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On a Hölder type estimate for quasisymmetric functions

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Abstract. We give a Hölder type estimate for normalized ρ-quasisymmetric functions, improving some results of J. Zając.

1. Introduction. It is known that a K-quasiconformal mapping $f(z)$ of $D = \{z \mid |z| < 1\}$ onto itself can be extended to a homeomorphism of the closed unit disk. It induces a topological mapping of the circumference. In view of the conformal invariance of quasiconformal mappings, the closed unit disk can be replaced by the upper halfplane and we can assume that ∞ corresponds to ∞. Thus, we can view f as a K-quasiconformal mapping of the upper halfplane on itself, sending ∞ to ∞. The boundary correspondence $f : \mathbb{R} \to \mathbb{R}$ is then a continuous increasing function such that $f(-\infty) = -\infty$ and $f(+\infty) = +\infty$. It is a ρ-quasisymmetric function satisfying the Beurling–Ahlfors condition

$$\text{(B-A)} \qquad \frac{1}{\rho} \leq \frac{f(x+t) - f(x)}{f(x) - f(x-t)} \leq \rho$$

for all $x \in \mathbb{R}$ and all $t > 0$ with $\rho = \lambda(K)$ (see [A], [BA]). The class of all increasing homeomorphisms $f : \mathbb{R} \to \mathbb{R}$ satisfying (B-A) with a constant $\rho \geq 1$ is denoted by $Q_{\mathbb{R}}(\rho)$. We let $Q^0_{\mathbb{R}}(\rho)$ denote the subset of $Q_{\mathbb{R}}(\rho)$ consisting of all functions normalized by $h(0) = 0$, $h(1) = 1$.

Beurling and Ahlfors introduced these boundary functions and characterized them by an explicit formula for extension to a K-quasiconformal mapping. Owing to these relations, quasisymmetric functions can be regarded as one-dimensional quasiconformal mappings, and they are expected to have properties analogous to those of two-dimensional quasiconformal mappings. Much research (see [Ke], [Kr1], [Z]) has been devoted to investigating properties of quasisymmetric functions, such as the distortion theorem, Hölder

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type inequality and compactness characterization. Kelingos [Ke] first established some Hölder inequalities for normalized ρ-quasisymmetric functions. His results are closely analogous to results for quasiconformal mappings of the plane under which the origin is a fixed point and the unit disk is invariant. A characterization of f in the case of K-quasiconformal automorphisms F of the unit disk D with fixed point at zero was given by Krzyż [Kr1]. Kelingos [Ke] proved

THEOREM 1.1. *Suppose $u(x)$ is normalized and K-quasisymmetric on the real line. Then*

$$2^{-\alpha}x^{\alpha} \le u(x) \le 2x^{\beta} \tag{1.1}$$

for $0 \le x \le 1$,

$$8^{-\alpha}(x_2 - x_1)^{\alpha} \le u(x_2) - u(x_1) \le 8^{\alpha}(x_2 - x_1)^{\beta} \tag{1.2}$$

for $0 \le x_1 \le x_2 \le 1$, *and*

$$x^{\beta/2} \le u(x) \le (2x)^{\alpha} \tag{1.3}$$

for $x \ge 1$, *where*

$$\alpha = \log_2(1+K), \qquad \beta = \log_2(1+1/K). \tag{1.4}$$

Furthermore, the exponents α and β are best possible.

Zając [Z] improved Kelingos' results by giving some sharp Hölder type estimates for normalized ρ-quasisymmetric functions. He proved

