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Involutions of real intervals

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Dedicated to Jorge Sotomayor for his 70th birthday

Abstract. This paper shows a simple construction of continuous involutions of real intervals in terms of continuous even functions. We also study smooth involutions defined by symmetric equations. Finally, we review some applications, in particular a characterization of isochronous potentials by means of smooth involutions.

1. Introduction. An *involution* is a function that is its own inverse. This is an important object in all mathematical fields. We are going to consider continuous involutions of real intervals only.

Proposition 1.1. *Let $h : J \to J$ be continuous function on the interval $J \subseteq \mathbb{R}$ which is the inverse of itself and does not coincide with the identity function id_J. Then h is strictly decreasing and has a unique fixed point $\bar{x} = h(\bar{x})$.*

Proof. The function h is strictly monotonic, being continuous and injective on an interval. Let us prove that h strictly decreases. Suppose it does not; then it is increasing. Since $h \neq \mathrm{id}_J$ we have $h(x_0) \neq x_0$ for some $x_0 \in J$. If $x_0 < h(x_0)$ we have $h(x_0) < h(h(x_0)) = x_0$, a contradiction; similarly, $x_0 > h(x_0)$ implies $h(x_0) > x_0$. Thus h decreases and the function $k(x) = x - h(x)$ strictly increases. The fixed points of h coincide with the zeros of k and there is one zero at most since k strictly increases. Consider a point $x_1 \in J$. If $k(x_1) = 0$ then x_1 is the unique fixed point of h. If $k(x_1) > 0$, then $x_1 > h(x_1)$ and $k(h(x_1)) = h(x_1) - h(h(x_1)) = h(x_1) - x_1 = -k(x_1) < 0$, so, by the continuity of k, there exists $\bar{x} \in (h(x_1), x_1)$ such that $k(\bar{x}) = 0$, that is, $\bar{x}$ is the fixed point of h. If $k(x_1) < 0$ we can argue similarly. ∎

These and other general properties are well known. Involutions are solutions to the celebrated Babbage functional equation $\phi^n(x) = x$, in the case

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$n = 2$; see the book [K, Chap. XV] by Kuczma, in particular Thms. 15.3 and 15.2, and Lemma 15.1. See also Kuczma, Choczewski and Ger [KCG, Chap. 11], and Section 1 of [PR, Chap. VIII] by Przeworska-Rolewicz where h as above is called a Carleman function.

For $\bar{x}$ as above, the function $x \mapsto h(x+\bar{x}) - \bar{x}$ is also an involution and has 0 as fixed point; conversely, $x \mapsto h(x-\bar{x}) + \bar{x}$ has fixed point $\bar{x}$ if $h(0) = 0$.

In the following we shall consider non-trivial involutions $h \neq \mathrm{id}_J$ with J an open interval and $h(0) = 0$; moreover we are going to study C^1 *smooth* involutions. By the chain rule we have $h'(h(x))h'(x) = 1$ so $h'(x) \neq 0$ at all x, more precisely $h'(x) < 0$ since we excluded the identity, and $h'(0) = -1$ necessarily. So, the present paper uses the following terminology:

Definition 1.2. A continuous function h of an open interval $J \subseteq \mathbb{R}$ onto itself is called an *involution* if

$$h^{-1} = h, \qquad 0 \in J, \qquad h(0) = 0, \qquad h \neq \mathrm{id}_J. \tag{1.1}$$

In particular it is called a *smooth involution* if $h \in C^1$, so it is a C^1 diffeomorphism with $h'(0) = -1$.

