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Weighted pluripotential theory on compact Kähler manifolds

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Abstract. We introduce a weighted version of the pluripotential theory on compact Kähler manifolds developed by Guedj and Zeriahi. We give the appropriate definition of a weighted pluricomplex Green function, its basic properties and consider its behavior under holomorphic maps. We also develop a homogeneous version of the weighted theory and establish a generalization of Siciak's H-principle.

Introduction. Recently there has been significant progress in weighted pluripotential theory on $\mathbb{C}^N$, which was originally developed in [Si1], [Si2] and generalized to parabolic manifolds in [Ze]. Specifically, we refer to [BL], [Bl1], [Bl2], [Bra], [MS]. Concurrently, pluripotential theory on a compact Kähler manifold X based on quasiplurisubharmonic functions has been explored in [GZ1], [GZ2], [Ko1], [Ko2] and [HKH] (see also applications in [Be1], [Be2], [BB]). In this article we try to connect the two theories by creating an analog of the plurisubharmonically-homogeneous pluripotential theory. Our starting point is an observation that a weighted pluripotential theory on $\mathbb{C}^N$ extends naturally to a pluripotential theory on $\mathbb{CP}^N$ with a suitably modified weight. In turn, this extends to a homogeneous pluripotential theory in the universal line bundle over $\mathbb{CP}^N$, whose charts are biholomorphic to $\mathbb{C}^{N+1}$. We will generalize these results to projective algebraic manifolds.

We define a weighted pluricomplex Green function on a compact complex manifold X with a Kähler form ω. The definition is formulated in terms of a mild function (see Definition 1). However, many results of our theory hold without requiring that Q be mild. For a mild function Q and a Borel set $K \subset X$ the weighted pluricomplex Green function is

$$V_{K,\omega,Q} = \sup\{\phi \in \mathrm{PSH}(X, \omega) : \phi \leq Q \text{ on } K\}.$$

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Basic properties of $V_{K,\omega,Q}$ are stated and proved in Section 1, followed by the extension of the weighted pluripotential theory in $\mathbb{C}^N$ to a suitable weighted pluripotential theory on $\mathbb{CP}^N$. In the case when X admits a positive line bundle (which by Kodaira's embedding theorem is equivalent to X being projective algebraic) we obtain more specific results, in particular generalizations of Siciak's H-principle and of some classical approximation results. Another interesting consequence is the following theorem (Theorem 5, Section 2):

THEOREM. *Let (X,ω) be a compact Kähler manifold and $f : X \to X$ a holomorphic surjection. Assume that there exist α and β, $1 < \alpha \le \beta$, such that*

$$\alpha f_*(\mathrm{PSH}(X,\omega)) \subset \mathrm{PSH}(X,\omega), \qquad f^*(\mathrm{PSH}(X,\omega)) \subset \beta \cdot \mathrm{PSH}(X,\omega).$$

Then for every Borel set $K \subset X$ and every mild function Q on X,

$$\alpha V_{f^{-1}(K),\omega,f^*Q/\alpha}(x) \le V_{K,\omega,Q} \circ f(x) \le \beta V_{f^{-1}(K),\omega,f^*Q/\beta}.$$

In fact, the similarity between Theorem 2.12 in [Bra] and Theorem 1 in [St1] (both of which are generalized versions of Theorem 5.3.1 in [Kl]) provided initial motivation for our work. These two results turn out to be special cases of the theorem above.

1. Weighted pluricomplex Green functions. Throughout the paper we assume that X is a connected compact Kähler manifold. Let ω be a closed real $(1,1)$-current on X with continuous local potentials. Following [Ko1] and [GZ1], the class of ω-plurisubharmonic functions is defined as

$$\mathrm{PSH}(X,\omega) = \{v \in L^1(X,\mathbb{R}\cup\{-\infty\}) : dd^c v \ge -\omega \text{ and } v \text{ is upper semicontinuous}\}.$$

(On $X = \mathbb{CP}^N$ such a class was introduced in [BT2].) The *ω-pluricomplex Green function* of a Borel set $K \subset X$ is defined as

$$V_{K,\omega}(x) = \sup\{v(x) : v \in \mathrm{PSH}(X,\omega),\ v|_K \le .0\}$$

Consider the class $\mathrm{PSH}(X,\omega)$, where ω is a Kähler form on X with local potentials $\phi_j : U_j \to \mathbb{R}$ for an open cover $\{U_j\}_{j=0}^m$ of X by coordinate neighborhoods.

DEFINITION 1. Let $Q : X \to \mathbb{R}\cup\{+\infty\}$ be a function such that the function $\exp(-Q+\phi_j)$ is continuous in U_j, $j = 1,\dots,m$, and $\{Q \ne +\infty\}$ is not a pluripolar subset of X. We will call Q satisfying these assumptions a *mild function.* Note that mild functions are necessarily lower semicontinuous.

DEFINITION 2. For a mild function Q on X and a Borel set $K \subset X$ let us define the *weighted ω-pluricomplex Green function as*

$$V_{K,\omega,Q} = \sup\{\phi \in \mathrm{PSH}(X,\omega) : \phi \le Q \text{ on } K\}.$$

The following properties are direct consequences of our definition of $V_{K,\omega,Q}$.

PROPOSITION 1. *Let K, K_1, K_2 be Borel subsets of X and Q, Q_1, Q_2 be mild functions.*

(i) *If $Q_1 \le Q_2$ on K then $V_{K,\omega,Q_1} \le V_{K,\omega,Q_2}$.*
(ii) *If $K_1 \subset K_2$ then $V_{K_2,\omega,Q} \le V_{K_1,\omega,Q}$.*
(iii) *Let Q be a mild function that belongs to the class* $\mathrm{PSH}(X,\omega)$. *Then $V_{X,\omega,Q} = Q$.*
(iv) *Let ω' be cohomologous to ω, that is, $\omega' = \omega + dd^c\xi$ for $\xi \in L^1(X)$. If ξ is mild and continuous, then $V_{K,\omega',Q} = V_{K,\omega,Q-\xi} + \xi$.*

We continue to establish basic properties of the weighted pluricomplex Green function in Propositions 2 and 3.

