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Differences of generalized weighted composition operators between growth spaces

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Abstract. Let φ and ψ be analytic self-maps of $\mathbb{D}$. Using the pseudo-hyperbolic distance $\rho(\varphi,\psi)$, we completely characterize the boundedness and compactness of the difference of generalized weighted composition operators between growth spaces.

1. Introduction. Let $\mathbb{D}=\{z:|z|<1\}$ be the unit disk in the complex plane and $H(\mathbb{D})$ be the space of all analytic functions on $\mathbb{D}$. For $\alpha>0$, the space $\mathcal{A}^{-\alpha}$ consists of analytic functions $f\in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{A}^{-\alpha}}=\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f(z)|<\infty. \tag{1.1}$$

Growth spaces are Banach spaces with the norm $\|\cdot\|_{\mathcal{A}^{-\alpha}}$, also called the *weighted Banach spaces of analytic functions*.

For $a\in\mathbb{D}$, let σ_a be the Möbius transformation of $\mathbb{D}$ defined by $\sigma_a(z)=(a-z)/(1-\overline{a}z)$. For $w,z\in\mathbb{D}$, the *pseudo-hyperbolic distance* $\rho(w,z)$ between z and w is given by $\rho(w,z)=|\sigma_w(z)|$.

For $w,z\in\mathbb{D}$, let $\gamma:[0,1]\to\mathbb{D}$ be a smooth curve connecting z and w. Then the *hyperbolic length* of γ is given by

$$l(\gamma)=\int_0^1\frac{|\gamma'(t)|\,dt}{1-|\gamma(t)|^2}.$$

The *hyperbolic distance* between z and w, denoted by $\beta(w,z)$, is defined as the infimum of $l(\gamma)$, where γ are smooth curves connecting z and w. It is known that

$$\beta(w,z)=l(\gamma_g)=\frac{1}{2}\ln\frac{1+\rho(w,z)}{1-\rho(w,z)},$$

where γ_g is the geodesic connecting z and w.

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Let $S(\mathbb{D})$ be the set of analytic self-maps of $\mathbb{D}$. For $\varphi \in S(\mathbb{D})$ and $u \in H(\mathbb{D})$, we denote by uC_φ the weighted composition operator, which is defined by

$$(uC_\varphi f)(z) = u(z)f(\varphi(z)), \quad z \in \mathbb{D}.$$

When $u(z) \equiv 1$, uC_φ becomes the composition operator C_φ. When $\varphi(z) = z$, uC_φ becomes the multiplication operator M_u.

During the past few decades much effort has been devoted to the research on such operators on different Banach spaces of analytic functions. The general idea is to describe the operator-theoretic behavior of uC_φ, such as boundendness and compactness, in terms of the function-theoretic properties of the symbols φ and u. For a comprehensive overview of the field, we refer to the books [S, CM].

Let n be a nonnegative integer, $\varphi \in S(\mathbb{D})$ and $u \in H(\mathbb{D})$. The *generalized weighted composition operator* $D^n_{\varphi,u}$ is defined by

$$D^n_{\varphi,u} f = uf^{(n)} \circ \varphi, \quad f \in H(\mathbb{D}), \tag{1.2}$$

where $f^{(n)}$ is the nth derivative of f and $f^{(0)} = f$. The operator $D^n_{\varphi,u}$ can be regarded as a product of a composition operator C_φ, a multiplication operator M_u and the nth differentiation operator D^n. When $u \equiv 1$, we denote the corresponding generalized weighted composition operator by D^n_φ. The generalized weighted composition operator $D^n_{\varphi,u}$ was probably studied for the first time in [Z1], and later in [Z2, Z3, Z4, S1, O, S2, JS, S3, SSB, YY, Sh1, Sh2, YwZ, Y, Z]. For some related operators in the setting of the unit ball see, for example, [S5, S4].