Of course $h(x) = -x$ is an involution on the whole $\mathbb{R}$. The following piecewise-linear example is taken from [PR, p. 177]:

$$h(x) = \begin{cases} -x/\lambda, & x \le 0, \\ -\lambda x, & x > 0, \end{cases} \qquad x \in \mathbb{R},\ \lambda > 0. \tag{1.2}$$

A very simple smooth involution which seems to be "new" is

$$h(x) = \ln(2 - e^x), \qquad x < \ln 2. \tag{1.3}$$

2. Constructing continuous involutions. Our main result is the following:

Theorem 2.1. *Let $h : J \to J$ be a (continuous) involution as in Definition 1.2. Then $k(x) := x - h(x)$ is a homeomorphism $J \to I$ with I a symmetric open interval, and the function $P : I \to \mathbb{R}$ defined by $P(y) = 2k^{-1}(y) - y$ satisfies $P(0) = 0$ and is even. Conversely, if $P : I \to \mathbb{R}$, with $P(0) = 0$, is a continuous even function on a symmetric open interval such that the function $K : I \to J$,*

$$K(y) = \tfrac{1}{2}(y + P(y)), \tag{2.1}$$

is a homeomorphism onto some J, then $h(x) := x - k(x)$, $k = K^{-1}$, is an involution on J.

Proof. If $h : J \to J$ is an involution then $k(x) := x - h(x)$ is strictly increasing as we already saw in the proof of Proposition 1.1, so a homeomorphism onto some open interval I, as is well known. The interval I is symmetric since

$$y = k(x) = x - h(x) \in I \;\Rightarrow\; -y = -k(x) = k(h(x)) \in I.$$

Next, $h(x) = k^{-1}(k(h(x))) = k^{-1}(-k(x))$ and

$$\begin{aligned} P(y) &:= 2k^{-1}(y) - y = -2y + 2k^{-1}(y) + y = -2k(k^{-1}(y)) + 2k^{-1}(y) + y \\ &= -2k^{-1}(y) + 2h(k^{-1}(y)) + 2k^{-1}(y) + y = 2k^{-1}(-y) + y = P(-y). \end{aligned}$$

This fact and $P(0) = 0$ prove the first assertion. To prove the second assertion let us plug $k(x) = K^{-1}(x)$ into (2.1), and then plug in $-k(x)$:

$$\begin{aligned} x &= \tfrac{1}{2}(k(x) + P(k(x))), \\ k^{-1}(-k(x)) &= \tfrac{1}{2}(-k(x) + P(-k(x))) = \tfrac{1}{2}(-k(x) + P(k(x))), \end{aligned}$$

so $x - k^{-1}(-k(x)) = k(x) = x - h(x)$. Thus $h(x) = k^{-1}(-k(x))$, which shows that h is a homeomorphism, $h \neq \mathrm{id}_J$ (otherwise $k = -k$ so $k \equiv 0$), and $h(h(x)) = x$ for all $x \in J$, that is, $h^{-1} = h$. Finally $P(0) = 0$ implies $k(0) = 0$ and $h(0) = 0$. ∎

Since $I = k(J)$ we have $I = (\inf J - \sup J, \sup J - \inf J)$ when $\inf J, \sup J \in \mathbb{R}$, and $I = \mathbb{R}$ otherwise.

Of course, if we consider an *arbitrary* C^1 even function P with $P(0) = 0$, then $P'(0) = 0$, and formula (2.1) restricted to the maximal symmetric open interval I where $K'(y) > 0$, defines a strictly increasing diffeomorphism onto an interval J, and $h(x) := x - k(x)$ with $k = K^{-1}$ is a smooth involution on J.