PROPOSITION 2. *Let E be a Borel set in X and Q a mild function on X. If E is not* $\mathrm{PSH}(X,\omega)$*-polar then* $V^*_{E,\omega,Q} \in \mathrm{PSH}(X,\omega)$.

Proof. By Choquet's lemma there exists an increasing sequence of functions $\phi_j \in \mathrm{PSH}(X,\omega)$ such that $\phi_j \le Q$ on E and

$$V^*_{E,\omega,Q} = (\lim_{j\to\infty} \phi_j)^*.$$

It follows from Proposition 2.6(2) in [GZ1] that $V^*_{E,\omega,Q} \in \mathrm{PSH}(X,\omega)$. ∎

PROPOSITION 3. *Let E be a Borel subset of X and P a* $\mathrm{PSH}(X,\omega)$*-polar set. Then*

$$V^*_{E\cup P,\omega,Q} = V^*_{E,\omega,Q}.$$

Proof. Recall that a set P is said to be $\mathrm{PSH}(X,\omega)$-*polar* if it is included in the $-\infty$-locus of some function $\psi \in \mathrm{PSH}(X,\omega)$ which is not identically $-\infty$ on X. By Prop. 1(ii) we have $V^*_{E\cup P,\omega,Q} \le V^*_{E,\omega,Q}$. We will show that $V^*_{E,\omega,Q} \le V^*_{E\cup P,\omega,Q}$. Suppose $u \in \mathrm{PSH}(X,\omega)$ with $u \le Q$ on E and let $v \in \mathrm{PSH}(X,\omega)$ be such that $P \subset \{v = -\infty\}$. We may assume $v \le Q$ on E. Then for each $\varepsilon > 0$,

$$(1-\varepsilon)u + \varepsilon v \le V_{E\cup P,\omega,Q} \le V^*_{E\cup P,\omega,Q}.$$

Therefore $u \le V^*_{E\cup P,\omega,Q}$ on X and by taking the supremum, $V^*_{E,\omega,Q} \le V^*_{E\cup P,\omega,Q}$. ∎

Now we will discuss how weighted pluripotential theory on $\mathbb{C}^N$ can be extended to a suitable weighted pluripotential theory on $\mathbb{CP}^N$. Recall that in the weighted theory on $\mathbb{C}^N$ one begins with an admissible weight function on a closed set $K \subset \mathbb{C}^N$. An admissible weight w is a nonnegative upper semicontinuous function w on $\mathbb{C}^N$ with $\{z \in K : w(z) > 0\}$ nonpluripolar and satisfying the boundedness condition $\lim_{|z|\to\infty} |z|w(z) = 0$ if K is an unbounded set (cf. [BL], [Bl1], [ST]). Let $Q = -\log w$. Then the weighted pluricomplex Green function of K is defined as

$$V_{K,Q} = \sup\{u \in \mathcal{L} : u \le Q \text{ on } K\}.$$

Let $[Z_0 : \ldots : Z_N]$ be homogeneous coordinates in $\mathbb{CP}^N$ and $z_{j,k} := Z_k/Z_j$ in $U_j = \{Z_j \neq 0\}$. (The set U_0 is identified with $\mathbb{C}^N$ and $z_{0,k} =: z_k$, $k = 1, \ldots, N$, are affine coordinates.) In these coordinates, let $\widetilde{w}(Z_0 : \ldots : Z_N) = w(z_1, \ldots, z_N)/|Z_0|$ in U_0, where w is nonnegative and upper semicontinuous with $\{w > 0\}$ nonpluripolar, but not necessarily satisfying the boundedness condition. The expression $W(Z) := \|Z\|\widetilde{w}(Z)$ defines a homogeneous function of order 0 in $\mathbb{C}^{N+1} \setminus \{Z_0 = 0\}$. We have $W(Z) = \varphi_0(z) + \log w(z)$ for $Z_0 \neq 0$, where $\varphi_(z) = (1/2)\log(1 + |z|^2)$. To obtain an upper semicontinuous function (still denoted by W) globally on $\mathbb{CP}^N$, with all values greater than or equal to 0, we take

$$\sqrt{|Z_1|^2 + \cdots + |Z_N|^2}\, \widetilde{w}(0 : Z_1 : \ldots : Z_N) = \limsup_{0 \neq Y_0 \to 0,\, Y_j \to Z_j} \|Y\|\widetilde{w}(Y)$$

for $Y = (Y_0, \ldots, Y_N)$.

The boundedness condition is equivalent to the property that this global function is identically zero on the hyperplane $\{Z_0 = 0\}$. This is because $\lim_{|z|\to\infty} |z|w(z) = \lim_{|z|\to\infty} \sqrt{1 + |z|^2} w(z)$. We will assume a weaker condition, namely that W is bounded in $\mathbb{CP}^N$. The following example demonstrates that the boundedness condition is too restrictive when constructing a weighted pluripotential theory on complex manifolds.

