embedding ψ of a neighborhood of $\bar{\Delta}(r)$ as a complex submanifold V of a neighborhood of $\overline{W}$ such that V is transversal to bW and $\psi^{-1}(W) = \Delta(r)$. Let U_1' be a bounded open subset of $\mathbb{C}^N$ whose intersection with V is $\psi(U_1)$, and let U_0' be a bounded open subset of $\mathbb{C}^N$ whose intersection with V is $\psi(U_0)$. We suppose $\widehat{\overline{U_1'}}$ to be contained in U_0'.

Let μ be a continuous function on bW such that $1 \le \mu \le 2$ and $\mu = 1$ on $bW \setminus U_0'$ and $\mu = 2$ on $U_1 \cap bW$ and $1 < \mu < 2$ on $(U_0 \setminus \overline{U}_1) \cap bW$. By a theorem of E. Løw [10] there is a function $\widetilde{g}$ bounded and holomorphic on W and vanishing at the point $\psi(p^+)$ with the property that the almost everywhere existent boundary values $\widetilde{g}^*$ of $\widetilde{g}$ satisfy $|\widetilde{g}^*| = \mu$ almost everywhere with respect to surface area measure on bW. For each $\alpha \ge 0$, let $\widetilde{\Sigma}_\alpha = \{z \in W : |\widetilde{g}(z)| = \alpha\}$.

To complete the proof of Lemma 2, we need a further lemma:

Lemma 3. *If $\alpha \in (1, 2)$ is sufficiently close to 2, then $\widetilde{\Sigma}_\alpha \subset U_0'$.*

Proof. Assume the lemma false, i.e., for a sequence $\{\alpha_j\}_{j=1}^\infty$ increasing to 2 the set $W \setminus U_0'$ contains a point w_j of $\widetilde{\Sigma}_\alpha$. The convexity of bW implies that if $1 \le s < 2$ and $C_s = \{z \in bW : \mu(z) \le s\}$, then for $\alpha > s \bar{\bar{\Sigma}}_\alpha \cap C_s = \emptyset$. This implies that when j is large $\bar{\bar{\Sigma}}_{\alpha_j} \cap bW \subset \overline{U}'_1$, which yields $\bar{\bar{\Sigma}}_{\alpha_j} \subset \widehat{\overline{U'_1}}$ when j is large. By hypothesis, $\widehat{\overline{U'_1}} \subset U'_0$. This completes the proof of Lemma 3.

To conclude the proof of Lemma 2, we take $g = \widetilde{g} \circ \psi$. The lemma is proved.

Fix permanently an α as in Lemma 2.

Notice that the surface Σ_α is fibered by the analytic hypersurfaces $g^{-1}(\zeta)$ for ζ with $|\zeta| = \alpha$.

As already noted, $\mathbb{S}^n \cup \overline{\Delta(s)}$ is polynomially convex.

If $s \in (0, r')$, there is a thin solid tube T in $\check{\mathbb{S}}^n$ over $\mathbb{S}^n \setminus \Delta(s)$ on which φ is injective. (The general principle here is that if $h : \mathscr{M} \to \mathscr{M}'$ is a local homeomorphism from the manifold $\mathscr{M}$ to the manifold $\mathscr{M}'$ that is injective on the compact set $K \subset \mathscr{M}$, then h is injective on a neighborhood of K.) Choose an $r'' \in (r', r)$. Let Ω_2 be a strictly pseudoconvex domain with

$$\mathbb{S}^n \cup \Delta(r'') \subset \Omega_2 \Subset (T \cup \Delta(r)).$$

The existence of such a domain follows from the polynomial convexity of $\mathbb{S}^n \cup \overline{\Delta(r')}$. We choose Ω_2 so large that $b\Omega_2 \cap \{z \in \Delta(r) : |g(z)| > \alpha\}$ is a neighborhood in $b\Omega_2$ of $b\Omega_2 \cap (\varphi^{-1}(\varphi(\mathbb{S}^n)) \setminus \mathbb{S}^n)$.

The map φ carries $(b\Omega_2 \cap \Delta(r)) \setminus \{z \in \Delta(r) : |g(z)| > \alpha\}$ injectively onto a set X in $\mathbb{C}^n$ that is at positive distance from $\varphi(S^n)$.