The study of the differences of two composition operators was started on Hardy spaces (see, for example, [B, SS]). The primary motivation for this is to understand the topological structure of $\mathcal{C}(H^2)$, the set of composition operators on the Hardy space H^2. After that, such related problems have been studied on several spaces of holomorphic functions by many authors: see, for example, [MOZ, M, N, HO, BLW, LW, W, DO, JS, Sa, SJ, L, YkZ] and the references therein. J. Moorhouse's results in [M] suggested that, on the standard weighted Bergman spaces, there might be some connection between the difference operator $C_\varphi - C_\psi$ and the corresponding weighted composition operators σC_φ and σC_ψ, where and henceforth

$$\sigma(z) = \frac{\varphi(z) - \psi(z)}{1 - \overline{\varphi(z)}\psi(z)}.$$

In [Sa], E. Saukko confirmed this connection.

In this paper, we investigate the boundedness and compactness of the differences of generalized weighted composition operators between $\mathcal{A}^{-\alpha}$ and $\mathcal{A}^{-\beta}$. Moreover, we find that the difference operator $D^n_\varphi - D^n_\psi$ is closely linked with the corresponding generalized weighted composition operators $D^n_{\varphi,\sigma}$ and $D^n_{\psi,\sigma}$ in the setting of growth spaces (see Corollaries 3.4 and 4.4).

Throughout this note, constants are denoted by C, they are positive and may differ from one occurrence to another. The notation $a \preceq b$ means that there is a positive constant C such that $a \le Cb$. Moreover, if both $a \preceq b$ and $b \preceq a$ hold, then one writes $a \asymp b$.

2. Prerequisites. In this section, we give some auxiliary results which will be used in proving the main results of the paper. They are incorporated in the lemmas which follow.

From the proofs of Propositions 7 and 8 in [Zh], we can directly deduce the following lemma.

LEMMA 2.1. *For every positive integer n, we have $f \in \mathcal{A}^{-\alpha}$ if and only if $f^{(n)} \in \mathcal{A}^{-(\alpha+n)}$, and the following asymptotic relationship holds:*

$$\|f\|_{\mathcal{A}^{-\alpha}} \asymp \sum_{k=0}^{n-1} |f^{(k)}(0)| + \sup_{z\in\mathbb{D}} (1-|z|^2)^{\alpha+n}|f^{(n)}(z)|.$$

LEMMA 2.2. *For every nonnegative integer n, if $z, w \in \mathbb{D}$ and $f \in \mathcal{A}^{-\alpha}$, then*

$$|(1-|z|^2)^{\alpha+n} f^{(n)}(z) - (1-|w|^2)^{\alpha+n} f^{(n)}(w)| \le C\|f\|_{\mathcal{A}^{-\alpha}}\rho(w,z). \tag{2.1}$$

Proof. Let $\gamma = \gamma(t)$ $(0 \le t \le 1)$ be the geodesic connecting w and z. Then $\beta(w,z) = l(\gamma)$ and

$$\begin{aligned}(1-|z|^2)^{\alpha+n} f^{(n)}(z) &- (1-|w|^2)^{\alpha+n} f^{(n)}(w)\\ &= \int_0^1 d\big((1-|\gamma(t)|^2)^{\alpha+n} f^{(n)}(\gamma(t))\big)\\ &= \int_0^1 (1-|\gamma(t)|^2)^{\alpha+n}\, df^{(n)}(\gamma(t)) + \int_0^1 f^{(n)}(\gamma(t))\, d(1-|\gamma(t)|^2)^{\alpha+n}.\end{aligned}$$