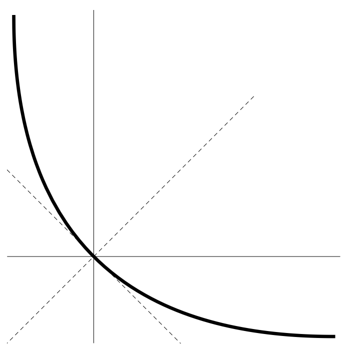

Fig. 1. Involution (2.2) given by $P(y) = y^2/8$

For instance, starting from $P(y) = y^2/8$, $y \in \mathbb{R}$, formula (2.1) defines $K(y) = y/2 + y^2/16$ and $K'(y) > 0$ if and only if $y > -4$. So K is injective on the symmetric open interval $I = (-4, 4)$ and a homeomorphism $I \to J = K(I) = (-1, 3)$. We have $k(x) = K^{-1}(x) = -4 + 4\sqrt{1 + x}$ and finally we get the involution $h(x) = x - k(x)$:

$$h : (-1, 3) \to (-1, 3), \qquad x \mapsto x + 4 - 4\sqrt{1 + x}. \tag{2.2}$$

If we start from $P(y) = y^6$ we have

$$k : J = \left(\frac{-5}{12 \cdot 6^{1/5}}, \frac{7}{12 \cdot 6^{1/5}}\right) \to I = \left(\frac{-1}{6^{1/5}}, \frac{1}{6^{1/5}}\right)$$

and $h : J \to J$ is the non-elementary algebraic function in Figure 2.

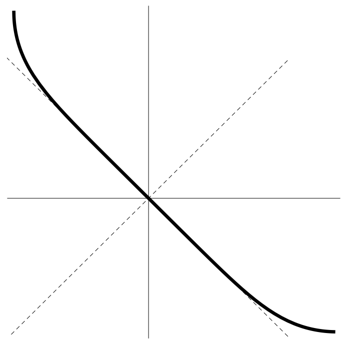

Fig. 2. Involution given by $P(y) = y^6$

To illustrate the first part of the theorem, consider the piecewise linear involution (1.2). It gives

$$k(x) = \begin{cases} (1+\lambda)x/\lambda, & x \le 0, \\ (1+\lambda)x, & x > 0, \end{cases} \qquad k^{-1}(y) = \begin{cases} \lambda\, y/(1+\lambda), & y \le 0, \\ y/(1+\lambda), & y > 0, \end{cases}$$

and the even function $P(y) = 2k^{-1}(y) - y$ is

$$P(y) = \frac{1-\lambda}{1+\lambda}|y|, \qquad y \in \mathbb{R},\ \lambda > 0. \tag{2.3}$$

Finally, the smooth involution (1.3) gives

$$k(x) = x - \ln(2 - e^x), \qquad k(-\infty, \ln 2) = \mathbb{R}, \qquad k^{-1}(y) = \ln \frac{2}{1+e^{-y}},$$

and the following even function:

$$P(y) = -y + 2\ln \frac{2}{1+e^{-y}} = -2\ln\cosh\frac{y}{2}, \qquad y \in \mathbb{R}. \tag{2.4}$$

Functional equations relating involutory and even functions are studied in Schwerdtfeger [S], but our simple Theorem 2.1 seems to be new.

3. Involutions given by symmetric equations. The condition $h^{-1} = h$ is equivalent to the symmetry of the graph of h with respect to the diagonal; indeed $(x, h(x))$ has $(h(x), x)$ as symmetric point and this coincides with the point $(h(x), h(h(x)))$ of the graph. For example, consider the hyperbola $y\,x = 1$, which is symmetric with respect to the diagonal. In order to fulfill the further condition $h(0) = 0$, we translate its point $(1, 1)$ to the origin. In this way we get $(y + 1)(x + 1) = 1$, which can be solved for y as $y = -x/(1+x)$. If we finally take the branch that goes through the origin we arrive at the involution

$$h(x) = -\frac{x}{1+x}, \qquad x \in J = (-1, \infty). \tag{3.1}$$

Involutions are preserved by homothety:

REMARK 3.1. Let $a \in \mathbb{R} \setminus \{0\}$ and h be an involution on (b, c). Then $\tilde{h}(x) = h(a\,x)/a$ is an involution on $(b/a, c/a)$ if $a > 0$, and on $(c/a, b/a)$ otherwise.