Let T' be a solid tube around $\mathbb{S}^n$ in $\check{\mathbb{S}}^n$ that is so thin that $T' \subset T \cup \Delta(r')$ and $\varphi(T')$ is disjoint from the set X. In addition, let r''' be a small positive number slightly greater than r. Then let Ω'_2 be a strictly pseudoconvex domain in $\check{\mathbb{S}}^n$ that contains $\mathbb{S}^n \cup \overline{\Delta(r)}$ and that is contained in $T' \cup \Delta(r''')$. The domain Ω'_2 can be chosen so that its boundary is transversal to the boundary of Ω_2. Using a process detailed in [12], we see that the intersection $b\Omega_2 \cap b\Omega'_2$ can be smoothed so as to obtain a strictly pseudoconvex domain Ω'_1 contained in $\Omega_2 \cap \Omega'_2$ and that agrees with this intersection outside a thin neighborhood of $b\Omega_2 \cap b\Omega'_2$. The strictly pseudoconvex domain Ω'_1 is contained in $\check{\mathbb{S}}^n$, contains $\mathbb{S}^n \cup \Delta(r')$, and satisfies

$$\Omega'_1 \cap \Delta(r) = \Omega_2 \cap \Delta(r).$$

Let Ω_1 be the domain

$$\Omega_1 = \Omega'_1 \setminus \{z \in \Delta(r) : |g(z)| \ge \alpha\}.$$

The domain Ω_1 is pseudoconvex and has the following extension property:

LEMMA 4. *If $V \subset \Omega_1$ is a connected open set such that $\overline{V}$ is a neighborhood of the set*

$$\Gamma = b\Omega_1 \setminus \{z \in \Delta(r) : |g(z)| = \alpha\}$$

in $\Omega_1 \cup \Gamma$, then each function f holomorphic on V continues holomorphically into Ω_1.

Proof. We consider first the set $\widetilde{\Omega}_1 = \Omega_1 \setminus \overline{U}_0 = \Omega_1' \setminus \overline{U}_0$. Its boundary is

$$b\widetilde{\Omega}_1 = (b\Omega_1 \setminus \overline{U}_0) \cup \overline{\Omega_1' \cap bU_0} = S \cup K.$$

The polynomially convex hull of the compact subset K of the boundary of $\widetilde{\Omega}_1$ is contained in the set $\overline{U}_0$ and so is disjoint from $\widetilde{\Omega}_1$. Accordingly, the principal result of the paper [9] implies that every CR-function on S extends holomorphically through all of $\widetilde{\Omega}_1$. If instead of a CR-function on S, we are given a function f holomorphic on a one-sided neighborhood of S that lies in $\widetilde{\Omega}_1$, then we apply this same extension result to the restriction of f to a surface S' lying in $\widetilde{\Omega}_1$ and obtained by pulling S in slightly, leaving it fixed at $bS = K \cap b\Omega_1$, so that f is defined on S'.

What we know, then, is that each function holomorphic on the domain V above extends holomorphically into $\widetilde{\Omega}_1$. We have to see that there actually is an extension into all of Ω_1.

To this end, notice that since the set $\overline{U}_0$ is polynomially convex, there is a Stein domain D that consists of $\Omega_1 \setminus \widetilde{\Omega}_1$ together with a thin neighborhood of $b\widetilde{\Omega}_1 \cap \Omega_1$. We can choose the domain D so that $bD \setminus \Sigma_\alpha$ is a smooth strictly pseudoconvex surface S'. A function defined on $\widetilde{\Omega}_1 \cup V$ is defined on $V \cap D$ and on a neighborhood of $bD \cap \widetilde{\Omega}_1$. The function g is defined on a Stein neighborhood of $\overline{D}$, viz. $\Delta(r)$, which is biholomorphically equivalent to a domain in $\mathbb{C}^n$.

At this point, it is convenient to treat the case $n \geq 3$ separately from the case $n = 2$. Suppose then that $n \geq 3$. Notice that the set $T_\alpha = bD \cap \Sigma_\alpha$ has the convexity property that if $z \in \overline{D} \setminus T_\alpha$, then there are analytic varieties of dimension $n - 1$ in a neighborhood of $\overline{D}$ that pass through the point z and miss T_α, e.g., the level set of g through z. This convexity property implies that each CR-function on $bD \setminus T_\alpha$ continues holomorphically into D, and indeed that any function defined and holomorphic on a one-sided neighborhood of $bD \setminus T_\alpha$ in $\overline{D}$ continues through D. For this relatively simple result, one can consult [3, Theorem 4.5.2] or [13, Theorem II.3].