By Lemma 2.1 we have

$$\begin{aligned}\Big|\int_0^1 (1-|\gamma(t)|^2)^{\alpha+n}\, df^{(n)}(\gamma(t))\Big| &\le \int_0^1 (1-|\gamma(t)|^2)^{\alpha+n}|\gamma'(t) f^{(n+1)}(\gamma(t))|\, dt\\ &\le C\|f\|_{\mathcal{A}^{-\alpha}} \int_0^1 (1-|\gamma(t)|^2)^{-1}|\gamma'(t)|\, dt \le C\|f\|_{\mathcal{A}^{-\alpha}}\beta(w,z)\end{aligned}$$

and

$$\Big|\int_0^1 f^{(n)}(\gamma(t))\,d(1-|\gamma(t)|^2)^{\alpha+n}\Big|$$
$$\le \int_0^1 (\alpha+n)(1-|\gamma(t)|^2)^{\alpha+n-1}\big|f^{(n)}(\gamma(t))\big(\overline{\gamma(t)}\gamma'(t)+\gamma(t)\overline{\gamma'(t)}\big)\big|\,dt$$
$$\le C\|f\|_{\mathcal{A}^{-\alpha}}\int_0^1 (1-|\gamma(t)|^2)^{-1}|\mathrm{Re}[(\overline{\gamma(t)}\gamma'(t)]|\,dt \le C\|f\|_{\mathcal{A}^{-\alpha}}\beta(w,z).$$

Thus,

$$|(1-|z|^2)^{\alpha+n}f^{(n)}(z)-(1-|w|^2)^{\alpha+n}f^{(n)}(w)| \le C\|f\|_{\mathcal{A}^{-\alpha}}\beta(w,z). \tag{2.2}$$

On the other hand, if $\rho(w,z)\le 1/2$, from the monotonicity of the function $\frac{1}{2x}\ln\frac{1+x}{1-x}$ in $[0,1)$, we have

$$\beta(w,z)=\frac{1}{2}\ln\frac{1+\rho(w,z)}{1-\rho(w,z)} \le (\ln 3)\rho(w,z). \tag{2.3}$$

If $\rho(w,z)>1/2$, then

$$|(1-|z|^2)^{\alpha+n}f^{(n)}(z)-(1-|w|^2)^{\alpha+n}f^{(n)}(w)| \le 2C\|f\|_{\mathcal{A}^{-\alpha}} \le C\|f\|_{\mathcal{A}^{-\alpha}}\rho(w,z). \tag{2.4}$$

From (2.2)–(2.4), we obtain inequality (2.1). ∎

Remark 2.3. From the proof of Lemma 2.2, in fact, it follows that for all $f\in\mathcal{A}^{-\alpha}$ and $z,w\in\mathbb{D}_r$, we have

$$|(1-|z|^2)^{\alpha+n}f^{(n)}(z)-(1-|w|^2)^{\alpha+n}f^{(n)}(w)| \le C\mathcal{A}_r^{-\alpha}(f)\rho(w,z) \tag{2.5}$$

where $\mathbb{D}_r=\{z\in\mathbb{D}:|z|\le r<1\}$ and

$$\mathcal{A}_r^{-\alpha}(f)=\max\Big\{\sup_{z\in\mathbb{D}_r}(1-|z|^2)^{\alpha+n}|f^{(n)}(z)|,\ \sup_{z\in\mathbb{D}_r}(1-|z|^2)^{\alpha+n+1}|f^{(n+1)}(z)|\Big\}.$$

The following criterion for compactness follows from standard arguments similar to those outlined in [CM, Proposition 3.11]. We omit the details.

Lemma 2.4. *Let n be a nonnegative integer, $\alpha,\beta>0$, $\varphi,\psi\in S(\mathbb{D})$ and $u,v\in H(\mathbb{D})$. Then $D^n_{\varphi,u}-D^n_{\psi,v}:\mathcal{A}^{-\alpha}\to\mathcal{A}^{-\beta}$ is compact if and only if $D^n_{\varphi,u}-D^n_{\psi,v}:\mathcal{A}^{-\alpha}\to\mathcal{A}^{-\beta}$ is bounded and for any bounded sequence $(f_k)_{k\in\mathbb{N}}$ in $\mathcal{A}^{-\alpha}$ which converges to zero uniformly on compact subsets of $\mathbb{D}$, we have $\|(D^n_{\varphi,u}-D^n_{\psi,v})f_k\|_{\mathcal{A}^{-\beta}}\to 0$ as $k\to\infty$.*