Example 1. Let ω_{FS} be the Fubini–Study Kähler form on $X = \mathbb{CP}^N$ with local potentials $\phi_j = (1/2)\log(1 + \sum_{k \neq j} |z_{j,k}|^2)$ in the coordinate neighborhoods $U_j = \{Z_j \neq 0\}$ with $j = 0, 1, \ldots, N$, and let K be a subset of $\mathbb{C}^N \subset \mathbb{CP}^N$. For $Z \in \mathbb{CP}^N$ define $Q_j(Z) = \phi_j(Z)$, $j = 0, \ldots, N$, so that $Q_0(z) = \log(\sqrt{1 + \|z\|^2})$ for $z \in \mathbb{C}^N$. The natural 1-to-1 correspondence between $\mathrm{PSH}(X, \omega_{\mathrm{FS}})$ and the class $\mathcal{L}(\mathbb{C}^N)$ of plurisubharmonic functions with logarithmic growth at infinity, presented explicitly in Example 1.2 in [GZ1], gives the following:

$$\begin{aligned} V_{K,Q_0}(x) &= \sup\{u(x) : u \in \mathcal{L}(\mathbb{C}^N),\ u(z) \le \log\sqrt{1 + \|z\|^2}\ \ \forall z \in K\} \\ &= \sup\{u(x) : u \in \mathcal{L}(\mathbb{C}^N),\ u(z) - (1/2)\log(1 + |z|^2) \le 0\ \forall z \in K\} \\ &= \sup\{v(x) + (1/2)\log(1 + |x|^2), v \in \mathrm{PSH}(\mathbb{CP}^N, \omega_{\mathrm{FS}}) : v|_K \le 0\} \\ &= V_{K,\omega_{\mathrm{FS}}}(x) + (1/2)\log(1 + |x|^2) \end{aligned}$$

for every $x \in \mathbb{C}^N$. Assume now that K is not $\mathrm{PSH}(\mathbb{CP}^N, \omega_{\mathrm{FS}})$-polar. Then $V^*_{K,\omega_{\mathrm{FS}}} \in \mathrm{PSH}(\mathbb{CP}^N, \omega_{\mathrm{FS}})$ and $V^*_{K,Q_0} \in \mathcal{L}(\mathbb{C}^N)$. For a point Z on the hyperplane at infinity $\{Z_0 = 0\}$ we get

$$V^*_{K,\omega_{\rm FS}}(Z) = \limsup_{x\to Z,\, x\in\mathbb{C}^N} (V^*_{K,Q_0}(x) - (1/2)\log(1+|x|^2)).$$

Note that the function $w(z) = \exp(-Q_0(z))$ in our example does not satisfy the boundedness condition in $\mathbb{C}^N$. Indeed, the function $\|Z\|\widetilde{w(z)} = \exp(-Q_j(Z)+\phi_j(Z))$ for $Z \in U_j$, $j = 0,\dots,N$, is a constant function 1 on $\mathbb{CP}^N$. We draw the reader's attention to the paper [Bl2], in which a relation between weighted theory in $\mathbb{C}^N$ and standard pluripotential theory in $\mathbb{C}^{N+1}$ is outlined. Examples considered in Section 5 of that paper deal with a weight function w which is given as the Hartogs radius of a domain with balanced fibers in $\mathbb{C}^{N+1}$ (for the definition and basic properties, see [Sh]). Such a function is upper semicontinuous, but as shown in [Bl2], does not have to satisfy the boundedness condition on $\mathbb{C}^N$. Furthermore, the results of [Si2] as well as [MS] were obtained without assuming the boundedness condition. It thus seems reasonable to weaken this condition when working on complex manifolds. In [Gu] a notion of a "convex" hull with respect to a closed real $(1,1)$-current T is considered where the functions f defining the hull satisfy the condition that $\exp(f+\phi)$ are continuous, with ϕ continuous local potentials for T. We adopted an analogous condition as part of our definition of a mild function.

The method demonstrated in Example 1 can also be used to prove the following:

PROPOSITION 4. *Let $K \subset \mathbb{C}^N \cong \{Z_0 \neq 0\}$. For a mild function Q on $\mathbb{CP}^N$ with respect to $\omega = \omega_{\rm FS}$ define*

$$q(z_1,\dots,z_N) = q(Z_1/Z_0,\dots,Z_N/Z_0) = Q(Z) - \log(\|Z\|/|Z_0|), \qquad Z_0 \neq 0.$$

Conversely, for a lower semicontinuous q on $\mathbb{C}^N$, consider

$$Q(Z) = q(Z_1/Z_0,\dots,Z_N/Z_0) + \log\|Z\| + \log|Z_0|,$$

together with its lower semicontinuous regularization as $Z_0 \to 0$. Then for all $x \in \mathbb{C}^N$,

$$V_{K,q}(x) = V_{K,\omega,Q}(x) - (1/2)\log(1+\|x\|^2).$$

Consider now a holomorphic line bundle L over a compact Kähler manifold X. Recall that a (singular) metric on L can be given (cf. [De], [DPS]) by a collection of real-valued functions $h = \{h_j\}$ on X, defined in a trivializing cover $\{U_j\}$, such that $h_j = h_i + \log|g_{ij}|$, where g_{ij} are transition functions for L. The metric is called *positive* if all h_j are plurisubharmonic. (The notion of positivity is used here in the weak sense.) In particular, a smooth metric $\{\phi_j\}$ such that $\omega = dd^c\phi_j$ is a Kähler form will be positive.

If L is a positive line bundle and $\omega = [c_1(L)]$, there is a 1-to-1 correspondence between the family of all positive metrics on L and the class

PSH(X,ω). In the case of $X = \mathbb{CP}^N$ with the Fubini–Study form ω, this correspondence is equivalent to the H-principle due to Siciak ([Si3]).