The case $n = 2$ requires something different. In essence, it seems to be necessary to revisit the ideas used in [9] and by other authors cited there. We begin with the remark that since $\Delta(r)$ is biholomorphically equivalent to a domain in $\mathbb{C}^2$, there are global holomorphic coordinates, say $z = (z_1, z_2)$, defined on Δ. As the function g is holomorphic on $\Delta(r)$, which is a domain of holomorphy in the z-space, there is a factorization

$$g(z) - g(w) = g_1(z, w)(z_1 - w_1) + g_2(z, w)(z_2, w_2) \tag{3}$$

with g_1, g_2 holomorphic but not necessarily bounded on $\Delta(r) \times \Delta(r)$.

Denote by K_{BM} the Bochner–Martinelli kernel so that

$$K_{\mathrm{BM}}(z,w) = c_2\,\frac{[(\bar z_2 - \overline{w}_2)d\bar z_1 - (\bar z_1 - \overline{w}_1)d\bar z_2]\wedge\omega(z)}{|z-w|^4},$$

in which $\omega(z) = dz_1 \wedge dz_2$ and c_2 is a suitable constant. This kernel has the property that if F is holomorphic on the smoothly bounded domain W and is continuous on the closure $\overline{W}$, then for $w \in D$,

$$F(w) = \int_{bD} f(z)K_{\mathrm{BM}}(z,w).$$

Direct calculation shows that

$$\bar\partial_z\left(c_2\,\frac{\bar z_1 - \overline{w}_1}{|z-w|^2}\right)\omega(z) = (z_2 - w_2)K_{\mathrm{BM}}(z,w), \tag{4}$$

$$\bar\partial_z\left(c_2\,\frac{\bar z_2 - \overline{w}_2}{|z-w|^2}\right)\omega(z) = -(z_1 - w_1)K_{\mathrm{BM}}(z,w). \tag{5}$$

Consequently, the form

$$\vartheta(z,w) = c_2\left(g_2(z,w)\,\frac{\bar z_1 - \overline{w}_1}{|z-w|^2} - g_1(z,w)\,\frac{\bar z_2 - \overline{w}_2}{|z-w|^2}\right)\omega(z)$$

satisfies

$$\bar\partial_z\vartheta(z,w) = (g(z) - g(w))K_{\mathrm{BM}}(z,w)$$

and thus, where $g(z) \neq g(w)$, we have

$$\bar\partial_z\left\{\frac{\vartheta(z,w)}{g(z)-g(w)}\right\} = K_{\mathrm{BM}}(z,w).$$

We now consider the domain D constructed above and a function f defined on a one-sided neighborhood W of $bD\setminus T_\alpha$. Our goal is to show that f continues holomorphically into the whole of D. We shall assume that, in fact, f is defined and holomorphic on a neighborhood of $bD \setminus T_\alpha$. This is a matter of convenience: If f is not defined on such a neighborhood, replace D by a domain D' obtained by pulling $bD \setminus T_\alpha$ in a little, leaving T_α fixed. The original f is now defined on a neighborhood of $bD' \setminus T_\alpha$, and we need only show that f continues into D'.

Accordingly, define a function H on D as follows. For $w \in D$, let $|g(w)| = \beta$. We have $\beta < \alpha$. Choose $\gamma \in (\beta,\alpha)$ such that the level set $\Sigma_\gamma = \{z \in \Delta(r) : |g(z)| = \gamma\}$ is a smooth hypersurface that meets $bD \setminus T_\alpha$ transversally. By Stokes's theorem, the quantity

$$H_\gamma(w) = \int_{bD\cap\{z\,:\,|g(z)|<\gamma\}} f(z)K_{\mathrm{BM}}(z,w) + \int_{bD\cap\Sigma_\gamma}\frac{f(z)\vartheta(z,w)}{g(z)-g(w)} \tag{6}$$

is independent of the choice of γ. (In the expression for $H_\gamma(w)$, the orientation of $bD \cap \{z : |g(z) < \gamma\}$ is that induced on bD as the boundary of the domain D. The orientation on $bD \cap \Sigma_\gamma$ is taken to be that induced on $bD \cap \Sigma_\gamma$ as the boundary of the manifold $\Sigma_\gamma \cap D$. The latter manifold is taken to be oriented as part of the boundary of $D \cap \{z : |g(z)| < \gamma\}$.) We define $H(w)$ to be $H_\gamma(w)$. The function H defined in this way depends in a real-analytic way on the point w in D.

Denote by m the minimum value of $|g|$ on $\bar{D}$. The set M on which $|g|$ assumes the value m is a compact subset of $bD \setminus T_\alpha$, and consequently, if $\varepsilon > 0$ is sufficiently small, f is defined and holomorphic on the set $B = \{z \in \bar{D} : |g(z)| \le m + \varepsilon\}$. If ε is chosen properly—invoke Sard's theorem—then the level set $\Sigma_{m+\varepsilon}$ will be transversal to bD, and we can use Stokes's theorem to write that, for $w \in B \cap D$,

$$H(w) = H_{m+\varepsilon}(w) = \int_{bB} f(z) K_{\mathrm{BM}}(z, w) = f(w).$$

That is to say, we have a real-analytic function H on D that agrees with f on an open set in D. It follows that H is holomorphic on D and that it gives the holomorphic continuation of f through D.