3. The boundedness of $D^n_{\varphi,u}-D^n_{\psi,v}$. In order to characterize the boundedness of $D^n_{\varphi,u}-D^n_{\psi,v}:\mathcal{A}^{-\alpha}\to\mathcal{A}^{-\beta}$, we will use the following three conditions in this section:

$$\sup_{z\in\mathbb{D}} |M_u^{\varphi}(z)|\rho(\varphi(z),\psi(z)) < \infty; \tag{3.1}$$

$$\sup_{z\in\mathbb{D}} |M_v^{\psi}(z)|\rho(\varphi(z),\psi(z)) < \infty; \tag{3.2}$$

$$\sup_{z\in\mathbb{D}} |M_u^{\varphi}(z) - M_v^{\psi}(z)| < \infty; \tag{3.3}$$

here and henceforth

$$M_u^{\varphi}(z) = \frac{(1-|z|^2)^{\beta}u(z)}{(1-|\varphi(z)|^2)^{\alpha+n}}, \qquad M_v^{\psi}(z) = \frac{(1-|z|^2)^{\beta}v(z)}{(1-|\psi(z)|^2)^{\alpha+n}}.$$

THEOREM 3.1. *Let n be a positive integer, $\alpha,\beta > 0$, $\varphi,\psi \in S(\mathbb{D})$ and $u,v \in H(\mathbb{D})$. Then the following statements are equivalent:*

(a) *$D^n_{\varphi,u} - D^n_{\psi,v} : \mathcal{A}^{-\alpha} \to \mathcal{A}^{-\beta}$ is bounded;*
(b) *(3.1) and (3.3) hold;*
(c) *(3.2) and (3.3) hold.*

Proof. (a)⇒(b). For some positive number $r < 1$, fix a point w in $\mathbb{D}$ such that $|\varphi(w)| \ge r$, and let

$$l_w(z) = \frac{1}{\tau(\alpha+n)\overline{\varphi(w)}^n} \frac{1-|\varphi(w)|^2}{(1-\overline{\varphi(w)}z)^{\alpha+1}},$$

$$f_w(z) = \frac{l_w(z)}{\lambda_n+1}(\sigma_{\varphi(w)}(z) + \lambda_n/\overline{\varphi(w)}),$$

where $\tau(\alpha) = 1$, $\tau(\alpha+n) = (\alpha+n)\tau(\alpha+n-1)$ and

$$\lambda_0 = 0, \qquad \lambda_n = \sum_{i=0}^{n-1} \frac{(n-i)!\tau(\alpha+i)}{\tau(\alpha+n)}.$$

It is easy to check that $l_w(z), f_w(z) \in \mathcal{A}^{-\alpha}$; moreover,

$$l_w^{(n)}(z) = \frac{1-|\varphi(w)|^2}{(1-\overline{\varphi(w)}z)^{\alpha+n+1}}$$