In this way (3.1) gives the following 1-parameter family of involutions:

$$h(x) = -\frac{x}{1+ax}, \qquad x \in J = \begin{cases} (-1/a, \infty), & a > 0, \\ (-\infty, \infty), & a = 0, \\ (-\infty, -1/a), & a < 0. \end{cases} \tag{3.2}$$

These are the only involutions that are rational functions of x, as shown in Cima, Mañosas, and Villadelprat [CMV].

Aczél [A] and Shisha and Mehr [SM] obtain injective functions $\hat{h} : \mathbb{R} \to \mathbb{R}$ such that $\hat{h}^{-1} = \hat{h}$ from symmetric functions $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = f(y, x)$. In [SM] it is assumed that for every $x \in \mathbb{R}$ there exists a unique y, to be denoted by $\hat{h}(x)$, such that $f(x, y) = 0$; then $\hat{h}$ satisfies $\hat{h}^{-1} = \hat{h}$. In particular, $f(x, y) = x^3 + y^3 - a$ gives $\hat{h}(x) = \sqrt[3]{a - x^3}$, which has the fixed point $\bar{x} = \sqrt[3]{a/2}$. The function $x \mapsto \hat{h}(x + \bar{x}) - \bar{x}$, i.e.

$$\mathbb{R} \to \mathbb{R}, \quad x \mapsto \sqrt[3]{a - (x + \sqrt[3]{a/2})^3} - \sqrt[3]{a/2}, \tag{3.3}$$

is an involution in the sense of Definition 1.2. For $a \neq 0$ it is non-differentiable at $x = \sqrt[3]{a} - \sqrt[3]{a/2}$.

We consider smooth symmetric equations in order to use the implicit function theorem:

PROPOSITION 3.2. *Let $f : \Omega \to \mathbb{R}$ be a C^1 function on the open set $\Omega \subseteq \mathbb{R}^2$ such that:* $(0, 0) \in \Omega$, $f(0, 0) = 0$, *and*

$$(x, y) \in \Omega \;\Rightarrow\; (y, x) \in \Omega \;\&\; f(y, x) = f(x, y). \tag{3.4}$$

Let Γ be the connected component of $f^{-1}(0)$ that contains the origin. Suppose that $\partial_2 f(x, y) \neq 0$ for all $(x, y) \in \Gamma$. Then Γ is the graph of a smooth involution h. All smooth involutions can be obtained this way.

Proof. By the implicit function theorem, Γ is the graph of a C^1 function h with $h(0) = 0$. Let J be the projection of Γ on the x-axis. It is an open interval since Γ is open and connected. From (3.4) we have $\partial_1 f(x, y) = \partial_2 f(y, x)$ for $(x, y) \in \Omega$ so $\partial_1 f$ never vanishes on Γ and has the same sign as $\partial_2 f$. We deduce that

$$h'(x) = -\frac{\partial_1 f(x, h(x))}{\partial_2 f(x, h(x))} < 0, \qquad x \in J,$$

and in particular $h'(0) = -1$. Finally, $f(h(y), y) = f(y, h(y)) = 0$ shows that $h^{-1} = h$.

Let us prove the last assertion. Let $h : J \to J$ be a smooth involution and define $f(x, y) := x + y - h(x) - h(y)$. This is a C^1 function on $J \times J$, $f(0, 0) = 0$, and $f(y, x) = f(x, y)$. We have $\partial_2 f(x, y) = 1 - h'(y) > 0$ for all $(x, y) \in J \times J$. The graph of h coincides with $f^{-1}(0)$. Indeed, if $y = h(x)$ then $x = h(y)$ and $f(x, y) = 0$; conversely, if $f(x, y) = 0$ then $k(y) = y - h(y) = -(x - h(x)) = -k(x)$, so $y = k^{-1}(k(y)) = k^{-1}(-k(x)) = k^{-1}(k(h(x)) = h(x)$. ■