We have now a complete proof of Lemma 4.

LEMMA 5. *The map φ is injective on the set Γ defined in the preceding lemma.*

Proof. The map φ is injective on $\Delta(r)$ and on T', so if $\varphi(z) = \varphi(z')$ for $z, z' \in \Gamma$, then $z \in \Delta(r)$ and $z' \in T' \setminus \Delta(r)$ or vice versa. Suppose the former case to obtain. As $z \in \Delta(r)$, we have $\varphi(z) \in X$. Finally, $z' \in T'$ implies that $\varphi(z') \notin X$. This completes the proof.

The fact that φ is injective on Γ implies that if Ω_0 is a thin one-sided neighborhood of Γ contained in Ω_1, then Ω_0 is carried injectively by φ onto a domain Ω in $\mathbb{C}^n$. As each f holomorphic on Ω_0 extends holomorphically into the pseudoconvex domain Ω_1, the envelope of holomorphy of Ω is the Riemann domain (Ω_1, φ). The manifold Ω_1 contains the totally real sphere $\mathbb{S}^n$.

Thus, for every $n = 2, 3, \ldots$, we have found a domain, say $\mathcal{D}_n$, in $\mathbb{C}^n$ whose envelope of holomorphy, $\widehat{\mathcal{D}}_n$, is a neighborhood of the n-sphere $\mathbb{S}^n$ in the complexified n-sphere $\check{\mathbb{S}}^n$.

There are various cases:

(1) $n = 3$. It was noted by Gromov that the three-sphere $\mathbb{S}^3$ admits totally real embeddings in $\mathbb{C}^n$; explicit embeddings were constructed by Ahern and Rudin [2]. Such an embedding, if chosen to be real-analytic, extends to a biholomorphic embedding of a neighborhood of $\mathbb{S}^3$ in $\check{\mathbb{S}}^3$ into $\mathbb{C}^3$, so if the domain $\mathcal{D}_3$ is chosen to be sufficiently thin, the envelope $\widehat{\mathcal{D}}_3$ is biholomorphically equivalent to a domain in $\mathbb{C}^3$.

(2) $n = 7$. Again, it was noted by Gromov that the seven-sphere $\mathbb{S}^7$ does not admit a totally real embedding into $\mathbb{C}^7$. Details of an argument establishing this are given in [17]. It follows that the envelope $\widehat{\mathcal{D}}_7$ is not biholomorphically equivalent to a domain in $\mathbb{C}^7$. It is, however, real-analytically equivalent to such a domain, for the complexification $\check{\mathbb{S}}^7$ is bianalytically equivalent to the product $\mathbb{S}^7 \times \mathbb{R}^7$, which, in turn, is bianalytically equivalent to $(\mathbb{R}^8 \setminus \{0\}) \times \mathbb{R}^6$. See [17].

(3) $n \neq 1, 3, 7$. For such n, the sphere $\mathbb{S}^n$ does not embed as a totally real submanifold of $\mathbb{C}^n$. The case of even n was treated by Wells [19] and by Aeppli [1]; the general case is in [17]. It follows that for $n \neq 1, 3, 7$, the envelope $\widehat{\mathcal{D}}_n$ is not biholomorphically equivalent to a domain in $\mathbb{C}^n$. In the case of the *even-dimensional* spheres more is true: If n is even, then results of Aeppli [1] imply that no Stein tube over $\mathbb{S}^n$ embeds homeomorphically in $\mathbb{C}^n$, so from this, when n is even, the envelope $\widehat{\mathcal{D}}_n$ is not homeomorphic to a domain in $\mathbb{C}^n$. The case of odd-dimensional spheres is not covered in the paper [1].