Proposition 5 (cf. [Gu, property (iv), p. 456]). *Let h be a logarithmically homogeneous plurisubharmonic nonnegative function on $\mathbb{C}^{N+1}$. Then h defines a positive metric on $\mathbb{CP}^N$. Conversely, each positive metric on $\mathbb{CP}^N$ defines a logarithmically homogeneous psh function on $\mathbb{C}^{N+1}$.*

Proof. By logarithmic homogeneity we have

$$v(Z_0/Z_k,\dots,1,\dots,Z_N/Z_k) = v(Z) - \log|Z_k| \quad \text{in } \{Z_k \neq 0\}.$$

Hence $v_k = v_i + \log|Z_k/Z_i|$ in $U_i \cap U_k$ and all v_i are plurisubharmonic. To prove the converse, take $h_0 = h|_{U_0}$. The function $v(Z) = h_0(Z) + \log|Z_0|$ in U_0, and $v(0,Z_1,\dots,Z_N) = \limsup_{Y_0\to 0,\, Y_j\to Z_j} v(Y_0,Y_1,\dots,Y_N)$ is plurisubharmonic. Since it also satisfies $v(\lambda Z) = v(Z) + \log|\lambda|$ for $\lambda \in \mathbb{C}$, our proof is complete. ∎

By Example 1.2 in [GZ1], the class $\mathcal{L}(\mathbb{C}^N)$ corresponds in a 1-to-1 manner to the class of PSH$(\mathbb{CP}^N,\omega)$ functions, which in turn correspond in a 1-to-1 manner to positive metrics on the (positive) hyperplane bundle over $\mathbb{CP}^N$. Thus Proposition 5 establishes a 1-to-1 correspondence between logarithmically homogeneous functions $\widetilde{v}$ on $\mathbb{C}^{N+1}$ and functions v in the class $\mathcal{L}(\mathbb{C}^N)$ so that $\widetilde{v}(1,z) = v(z)$ for $z \in \mathbb{C}^N$, that is, the H-principle. This well-known correspondence has been utilized in many works, most recently in [Be3].

If L is a positive line bundle over X, then its dual L' is negative ([GF, Prop. VI.6.1 and VI.6.2]). Hence there exists a system of trivializations $\theta_i : L'|_{U_i} \to U_i \times \mathbb{C}$ with transition functions $G_{ik} = g_{ik}^{-1} = g_{ki}$ and a smooth metric $\{h_i\}$ on L such that the smooth function $\chi_h : L' \to \mathbb{R}$, defined as $\chi_h \circ \theta_i^{-1}(x,t) = H_i(x)\cdot|t|^2$, is strictly plurisubharmonic outside the zero section of L', where $H_i(x) = \exp 2h_i(x)$, $x \in U_i$. As a simple example of a negative line bundle we can take the universal line bundle over $\mathbb{CP}^N$, $\mathcal{O}(-1) := \{([Z],\xi) \in \mathbb{CP}^N \times \mathbb{C}^N : \xi \in \mathbb{C}\cdot Z,\ Z \in \mathbb{C}^{N+1}\setminus\{0\},\ [Z] = \mathbb{C}^*\cdot Z\}$. That is, the fiber of $\mathcal{O}(-1)$ over a point $[Z] \in \mathbb{CP}^N$ is the complex line in $\mathbb{C}^{N+1}$ generated by $(Z_0,\dots,Z_N)$. The function $\chi\circ\theta_i^{-1}(Z,t) = |t|^2|Z_i|^{-2}\|Z\|^2$ for $Z_i \neq 0$, associated with the Fubini–Study metric on the dual line bundle $\mathcal{O}(1)$ over $\mathbb{CP}^N$, is plurisubharmonic.

The above characterization of negative line bundles as those whose zero section has a strongly pseudoconvex neighborhood (due to Grauert, see [Gr]) leads to the following generalization of Siciak's H-principle:

Theorem 1 (cf. [GF, Prop. VI.6.1]). *Let L be a positive line bundle over a compact Kähler manifold X and let $d > 0$. Let $\mathcal{H}_d^+$ denote the family of all functions $\chi \in$ PSH(L') which are nonnegative, not identically 0 and* *absolutely homogeneous of order d in each fiber. Then there is a one-to-one correspondence between $\mathcal{H}_d^+$ and the class of positive metrics on L.*

Proof. Consider a system of trivializations $\theta_i : L'|_{U_i} \to U_i \times \mathbb{C}$ with transition functions $G_{ik} = g_{ki} = 1/g_{ik}$. Let $\chi \in \mathcal{H}_d^+$. For $x \in U_i$, $t \neq 0$ define

$$H_i(x) := \chi \circ \theta_i^{-1}(x,t)/|t|^d.$$

Note that this expression does not depend on t. We have $\chi \circ \theta_i^{-1}(x,t) = \chi \circ \theta_k^{-1}(x, G_{ki}(x)t)$, hence by absolute homogeneity of order d, $H_k(x) = |G_{ki}(x)|^d H_i(x)$ in $U_i \cap U_k$. Taking $h_i = (1/d)\log H_i$ in U_i we get a collection of plurisubharmonic functions satisfying $h_k = \log|g_{ik}| + h_i$, i.e., a positive metric on L. Conversely, let $\{h_i\}$ be a metric on L. The function χ on L' defined as $\chi \circ \theta_i^{-1}(x,t) = \exp(dh_i(x)) \cdot |t|^d$ is plurisubharmonic if and only if the h_i are, so for a positive metric the associated function χ is in $\mathcal{H}_d^+$. ∎

Unless otherwise indicated, we will work with $\mathcal{H}^+ := \mathcal{H}_1^+$. Note that if we take L' in Theorem 1 to be the universal line bundle $\mathcal{U}$ over $\mathbb{CP}^N$, then the trivialization $\theta_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}$ is given as $\theta_i(t(Z)) = ([Z_0 : \ldots : Z_N], tZ_i)$. Hence for a function $\chi \in \mathcal{H}^+$ we have $\chi \circ \theta_i^{-1}([Z_0 : \ldots : Z_N], t) = h_i([Z_0 : \ldots : Z_N]) + \log|Z_i| + \log|t|$ for $Z_i \neq 0$, where the h_i define a metric on $\mathbb{CP}^N$. By Proposition 5, over the chart $Z_0 \neq 0$ we get $\chi(tZ) = v(Z_1/Z_0, \ldots, Z_N/Z_0) + \log|t|$ for $t \neq 0$ with v plurisubharmonic. That is, χ defines a logarithmically homogeneous psh function on $\mathbb{C}^{N+1}$.