THEOREM 1.2. *Suppose that $\rho \ge 1$ and f is a normalized ρ-qs function on $\mathbb{R}$. Then for each $m \in \mathbb{N}$,*

$$\left(1 - \left(\frac{\rho}{\rho+1}\right)^m\right)t^{\alpha_m} \le f(t) \le \left(1 + \frac{1}{(\rho+1)^m - 1}\right)t^{\beta_m} \tag{1.5}$$

for $0 \le t \le 1$,

$$\begin{aligned}\left(\frac{2}{\rho} - 1\right)\left(1 - \left(\frac{\rho}{\rho+1}\right)^m\right)(t_2 - t_1)^{\alpha_m} &\le f(t_2) - f(t_1) \\ &\le (2\rho - 1)\left(1 + \frac{1}{(\rho+1)^m - 1}\right)(t_2 - t_1)^{\beta_m}\end{aligned} \tag{1.6}$$

for $0 \le t_1 \le t_2 \le 1$ (*the left-hand bound in* (1.6) *is only significant for* $1 \le \rho \le 2$), *and*

$$\left(1 + \frac{1}{(\rho+1)^m - 1}\right)t^{\beta_m} \le f(t) \le \left(1 - \left(\frac{\rho}{\rho+1}\right)^m\right)^{-1} t^{\alpha_m} \tag{1.7}$$

for $t \ge 1$, *where*

$$\alpha_m = \log_{1-2^{-m}}\left(1-\left(\frac{\rho}{\rho+1}\right)^m\right),\tag{1.8}$$
$$\beta_m = \log_{1-2^{-m}}\left(1-\left(\frac{1}{\rho+1}\right)^m\right).$$

As pointed out by Zając, if $m = 1$, (1.5) and (1.7) reduce to those of Kelingos while (1.6) is better. However, the left-hand bound in (1.6) is only significant for $1 \le \rho \le 2$, in order to have $2/\rho - 1 \ge 0$. In this note, we shall give better upper and lower bound estimates for these Hölder type inequalities without the restriction $1 \le \rho \le 2$.

2. Main theorem and its proof. Our main result can be stated as follows:

THEOREM 2.1. *Suppose that $\rho \ge 1$ and f is a normalized ρ-qs function on $\mathbb{R}$. Then for each $m \in \mathbb{N}$,*

$$\begin{aligned} M_\rho\left(1-\left(\frac{\rho}{\rho+1}\right)^m\right)(t_2-t_1)^{\alpha_m} &\le f(t_2)-f(t_1) \\ &\le \left(\rho+2\frac{\rho-1}{\rho+1}\right)\left(1+\frac{1}{(\rho+1)^m-1}\right)(t_2-t_1)^{\beta_m} \end{aligned}\tag{2.1}$$

for $0 \le t_1 \le t_2 \le 1$, where

$$M_\rho = \begin{cases} \frac{1}{\rho} - 4\frac{\rho-1}{(\rho+1)^2}, & 1 \le \rho < \frac{5+\sqrt{41}}{8}, \\ \frac{1}{1+\rho}, & \frac{5+\sqrt{41}}{8} \le \rho < \frac{3}{2}, \\ \frac{3-\rho}{\rho(1+\rho)}, & 3/2 \le \rho < 2, \\ \frac{1}{\rho(1+\rho)}, & 2 \le \rho < \infty, \end{cases}\tag{2.2}$$

$$\alpha_m = \log_{1-2^{-m}}\left(1-\left(\frac{\rho}{\rho+1}\right)^m\right),\tag{2.3}$$
$$\beta_m = \log_{1-2^{-m}}\left(1-\left(\frac{1}{\rho+1}\right)^m\right).$$

In order to prove Theorem 2.1, we need the following Theorem 2.2 due to Krzyż [Kr2].

THEOREM 2.2. *If h is ρ-quasisymmetric on $\mathbb{R}$ and $h(x) - x$ vanishes at the end-points of an interval I then*

$$|h(x)-x| \le |I|\frac{\rho-1}{\rho+1}\tag{2.4}$$

for any $x \in I$, where $|I|$ denotes the length of I.