and

$$\begin{aligned} f_w^{(n)}(z) &= \sum_{i=0}^{n-1} \frac{l_w^{(i)}(z)(\sigma_{\varphi(w)}(z)+\lambda_n/\overline{\varphi(w)})^{(n-i)}}{\lambda_n+1} + \frac{l_w^{(n)}(z)(\sigma_{\varphi(w)}(z)+\lambda_n/\overline{\varphi(w)})}{\lambda_n+1} \\ &= \sum_{i=0}^{n-1} \frac{l_w^{(i)}(z)}{\lambda_n+1}(\sigma_{\varphi(w)}(z))^{(n-i)} + \frac{(1-|\varphi(w)|^2)(\sigma_{\varphi(w)}(z)+\lambda_n/\overline{\varphi(w)})}{(\lambda_n+1)(1-\overline{\varphi(w)}z)^{\alpha+n+1}} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=0}^{n-1} \frac{1-|\varphi(w)|^2}{(1-\overline{\varphi(w)}z)^{\alpha+i+1}} \frac{(\alpha+1)\cdots(\alpha+i)}{(\lambda_n+1)\tau(\alpha+n)\overline{\varphi(w)}^{n-i}} (\sigma_{\varphi(w)}(z))^{(n-i)} \\
&\quad + \frac{1-|\varphi(w)|^2}{(\lambda_n+1)(1-\overline{\varphi(w)}z)^{\alpha+n+1}} (\sigma_{\varphi(w)}(z) + \lambda_n/\overline{\varphi(w)}) \\
&= \sum_{i=0}^{n-1} \frac{1-|\varphi(w)|^2}{(1-\overline{\varphi(w)}z)^{n+\alpha+2}} \frac{(n-i)!(|\varphi(w)|^2-1)\tau(\alpha+i)}{(\lambda_n+1)\tau(\alpha+n)\overline{\varphi(w)}} \\
&\quad + \frac{1-|\varphi(w)|^2}{(\lambda_n+1)(1-\overline{\varphi(w)}z)^{\alpha+n+1}} \frac{|\varphi(w)|^2-\overline{\varphi(w)}z+\lambda_n(1-\overline{\varphi(w)}z)}{(1-\overline{\varphi(w)}z)\overline{\varphi(w)}} \\
&= \frac{(1-|\varphi(w)|^2)\big[\lambda_n(|\varphi(w)|^2-1)+|\varphi(w)|^2-\overline{\varphi(w)}z+\lambda_n(1-\overline{\varphi(w)}z)\big]}{\overline{\varphi(w)}(\lambda_n+1)(1-\overline{\varphi(w)}z)^{\alpha+n+2}} \\
&= \frac{(1-|\varphi(w)|^2)(|\varphi(w)|^2-\overline{\varphi(w)}z)}{\overline{\varphi(w)}(1-\overline{\varphi(w)}z)^{\alpha+n+2}} = \frac{(1-|\varphi(w)|^2)}{(1-\overline{\varphi(w)}z)^{\alpha+n+1}}\sigma_{\varphi(w)}(z),
\end{aligned}
$$

and then

$$
l_w^{(n)}(\varphi(w)) = \frac{1}{(1-|\varphi(w)|^2)^{\alpha+n}}, \qquad l_w^{(n)}(\psi(w)) = \frac{1-|\varphi(w)|^2}{(1-\overline{\varphi(w)}\psi(w))^{\alpha+n+1}},
$$

$$
f_w^{(n)}(\varphi(w)) = 0, \qquad f_w^{(n)}(\psi(w)) = \frac{(1-|\varphi(w)|^2)\sigma_{\varphi(w)}(\psi(w))}{(1-\overline{\varphi(w)}\psi(w))^{\alpha+n+1}}.
$$

Since the operator $D^n_{\varphi,u} - D^n_{\psi,v} : \mathcal{A}^{-\alpha} \to \mathcal{A}^{-\beta}$ is bounded, by Lemma 2.2 we have

$$
\begin{aligned}
\text{(3.4)} \qquad \infty &> \|(D^n_{\varphi,u} - D^n_{\psi,v})l_w\|_{\mathcal{A}^{-\beta}} \\
&= \sup_{z\in\mathbb{D}} (1-|z|^2)^\beta |l_w^{(n)}(\varphi(z))u(z) - l_w^{(n)}(\psi(z))v(z)| \\
&\geq (1-|w|^2)^\beta |l_w^{(n)}(\varphi(w))u(w) - l_w^{(n)}(\psi(w))v(w)| \\
&= \left| M_u^\varphi(w) - M_v^\psi(w) \frac{(1-|\psi(w)|^2)^{\alpha+n}(1-|\varphi(w)|^2)}{(1-\overline{\varphi(w)}\psi(w))^{\alpha+n+1}} \right|
\end{aligned}
$$