For a positive holomorphic line bundle L over a compact Kähler manifold X there is a precise relation between the weighted pluricomplex Green function with respect to $\omega = [c_1(L)]$ of a Borel set K in X and an $\mathcal{H}^+$-envelope of some associated set $\widetilde{K}$ in the dual bundle L'. It generalizes the formulas obtained by Bloom in [Bl2].

For the weight Q on X consider the collection $q_i = Q - \phi_i$, where $\omega = dd^c\phi_i$ in U_i and the U_i form a trivializing cover for L. For $K \subset X$ define $\widetilde{K} \subset L'$ by taking

$$\widetilde{K} \cap \pi^{-1}(U_i) = \{\theta_i^{-1}(x,t) : x \in U_i \cap K,\ |t| = \exp(-q_i(x))\}.$$

This set is well defined, since $\theta_k^{-1}(x,t) = \theta_i^{-1}(x, G_{ki}(x)t)$. Hence if $x \in U_i \cap U_k \cap K$, then $|G_{ki}(x)t| = \exp(-q_i(x))$ if and only if $|t| = \exp(-q_k(x))$. Consider

$$H_{\widetilde{K}} = \sup\{u \in \mathrm{PSH}(L') : \exp u \in \mathcal{H}^+,\ u|_{\widetilde{K}} \leq 0\}.$$

The following theorem gives the relationship between the functions $H_{\widetilde{K}}$ and $V_{K,\omega,Q}$.

Theorem 2 (cf. [Bl2, Thm. 2.1]). *For all i,*

$$H_{\widetilde{K}} \circ \theta_i^{-1}(x,t) = V_{K,\omega,Q}(x) + \log|t| + \phi_i(x).$$

Proof. By Theorem 1,

$$H_{\widetilde{K}} = \sup\{u : u \circ \theta_i^{-1}(x,t) = h_i(x) + \log|t|,\ u|_{U_i\cap K} \le 0\}$$
$$= \sup\{u : u \circ \theta_i^{-1}(x,t) = h_i(x) + \log|t|,\ h_i(x) \le q_i,\ \forall i\}$$

where the h_i define a positive metric on L. Hence, for such h_i,

$$H_{\widetilde{K}} \circ \theta_i^{-1}(x,t) = \sup\{h_i(x) : h_i(x)|_{K\cap U_i} \le q_i\} + \log|t|$$
$$= \sup\{v(x) + \phi_i(x) : v \in \mathrm{PSH}(X,\omega),\ v|_K \le Q\} + \log|t|$$
$$= V_{K,\omega,Q}(x) + \log|t| + \phi_i(x),\ \forall i. \ \blacksquare$$

Theorem 2 allows us to study the behavior of the weighted pluricomplex Green functions as we vary the weight. Namely, we have the following:

PROPOSITION 6 (cf. [Bl2, Cor. 2.2]). *Let $K \subset X$ be a Borel set. Suppose $\{Q_n\}, Q_0$ are mild functions with $Q_n \nearrow Q_0$. Then* $\lim_{n\to\infty} V_{K,\omega,Q_n} = V_{K,\omega,Q_0}$.

Proof. Consider the sets $K_n, M_n \subset L'$, where

$$M_n \cap \pi^{-1}(U_i) = \{\theta_i^{-1}(x,t) : x \in U_i \cap K, |t| \le \exp(-q_i^{(n)}(x))\},$$
$$K_n \cap \pi^{-1}(U_i) = \{\theta_i^{-1}(x,t) : x \in U_i \cap K, |t| = \exp(-q_i^{(n)}(x))\},$$

and $q_i^{(n)} = Q_n - \phi_i, n \ge 0$. The sequence M_n is decreasing, with $\bigcap_{n=1}^{\infty} M_n = \widetilde{M_0}$. By the maximum principle (applied in each fiber), $H_{M_n} = H_{K_n}$ for $n \ge 0$ (here we use the assumption of all Q_n being mild). For a function $u \in \mathcal{H}^+$ such that $u \le 0$ on M_0 and an arbitrary $\varepsilon > 0$, there exists an n_0 such that for all $n \ge n_0$ we have $M_n \subset \{u < \varepsilon\}$. The function $u - \varepsilon$ is in $\mathcal{H}^+$ and for $n \ge n_0$ it satisfies $u - \varepsilon \le H_{M_n} \le \lim_{n\to\infty} H_{M_n} \le H_{M_0}$, hence $\lim_{n\to\infty} H_{M_n} = H_{K_0}$. By Theorem 2, $\lim_{n\to\infty} V_{K,\omega,Q_n} = V_{K,\omega,Q_0}$. ■

PROPOSITION 7 (cf. [Bl2, Cor. 2.4]). *Let Q_n, $n \ge 0$, be mild functions on X such that $Q_n \searrow Q_0$. Then $V_{K,\omega,Q_0} = \lim_{n\to\infty} V_{K,\omega,Q_n}$.*

Proof. Since the potentials ϕ_i of ω are continuous, we have $H^*_{\widetilde{K}} \circ \theta_i^{-1}(x,t) = V^*_{K,\omega,Q_0} + \log|t| + \phi_i$ for all i. We can assume that the set M_1 (see Proposition 6) is not ω-polar. By Proposition 2, $H^*_{K_0}$ is plurisubharmonic on L'. Let $H = \lim_{n\to\infty} H_{M_n}$. The function H is in $\mathcal{H}^+$ and satisfies $H \le 0$ on $K_0 \setminus P$, where P is some pluripolar set. Hence $H \le H^*_{K_0}$. ■

COROLLARY 1. *The conclusion of Proposition* 6 *holds when the convergence $Q_n \nearrow Q_0$ takes place quasi-everywhere on X, that is, outside some ω-polar set.*