Proof of Theorem 2.1. Suppose that f is a normalized ρ-qs function on $\mathbb{R}$ satisfying (B-A) with a constant $\rho \ge 1$. Then for every $t_1 \in [0,1]$ the function

$$g_{t_1}(t) = \frac{f(t+t_1)-f(t_1)}{f(1+t_1)-f(t_1)}$$

belongs to $Q^0_{\mathbb{R}}(\rho)$. Therefore, by (1.5) of Theorem 1.2 with $t = t_2 - t_1$, $0 \le t_1 \le t_2 \le 1$, we have

$$f(t_2)-f(t_1) \le \big(f(1+t_1)-f(t_1)\big)\left(1+\frac{1}{(\rho+1)^m-1}\right)(t_2-t_1)^{\beta_m}, \tag{2.5}$$

$$f(t_2)-f(t_1) \ge \big(f(1+t_1)-f(t_1)\big)\left(1-\left(\frac{\rho}{\rho+1}\right)^m\right)(t_2-t_1)^{\alpha_m} \tag{2.6}$$

for any $m \in \mathbb{N}$.

We will need the fact that if $f \in Q^0_{\mathbb{R}}(\rho)$, then $f(x)-x$ vanishes at the end-points of $I=[0,1]$ so by Theorem 2.2,

$$|f(x)-x| \le \frac{\rho-1}{\rho+1} \quad \text{for } x \in [0,1]. \tag{2.7}$$

Now, we shall prove the upper estimate of (2.1). By (B-A), we have

$$N = f(1+t_1)-f(t_1) \le \rho(1-f(1-t_1))+1-f(t_1). \tag{2.8}$$

Using (2.7), we deduce from (2.8) that

$$\begin{aligned} N &\le (\rho-1)(1-f(1-t_1))+1-t_1-f(1-t_1)+t_1-f(t_1)+1 \\ &\le (\rho-1)+2\frac{\rho-1}{\rho+1}+1 = \rho+2\frac{\rho-1}{\rho+1}. \end{aligned} \tag{2.9}$$

By (2.5) and (2.9), we obtain

$$f(t_2)-f(t_1) \le \left(\rho+2\frac{\rho-1}{\rho+1}\right)\left(1+\frac{1}{(\rho+1)^m-1}\right)(t_2-t_1)^{\beta_m}.$$

Note that for $\rho \in [1,\infty)$, we have

$$\rho+2\frac{\rho-1}{\rho+1} \le 2\rho-1.$$

Next, for the lower estimate of (2.1), if $1 \ge t_1 \ge \frac{4\rho}{(1+\rho)^2}$, then, by (B-A) and (2.7), we have

$$\begin{aligned} N &\ge \frac{1}{\rho}(f(t_1)-f(1-t_1)) \\ &= \frac{1}{\rho}\big(f(t_1)-t_1+(1-t_1)-f(1-t_1)+2t_1-1\big) \\ &\ge \frac{1}{\rho}\left(-2\frac{\rho-1}{\rho+1}+2t_1-1\right) \ge \frac{1}{\rho}\left(-2\frac{\rho-1}{\rho+1}+\frac{8\rho}{(1+\rho)^2}-1\right) \\ &= \frac{1}{\rho}-4\frac{\rho-1}{(1+\rho)^2}. \end{aligned} \tag{2.10}$$

If $0 \le t_1 \le \frac{4\rho}{(1+\rho)^2}$, (B-A) and (2.7) imply that

$$N \ge \frac{1}{\rho}(1 - f(1-t_1)) + 1 - f(t_1) \tag{2.11}$$
$$= \frac{1}{\rho}(t_1 + 1 - t_1 - f(1-t_1)) + (1-t_1) + t_1 - f(t_1)$$
$$\ge \frac{1}{\rho}\left(t_1 - \frac{\rho-1}{\rho+1}\right) + (1-t_1) - \frac{\rho-1}{\rho+1}$$
$$= \left(\frac{1}{\rho} - 1\right)t_1 + \frac{1}{\rho} \ge \frac{1}{\rho} - 4\frac{\rho-1}{(1+\rho)^2}.$$

It is easy to check that for every $\rho \in [1, \infty)$,

$$\frac{1}{\rho} - 4\frac{\rho-1}{(1+\rho)^2} \ge \frac{2}{\rho} - 1.$$

On the other hand, we need the following estimate. First, for $f \in Q^0_{\mathbb{R}}(\rho)$, from [A] we have

$$\frac{1}{1+\rho} \le f\left(\frac{1}{2}\right) \le \frac{\rho}{1+\rho}. \tag{2.12}$$

We consider two cases.