For instance, let us consider the following function on the whole $\mathbb{R}^2$:

$$\tilde{f}(x, y) = ((x + 1)^3 + (y + 1)^3 - 2)(x + y + 2),$$

let $\tilde{\Gamma}$ be the cubic plane curve $\big(x + 1)^3 + \big(y + 1)^3 - 2 = 0$ and let L be the straight line $x+y+2 = 0$. The connected sets $\tilde{\Gamma}$ and L are disjoint, moreover $(0, 0) \in \tilde{\Gamma}$. We have $\partial_2 \tilde{f}(x, y) = 0$ for $(x, y) \in \tilde{\Gamma}$ when $y = -1$, that is, at the point $(c, -1) \in \tilde{\Gamma}$ with $c = \sqrt[3]{2} - 1$. So we define $\Omega = (-1, c) \times (-1, c)$, an open square with $(0, 0) \in \Omega$, and we see that the restriction $f = \tilde{f}|\Omega$ satisfies (3.4). The connected component of $f^{-1}(0)$ that contains the origin is $\Gamma = \tilde{\Gamma} \cap \Omega$, and $\partial_2 f(x, y) \neq 0$ for all $(x, y) \in \Gamma$. Therefore Γ is the graph of a smooth involution h. In this case we can even write an explicit formula which is the following restriction of the involution (3.3) for $a = 2$:

$$h : (-1, \sqrt[3]{2} - 1) \to (-1, \sqrt[3]{2} - 1), \qquad x \mapsto \sqrt[3]{2 - (x + 1)^3} - 1. \tag{3.5}$$

The thick curve in Figure 4 is the graph of this smooth involution; it is a piece of the non-smooth graph $\tilde{\Gamma}$ of Figure 3. The straight line below $\tilde{\Gamma}$ is L.

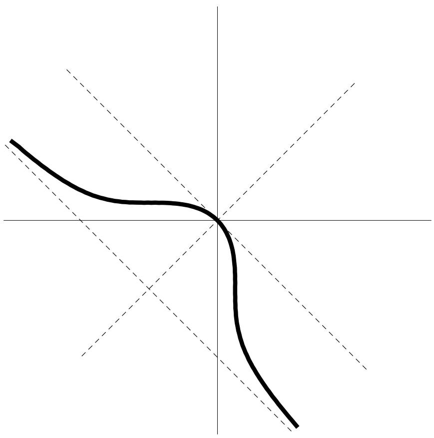

Fig. 3. Global involution (3.3) for $a = 2$

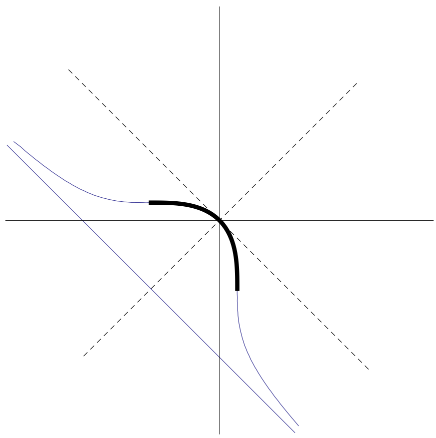

Fig. 4. Smooth involution in (3.5)

4. Isochronous potentials by involutions. An equilibrium point of a planar vector field is called a (local) *center* if all orbits in a neighborhood are periodic and enclose it. The center is *isochronous* if all periodic orbits have the same period. Smooth involutions can be used to construct isochronous centers for the scalar equation $\ddot{x} = -g(x)$, as proved in the 1989 paper [Z2] by the present author. There are other different approaches to such isochronous centers, which do not involve involutions, in particular Urabe's 1961 paper [U1] (see also [U2]).