COROLLARY 2. *The conclusion of Proposition* 7 *holds when the convergence $Q_n \searrow Q_0$ takes place quasi-everywhere on X.*

2. Approximation and pullbacks by holomorphic maps. In standard pluripotential theory in $\mathbb{C}^N$ and its weighted generalization there is a function Φ_K such that $\log \Phi_K = V_{K,Q}$. The function Φ_K is given as the supremum of certain functions with "regular" growth, that is, polynomials (when $Q \equiv 0$) or weighted polynomials (see Theorem 6.2 in [Si1], Theorem 2.8 in [Bl1], and Théorème 5.1 in [Ze]). In [GZ1] it is proven that $V_{K,\omega}(x) = \sup\{(1/n)\log \|s\|_{n\varphi}(x) : n \geq 1,\ s \in \Gamma(X, L^n),\ \sup_K \|s\|_{n\varphi} \leq 1\}$, where L is a positive holomorphic line bundle over a compact manifold X, $\omega = dd^c\varphi_j$ in a trivializing cover U_j is a (global) Kähler form and the norm $\|s\|_{n\varphi}$ of a section s of the tensor power L^n is computed as follows: $\|s\|_{n\varphi} = |s_j| \exp(-n\varphi_j)$ in U_j. All such theorems are based on the possibility of approximation of general plurisubharmonic functions by so-called Hartogs functions, which are obtained by certain operations from functions of the type $\log |f|$ with f holomorphic (cf. [Kl, Theorem 5.1.6]). Such approximation may not always be possible, but is possible for example in pseudoconvex domains in $\mathbb{C}^N$, as shown in [Bre]. Below, we will work in pseudoconvex neighborhoods of the zero section of L' to prove the following:

THEOREM 3. *Let X, L, φ, ω be as above. Let Q be a mild function on X and let K be a compact subset of X. Then*

$$V_{K,\omega,Q} = \log \Phi_{K,\omega,Q} \quad \textit{where} \quad \Phi_K(x) = \sup_{n\geq 1} (\Phi_n(x))^{1/n}$$

with

$$\Phi_n(x) = \sup\{\|s\|_{n\varphi}(x) : n \geq 1,\ s \in \Gamma(X, L^n),\ \sup_K \exp(-nQ)\|s\|_{n\varphi} \leq 1\}.$$

Unlike [GZ1], in which the theorem was proved for $Q \equiv 0$, we will not use L^2-estimates for the $\bar{\partial}$-operator. Instead, we will apply the Approximation Lemma (see below), which we will prove using an argument that can be traced back to [Bre]. Similar lemmas, e.g. [Ze, Lemme 5.2] and [Be1, Lemma 2.1 and 3.2], were proved for Stein manifolds. However, a neighborhood of the zero section of a negative line bundle L' cannot be Stein (since it contains the zero section as a compact complex submanifold), so an extra effort is needed to make the argument work in our case. This is achieved by blowing down the zero section, as shown below.

APPROXIMATION LEMMA. *Let X, ω, L be as above and let $v \in \mathrm{PSH}(X,\omega) \cap \mathcal{C}^\infty$ be such that $dd^c v + \omega$ is strictly positive. Then for every $\varepsilon > 0$ and every compact $K \subset X$ there exist $N_1, \ldots, N_m$ and $s_1, \ldots, s_m$ such that $s_j \in \Gamma(X, L^{N_j})$, $j = 1, \ldots, m$, and*

$$v(x) - \varepsilon \leq \sup_{1\leq j\leq m} (1/N_j) \log \|s_j(x)\|_{N_j\varphi} \leq v(x) \quad \textit{for all } x \in K,$$

where the norm of the section s_j is computed as above.

Proof. Let φ_i be local potentials for the Kähler form ω, $[\omega] = [c_1(L)]$, and let $h = \{h_i = v + \varphi_i\}$ be the positive metric corresponding to v. The inequality in the statement of the lemma is equivalent to

$$h_i - \varepsilon \leq \sup_{1\leq j\leq m} (1/N_j)\log|s_j(x)| \leq |h_i(x)|, \qquad x \in K \cap U_i,\ i = 1,\dots,l,$$

where $|\cdot|$ is the usual absolute value of a complex number. Let $r \in (0,1)$ and let χ_r be the function in the class $\mathcal{H}^+$ on L' associated with the metric $r\cdot h$. For every r the set $\Omega_r = \{\chi_r < 1\}$ is a strictly pseudoconvex neighborhood of the zero section in L' (cf. [GF, VI.6.1]). Fix a point $x_0 \in K$ and $\zeta_0 = \theta_i^{-1}(x_0, 1)$. Then $|t| < \chi_r(\zeta_0)$ if and only if $(x_0, t) \in \Omega := \Omega_r$. The function $f(t) = \sum_{n=1}^{\infty}(\chi_r(\zeta_0))^n t^n$, $|t| < 1/\chi_r(\zeta_0)$, $f(0) = 0$, is holomorphic on the analytic set $(\Omega \cap L'_{x_0}) \cup X$ and is identically 0 on X.

Let us consider the Remmert reduction of Ω (see [Gr, Theorem 1], or [P, Theorem 2.7 and preceding discussion]). That is, we have a Stein space Y and a proper surjective holomorphic map $\Phi : \Omega \to Y$ with the following properties: (i) Φ has connected fibers; (ii) $\Phi_*(\mathcal{O}_\Omega) = \mathcal{O}_Y$; (iii) the canonical map $\mathcal{O}_Y(Y) \to \mathcal{O}_\Omega(\Omega)$ is an isomorphism; (iv) if $\sigma : \Omega \to Z$ is a holomorphic map into a Stein space Z then there exists a uniquely determined holomorphic map $\tau : Y \to Z$ such that $\tau \circ \Phi = \sigma$. The map Φ blows down the zero section of L'. Note that the set $A = \Phi(L'_{x_0} \cup X) = \Phi(L'_{x_0})$ is analytic in Y (by Remmert's proper mapping theorem) and the function $\widetilde{f}(\Phi(t)) := f(t)$ is holomorphic on A (by property (ii) of Remmert's reduction). Every analytic set in a complex space is the support of a closed complex subspace (cf. [GR, A.3.5]), so we can apply Theorem V.4.4 of [GR] to conclude that the function $\widetilde{f}$ is the restriction to A of a function $\widetilde{F}$ that is holomorphic on the Stein space Y.