It is true, though, that for odd n, the envelope $\widehat{\mathcal{D}}_n$ is not diffeomorphic to a domain in $\mathbb{C}^n$. This is an immediate consequence of the known result—see Kervaire [8] and the references cited there—that *the normal bundle of a smoothly embedded n-sphere in $\mathbb{R}^{2n}$ is trivial.* Suppose then that $\widehat{\mathcal{D}}_n$ is diffeomorphic to a domain in $\mathbb{C}^n$ under, say, the diffeomorphism ψ. Then the normal bundle to the embedded sphere $\psi(\mathbb{S}^n)$ in $\mathbb{C}^n$ is trivial, which implies that the normal bundle of the embedded sphere $\mathbb{S}^n$ in $\widehat{\mathcal{D}}_n$ (or $\check{\mathbb{S}}^n$) is trivial. The complex structure J on $\check{\mathbb{S}}^n$ effects an isomorphism of the normal bundle to $\mathbb{S}^n$ with the tangent bundle to $\mathbb{S}^n$. Consequently, $\mathbb{S}^n$ is parallelizable, so $n = 1, 3$, or 7. (This argument was already used in [17].)

This discussion is again in the domain of differential topology; whether $\mathcal{D}_n$, n odd, not $1, 3, 7$, is *homeomorphic* to a domain in $\mathbb{C}^n$ is still not evident.

3. The envelope of holomorphy constructed in [14] is not homeomorphic to a domain in $\mathbb{C}^n$. In the paper [14] a domain Ω in $\mathbb{C}^n$, $n \geq 2$, is exhibited whose envelope of holomorphy $\widetilde{\Omega}$ is not diffeomorphic—even of class $\mathscr{C}^1$—to a domain in $\mathbb{C}^n$. At the time that paper was written, it was not evident to the author that the Riemann domain $\widetilde{\Omega}$ is not *homeomorphic* to a domain in $\mathbb{C}^n$. The object of the present paragraph is to observe that, in fact, $\widetilde{\Omega}$ is not topologically equivalent to a domain in $\mathbb{C}^n$.

We begin by recalling the principle involved in the example given in [14]. There the counterexample hinges on the construction of a domain Ω in $\mathbb{C}^n$ such that if $(\widetilde{\Omega}, \pi)$ is the envelope of holomorphy of Ω, then the Riemann domain $\widetilde{\Omega}$ contains a pair of smoothly embedded orientable n-manifolds M_1

and M_2 that intersect in a single point and whose intersection is transversal. By intersection theory in the setting of differential topology (see [7, p. 132]) this configuration cannot exist in $\mathbb{C}^n$. This is an argument in differential topology and does not exclude the possibility that $\widetilde{\Omega}$ might be *homeomorphic* to a domain in $\mathbb{C}^n$.

There is a topological theory of intersection that can be brought to bear on the matter at hand and that yields the result we seek: *The manifold $\widetilde{\Omega}$ is not homeomorphic to a domain in $\mathbb{C}^n$.* The intersection theory necessary for this conclusion is written out in the book of Dold [4, pp. 197–201 and 342–345].

In our situation, this theory attaches to each pair $\xi \in H_i(M_1)$ and $\eta \in H_j(M_2)$ of homology classes a homology class $\xi \bullet \eta \in H_{i+j-2n}(M_1 \cap M_2)$. With $i = j = n$ and with ξ and η the fundamental classes $o_{M_1} \in H_n(M_1)$ and $o_{M_2} \in H_n(M_2)$, the resulting product $o_{M_1} \bullet o_{M_2}$ lies in $H_0(M_1 \cap M_2) = H_0(\{p\}) = \mathbb{Z}$. Moreover, because the manifolds M_1 and M_2 meet transversally at the point p, we have $o_{M_1} \bullet o_{M_2} = \pm o_{\{p\}}$. In particular, this product is not zero.

On the other hand, these intersection numbers are altered at most by a sign by a homeomorphism of the manifold $\widetilde{\Omega}$, so because in $\mathbb{R}^n$ all intersection products vanish (see [4, p. 198]), the manifold $\widetilde{\Omega}$ cannot be homeomorphic to a domain in $\mathbb{C}^n$.

4. Another example. To conclude, we give an example that was brought to our attention by William R. Zame. The paper [16] contains an example of a domain D in $\mathbb{C}^n$ whose universal covering space D^* is not biholomorphic to a domain in $\mathbb{C}^n$. The obstruction is that by construction D^* contains a pair of smoothly embedded n-manifolds Σ and Σ_1 that intersect transversally at one point and that have no other intersection. The existence of these manifolds precludes the possibility that D^* is biholomorphic or even diffeomorphic to a domain in $\mathbb{C}^n$. And, as in the preceding section, we recognize that D^* is not topologically a domain in $\mathbb{C}^n$. If we now recall that according to [6], there is a domain D_0 in $\mathbb{C}^n$ whose envelope of holomorphy is the manifold D^*, we have another example of a domain in $\mathbb{C}^n$ whose envelope of holomorphy is not homeomorphic to a domain in $\mathbb{C}^n$.

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Received 1.7.2008
and in final form 17.9.2008 (1897)