THEOREM 4.1. *Let $h : J \to J$ be a smooth involution, let $\omega > 0$, and define*

$$V(x) = \frac{\omega^2}{8}(x - h(x))^2, \qquad x \in J. \tag{4.1}$$

Then the origin is an isochronous center for $\ddot{x} = -g(x)$, where $g(x) = V'(x)$, that is, all orbits which intersect the J interval of the x-axis in the $x, \dot{x}$-plane are periodic and have the same period $2\pi/\omega$. Conversely, let g be continuous on a neighborhood of $0 \in \mathbb{R}$, $g(0) = 0$, suppose $g'(0)$ exists and $g'(0) > 0$, and the origin is an isochronous center for $\ddot{x} = -g(x)$. Then there exist an open interval J with $0 \in J$, which is a subset of the domain of g, and an involution $h : J \to J$ such that (4.1) holds with $V(x) = \int_0^x g(s)\, ds$ and $\omega = \sqrt{g'(0)}$.

The potential V of an isochronous center is called an *isochronous potential.* The proof is included in the proof of Proposition 1 in [Z2] as a particular case. Formula (4.1) corresponds to formula (6.2) in [Z2]. A detailed proof can also be found in the recent paper [Z3]; see Theorem 2.1 and Corollary 2.2 there. This last paper also contains the following necessary conditions for a smooth enough potential V to be isochronous:

$$V^{(4)}(0) = \frac{5V'''(0)^2}{3V''(0)}, \quad V^{(6)}(0) = \frac{7V'''(0)V^{(5)}(0)}{V''(0)} - \frac{140V'''(0)^4}{9V''(0)^3}, \tag{4.2}$$

which can be deduced by taking successive derivatives of the involution relation $h(h(x)) \equiv x$ at $x = 0$. We can consider the necessary condition at any even order derivative, provided that V admits that derivative.

Inserting the involution (3.2) into formula (4.1) we obtain the following isochronous potential:

$$V(x) = \frac{\omega^2}{8}\,x^2\left(\frac{2+ax}{1+ax}\right)^2, \qquad x \in J = \begin{cases} (-1/a, \infty), & a > 0,\\ (-\infty, \infty), & a = 0,\\ (-\infty, -1/a), & a < 0. \end{cases} \tag{4.3}$$

This is the only isochronous rational potential, as proved in [CV].

The paper [GZ], by Gorni and the present author, studies global isochronous potentials $V : \mathbb{R} \to \mathbb{R}$ in terms of smooth involutions. In particular it gives implicit examples and new explicit ones. Also, [GZ] revisits Stillinger's and Dorignac's global isochronous potentials in terms of involutions which are given by hyperbolas in Stillinger's case.

5. Instability under some attractive central forces. The paper [Z1] considers the differential system

$$\ddot{x} = -xf(x), \qquad \ddot{y} = -yf(x), \qquad f(0) = 1, \tag{5.1}$$

where f is continuous near 0. It represents the motion under a particular *attractive central force* which is not a gradient. The origin of $\mathbb{R}^2$ is a (local) center for the first equation $\ddot{x} = -xf(x)$. Let us introduce the potential $V(x) = \int_0^x sf(s)\,ds$. For a suitable open interval $J \ni 0$, the potential V is strictly increasing on $J \cap \mathbb{R}_+$, strictly decreasing on $J \cap \mathbb{R}_-$, and for each $x \in J$ there is a unique $h(x) \in J$ with $V(h(x)) = V(x)$, and $xh(x) < 0$ for $x \neq 0$. We easily see that h is a smooth involution. The origin in $\mathbb{R}^4$ is Lyapunov stable for (5.1) if and only if for $x \neq 0$ in a neighborhood of 0 we have

$$\frac{1}{V(x)} = \frac{1}{2}\left(\frac{1}{x} - \frac{1}{h(x)}\right)^2 \tag{5.2}$$

(see [Z1, (4.3)]). In particular, if f is even then so is $V(x)$ and $h(x) = -x$, formula (5.2) is equivalent to $V(x) = x^2/2$ and we have stability if and only if f is constant in a neighborhood of 0. This particular case was studied in [ZB] with a different approach.