By the properties (ii) and (iii) above, there exists a function F holomorphic on Ω such that $\widetilde{F} \circ \Phi = F$. For $t \neq 0$ one can represent F as $F \circ \theta_i^{-1}(x,t) = \sum_{n=1}^{\infty} F_n^{(i)}(x)t^n$, with $F_n^{(i)}$ holomorphic in U_i. We have $F \circ \theta_k^{-1}(x,t) = F \circ \theta_i^{-1}(x, G_{ik}(x)t)$, which gives $F_n^{(i)}(x) = (g_{ik}(x))^n F_n^{(k)}(x)$, i.e., F_n are cocycles corresponding to holomorphic sections of the tensor product L^n over Ω_r.

Considering the domain of convergence of the representation for $F \circ \theta_k^{-1}$, $k = 1,\dots,l$, we get $\limsup_{n\to\infty} |F_n(x)|^{1/n} \leq \exp rh(x)$, $x \in X$. Let $\delta > 0$. By Hartogs's lemma, there exists an $n_\delta > 1$ such that $(1/n)\log|F_n(x)| \leq r\cdot h(x) + \delta$, $x \in K$, $n \geq n_\delta$. For the estimate from below, note that $F_n(x_0) = \chi_r(\zeta_0) = rh(x_0)$ for all n. Since $rh = r(v + \varphi)$ is continuous, there exists an $n_0 \geq n_\delta$ and a neighborhood W_{x_0} of x_0 such that $(1/n_0)\log|F_{n_0}(x)| > rh(x) - \delta$, $x \in W_{x_0}$. Compactness of K and suitable relations between ε, δ and r give holomorphic sections satisfying the conclusion of the lemma. ∎

Proof of Theorem 3. We mimic the method of proof of Theorem 2.8(i) in [Bl1]. Let $u \in \mathrm{PSH}(X,\omega)$, $u|_K \le Q$. By Theorem 7.1 in [GZ1], there is a sequence $u_k \in \mathrm{PSH}(X,\omega) \cap \mathcal{C}^\infty(X)$ such that $u_k \searrow u$. Let $\varepsilon > 0$. By Dini's theorem, there exists an integer k_0 such that $u(x) \le u_k(x) \le Q(x) + \varepsilon$ for all $x \in K$, $k \ge k_0$. By adding a small multiple of a local Kähler potential in some coordinate neighborhood, we can assume that $dd^c u_k + \omega$ is strictly positive. By the Approximation Lemma, there exist $s_j^{(k)} \in \Gamma(X, L^{N_j^{(k)}})$, $j = 1, \dots, m_k$, such that

$$u_k - 3\varepsilon \le \sup_{j=1,\dots,m_k} \log|\exp(-2N_j^{(k)}\varepsilon s_j^{(k)})|/N_j^{(k)} \le (1/n)\log \varPhi_n(x),$$

where $n = \max_j N_j^{(k)}$, $j = 1, \dots, m_k$. Hence $u - 4\varepsilon \le \log \varPhi$. The reverse inequality is obvious, since $(1/N)\log\|s\|_{N\varphi}$ defines a positive singular metric on L. ∎

Under the assumptions of Theorem 3 we also have the following:

Proposition 8. *Let* $\varPsi(x) = \lim_{n\to\infty} \psi_n(x) = \sup_{n\ge 1} \psi_n(x)$, *with*

$$\psi_n(x) = \sup\{\|s\|_{n\varphi}(x), s \in \Gamma(X, L^n), \sup_K{}^{\circ}(\exp(-nQ)\|s\|_{n\varphi}) \le 1\}$$

and $\sup_K^{\circ}(f) := \inf\{\sup_{K\setminus P}(f) : P \subset K,\ P$ *is* $\mathrm{PSH}(X,\omega)$*-polar*$\}$. *Then*

$$V^*_{K,\omega,Q} = (\log \varPsi_K)^*.$$

The proof proceeds exactly like that of [Bl1, Theorem 2.8(ii)], provided we have the domination principle on a compact Kähler manifold of dimension N (cf. [Kl, Cor. 3.7.5 and Prop. 5.5.1], [BT1, Cor. 4.5], [Ta] for versions on open subsets of $\mathbb{C}^N$). In our proof we will assume that one of the functions is in $L^\infty(X)$, since this is the case we need. A more general version was recently proved independently as Proposition 2.7 in [BB]. Proofs of the domination principle rely on the comparison principle, which was established in [GZ2] for the class of functions $\mathcal{E}(X,\omega)$ defined therein, which contains $L^\infty(X)$ (cf. also [Ko1], [Ko2], [HKH] for proofs in the case of L^∞-functions on manifolds). Recall the following result, which allows us to apply the comparison and domination principles in the weighted theory:

Proposition 9. *If K is not* $\mathrm{PSH}(X,\omega)$*-polar and Q is continuous*, *then* $V^*_{K,\omega,Q} \in \mathrm{PSH}(X,\omega) \cap L^\infty(X)$. *In particular*, *the complex Monge–Ampère operator* $(\omega_{V_{K,Q}})^n$ *is well defined and satisfies* $(\omega_{V_{K,Q}})^N = 0$ *in* $X \setminus \overline{K}$.

Proof. The proof proceeds as that of [GZ1, Theorem 4.2.2], and uses Proposition 2. ∎

Now we may state and prove the required domination principle.