(i) If $1/2 \le t_1 \le 1$, then by (B-A), (2.12) and the monotonicity of $f(t)$, we have

$$f(1+t_1) - f(t_1) \ge \frac{1}{\rho}(f(t_1) - f(t_1 - 1)) \ge \frac{1}{\rho}f(t_1) \ge \frac{1}{\rho}f\left(\frac{1}{2}\right) \tag{2.13}$$
$$\ge \frac{1}{\rho(1+\rho)},$$

and again by (B-A) and (2.4),

$$f(1+t_1) - f(t_1) \ge \frac{1}{\rho}(1 - t_1 + t_1 - f(1-t_1)) + 1 - f(t_1) \tag{2.14}$$
$$\ge \frac{t_1}{\rho} - \frac{\rho-1}{\rho(\rho+1)} + \frac{1-f(t_1)}{\rho}$$
$$\ge \frac{3-\rho}{\rho(1+\rho)}.$$

(ii) If $0 \le t_1 \le 1/2$, then

$$f(1+t_1) - f(t_1) \ge \frac{1}{\rho}(1 - f(1-t_1)) + 1 - f(t_1) \tag{2.15}$$
$$\ge 1 - f\left(\frac{1}{2}\right) \ge \frac{1}{1+\rho}.$$

We see that $\rho_0 = \frac{5+\sqrt{41}}{8}$ is one of the roots of the equation

$$\frac{1}{\rho} - 4\frac{\rho-1}{(1+\rho)^2} = \frac{1}{1+\rho},$$

and if $\rho \in \big[1, \frac{5+\sqrt{41}}{8}\big)$, then

$$\frac{1}{\rho} - 4\frac{\rho-1}{(1+\rho)^2} \geq \frac{1}{1+\rho}.$$

By (2.10) and (2.11), we have

$$f(1+t_1) - f(t_1) \geq \frac{1}{\rho} - 4\frac{\rho-1}{(1+\rho)^2}, \quad \rho \in \left[1, \frac{5+\sqrt{41}}{8}\right). \tag{2.16}$$

If $\rho \in \big[\frac{5+\sqrt{41}}{8}, \frac{3}{2}\big)$, then

$$\frac{3-\rho}{\rho(1+\rho)} \geq \frac{1}{1+\rho},$$

and $\frac{3}{2}$ is a root of the equation

$$\frac{3-\rho}{\rho(1+\rho)} = \frac{1}{1+\rho}.$$

By (2.14) and (2.15) we have

$$f(1+t_1) - f(t_1) \geq \frac{1}{1+\rho}, \quad \rho \in \left[\frac{5+\sqrt{41}}{8}, \frac{3}{2}\right). \tag{2.17}$$

If $\rho \in [3/2, 2)$, then

$$\frac{1}{1+\rho} \geq \frac{3-\rho}{\rho(1+\rho)},$$

and by (2.14) and (2.15) we obtain

$$f(1+t_1) - f(t_1) \geq \frac{3-\rho}{\rho(1+\rho)}, \quad \rho \in [3/2, 2). \tag{2.18}$$

If $\rho \in [2, \infty)$, then 2 is one of the roots of the equation

$$\frac{3-\rho}{\rho(1+\rho)} = \frac{1}{\rho(1+\rho)}.$$

By (2.13) and (2.15), we have

$$f(1+t_1) - f(t_1) \geq \frac{1}{\rho(1+\rho)}, \quad \rho \in [2, \infty). \tag{2.19}$$

Combining (2.6) and (2.16)–(2.19), we have proved that

$$f(t_2) - f(t_1) \geq M_\rho\left(1 - \left(\frac{\rho}{\rho+1}\right)^m\right)(t_2 - t_1)^{\alpha_m}$$

with M_ρ as stated in the theorem. ∎

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