In Figure 5 you can see the projection on the x, y-plane of a solution to (5.1) for $f(x) = 1 + x^2$. In this case, the origin is an unstable equilibrium for (5.1). The initial condition for the solution in Figure 5 is

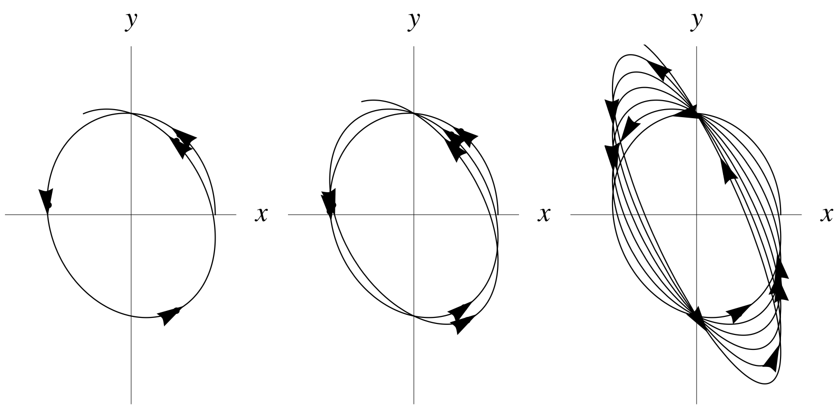

Fig. 5. Projection on the x, y-plane of a solution to (5.1)

$(x(0), \dot{x}(0), y(0), \dot{y}(0)) = (0.4, 0, 0, 0.5)$, on the left $t \in [0, 8]$, in the middle $t \in [0, 14]$, and on the right $t \in [0, 38]$. It is an unbounded motion (see [Z1] for details).

6. Functional-differential equations with involutions. Consider the following problem which involves the involution (3.1) on the interval $(-1, \infty)$, the parameter $a \in \mathbb{R}$, and the initial datum $y_0 \in \mathbb{R}$ at $t = 0$:

$$\begin{cases} y'(t) = ay(h(t)), \\ y(0) = y_0, \end{cases} \qquad h(t) = -\frac{t}{1+t}, \qquad t > -1. \tag{6.1}$$

If $y(t)$ is a C^1 solution then it is C^2. By differentiation we get

$$y''(t) = ah'(t)y'(h(t)) = -\frac{a^2}{(1+t)^2}y(h(h(t))) = -\frac{a^2}{(1+t)^2}y(t).$$

So (6.1) is equivalent to the ordinary Cauchy problem

$$\begin{cases} y''(t) = -\dfrac{a^2}{(1+t)^2}y(t), \quad t > -1, \\ y(0) = y_0, \\ y'(0) = ay_0. \end{cases} \tag{6.2}$$

The solution is defined on the whole $(-1, \infty)$. For $|a| > 1/2$,

$$y(t) = y_0\sqrt{1+t}\left(\cos(c\ln(1+t)) + \frac{2a-1}{2c}\sin(c\ln(1+t))\right),$$

where $c := \sqrt{4a^2-1}/2$. For $a = 1/2$ we have

$$y(t) = y_0\sqrt{1+t};$$

for $a = -1/2$,

$$y(t) = y_0\sqrt{1+t}(1 - \ln(1+t));$$

and for $|a| < 1/2$,

$$y(t) = \frac{y_0}{2b}(1+t)^{(1-b)/2}\big(b + 1 - 2a + (b - 1 + 2a)(1+t)^b\big),$$

where $b := \sqrt{1-4a^2}$. This is just an example of functional-differential equations of Carleman type; their general theory is treated in Chapter VIII of Przeworska-Rolewicz [PR] where other references are given. Equations with involutions are also studied in [BT], [CI], [DI], [SW], [W], [W1], [W2], [WW].

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