THEOREM 4 (Domination principle). *Let* $u, v \in \mathrm{PSH}(X, \omega)$ *with* $v \in L^\infty(X)$ *be such that*

$$\int_{\{u<v\}} (\omega + dd^c u)^N = 0.$$

Then $u \ge v$ *in* X.

Proof. The following argument was communicated to us by Ahmed Zeriahi. It is enough to prove that $u \ge v$ on a set of full ω-volume in X. We can assume that v is negative everywhere on X. Then for all $s, t > 0$, $\{u - v \le -s - t\} \subset \{u - v \le -s - tv\}$, which for small t is still a subset of $\{u - v < 0\}$. Then, by Lemma 2.2 in [EGZ],

$$0 = \int_{\{u-v<-s-tv\}} (\omega + dd^c u)^N \ge t^N \mathrm{Cap}\{u - v \le -s - t\},$$

where Cap is the Monge–Ampère capacity defined in [Ko1] (see also [GZ1, Definition 2.4]. Proposition 2.5(1) in [GZ1] implies that $\mathrm{Vol}\{u-v \le -s-t\} = 0$ for $s, t > 0$, t small, hence $\mathrm{Vol}\{u - v < 0\} = 0$. ∎

In Section 1 we referred to the definition of "polynomial convexity" with respect to a positive closed current ω of bidegree $(1, 1)$ on a complex manifold X introduced in [Gu]. When X is projective algebraic and $[\omega] = [c_1(L)]$, $\omega = dd^c\phi$ for a positive holomorphic line bundle L over X, this definition is equivalent to the following one:

DEFINITION 3 ([Gu, Definition 3.1 and Proposition 3.2]). Let K be a compact subset of X. The *ω-polynomial hull* of K is

$$\widehat{K}^\omega = \{x \in X : \|s\|_{n\phi}(x) \le \sup_K \|s\|_{n\phi} \ \forall s \in \Gamma(X, L^n)\ \forall n \in \mathbb{N}\}.$$

Directly from Theorem 3 we obtain the following:

COROLLARY 3. *For every compact* $K \subset X$ *and for every mild function* Q *on* X *we have*

$$V_{K,Q,\omega} = V_{\widehat{K}^\omega,Q,\omega}.$$

Finally, we are interested in how weighted pluricomplex Green functions change under a holomorphic map $f : X \to X$, where X is a compact Kähler manifold (not necessarily projective algebraic) with a closed real $(1, 1)$-current ω on X with continuous local potentials (not necessarily a Kähler form). Proposition 4.4.5 in [GZ1] states that if $f : X \to X$ is holomorphic, and $K \subset X$ is a Borel set, then $V_{f(K),\omega} \circ f \le V_{K,f^*\omega}$. The proof applies also to the weighted pluricomplex Green function and gives the following:

PROPOSITION 10. *Let* X, ω, K *be as above and let* Q *be a mild function on* X. *Then* $V_{f(K),\omega,Q} \circ f \le V_{K,f^*\omega,Q\circ f}$ *in* X.

Below, we establish a relation between the pullback of $V_{K,\omega,Q}$ by a surjective holomorphic map $f : X \to X$ and $V_{f^{-1}(K),\omega,\widetilde{Q}}$ with an appropriate function $\widetilde{Q}$. For a function $u : X \to \mathbb{R} \cup \{-\infty\}$ let us define $f_* u(x) = \sup\{u(y) : y \in f^{-1}(x)\}$. This is a well-defined function, since $f^{-1}(x)$ is compact. Also, let $f^* u = u \circ f$. The following theorem generalizes Theorem 2.12 in [Bra] and Theorem 1 in [St1] (it yields both as special cases):

THEOREM 5. *Assume that there exist α and β, $1 < \alpha \le \beta$, such that*

$$\alpha f_*(\mathrm{PSH}(X,\omega)) \subset \mathrm{PSH}(X,\omega), \qquad f^*(\mathrm{PSH}(X,\omega)) \subset \beta \cdot \mathrm{PSH}(X,\omega).$$

Then for every Borel set $K \subset X$ and every mild function Q on X,

$$\alpha V_{f^{-1}(K),\omega,f^*Q/\alpha}(x) \le V_{K,\omega,Q} \circ f(x) \le \beta V_{f^{-1}(K),\omega,f^*Q/\beta}.$$

Proof. Let $u \in \mathrm{PSH}(X,\omega)$ be such that $\alpha u \le f^* Q$ on $f^{-1}(K)$. Then $v = \alpha f_* u$ is in $\mathrm{PSH}(X,\omega)$ and satisfies $v \le Q$ on X. Moreover, $\alpha u(x) \le v(f(x)) \le V_{K,\omega,Q}(f(x))$, which gives the first inequality. For the second one, if $u \in \mathrm{PSH}(X,\omega)$ satisfies $u \le Q$ on K, then by assumption $(1/\beta) f^* u$ is in $\mathrm{PSH}(X,\omega)$ and $(1/\beta) f^* u \le (1/\beta) f^* Q$ on $f^{-1}(K)$, which gives the conclusion. ∎

On $X = \mathbb{CP}^N$, the assumptions of Theorem 5 are equivalent to assumptions about growth of f made in Theorem 2.12 in [Bra] or its unweighted counterpart, Theorem 5.3.1 in [Kl]. Details may be found in Theorem 1 in [St1] and its proof. The main theorem in [St2] has conditions equivalent to the assumption $\alpha f_* \mathrm{PSH}(X,\omega) \subset \mathrm{PSH}(X,\omega)$ when $X \hookrightarrow \mathbb{CP}^N$ is a projective algebraic manifold and ω is the pullback of the Fubini–Study form by the embedding $\hookrightarrow$. One of the conditions is that f has an attracting divisor in X, so in fact the assumption is quite strong.

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