and

$$
\begin{aligned}
\text{(3.5)} \qquad \infty &> \|(D^n_{\varphi,u} - D^n_{\psi,v})f_w\|_{\mathcal{A}^{-\beta}} \\
&= \sup_{z\in\mathbb{D}} (1-|z|^2)^\beta |f_w^{(n)}(\varphi(z))u(z) - f_w^{(n)}(\psi(z))v(z)| \\
&\geq (1-|w|^2)^\beta |f_w^{(n)}(\varphi(w))u(w) - f_w^{(n)}(\psi(w))v(w)| \\
&= (1-|w|^2)^\beta \frac{|v(w)|(1-|\varphi(w)|^2)\rho(\varphi(w),\psi(w))}{|1-\overline{\varphi(w)}\psi(w)|^{\alpha+n+1}} \\
&= \left| M_v^\psi(w) \frac{(1-|\psi(w)|^2)^{\alpha+n}(1-|\varphi(w)|^2)\rho(\varphi(w),\psi(w))}{(1-\overline{\varphi(w)}\psi(w))^{\alpha+n+1}} \right|.
\end{aligned}
$$

Since the pseudo-hyperbolic metric is bounded by one, multiplying (3.4) by $\rho(\varphi(w),\psi(w))$ and applying (3.5), we obtain

$$\sup_{w\in\mathbb{D},\,|\varphi(w)|\geq r} M_u^{\varphi}(w)\rho(\varphi(w),\psi(w))<\infty. \tag{3.6}$$

If $|\varphi(w)|<r$, using the test function

$$k_w(z)=\frac{(z-\psi(w))^{n+1}}{(n+1)!},$$

we see that

$$\begin{aligned}\infty &> \|(D^n_{\varphi,u}-D^n_{\psi,v})k_w\|_{\mathcal{A}^{-\beta}}\\ &\geq (1-|w|^2)^{\beta}|k_w^{(n)}(\varphi(w))u(w)-k_w^{(n)}(\psi(w))v(w)|\\ &= (1-|w|^2)^{\beta}\big|u(w)(\varphi(w)-\psi(w))\big|,\end{aligned}$$

therefore,

$$\frac{(1-|w|^2)^{\beta}\big|u(w)(\varphi(w)-\psi(w))\big|}{(1-|\varphi(w)|^2)^{\alpha+n}\big|1-\overline{\varphi(w)}\psi(w)\big|}\leq C\|(D^n_{\varphi,u}-D^n_{\psi,v})k_w\|_{\mathcal{A}^{-\beta}}. \tag{3.7}$$

From (3.6) and (3.7), we get (3.1).

Using another triple of test functions which come from $l_w(z)$, $f_w(z)$ and $k_w(z)$ by exchanging φ and ψ, we get (3.2).

Next, we prove (3.3). For $|\varphi(w)|\geq r$, by (3.4), we also have

$$\begin{aligned}\infty &> \|(D^n_{\varphi,u}-D^n_{\psi,v})l_w\|_{\mathcal{A}^{-\beta}}\\ &\geq \left|M_u^{\varphi}(w)-M_v^{\psi}(w)\frac{(1-|\psi(w)|^2)^{\alpha+n}(1-|\varphi(w)|^2)}{(1-\overline{\varphi(w)}\psi(w))^{\alpha+n+1}}\right|\\ &= \Bigg|M_u^{\varphi}(w)-M_v^{\psi}(w)\\ &\qquad\qquad + M_v^{\psi}(w)\left(1-\frac{(1-|\psi(w)|^2)^{\alpha+n}(1-|\varphi(w)|^2)}{(1-\overline{\varphi(w)}\psi(w))^{\alpha+n+1}}\right)\Bigg|\\ &\geq |M_u^{\varphi}(w)-M_v^{\psi}(w)|-|M_v^{\psi}(w)|\\ &\quad\times |l_w^{(n)}(\varphi(w))(1-|\varphi(w)|^2)^{\alpha+n}-l_w^{(n)}(\psi(w))(1-|\psi(w)|^2)^{\alpha+n}|.\end{aligned} \tag{3.8}$$

From Lemma 2.2 and (3.2), we see that

$$\begin{aligned}|M_v^{\psi}(w)|\cdot|l_w^{(n)}(\varphi(w))(1-|\varphi(w)|^2)^{\alpha+n}-l_w^{(n)}(\psi(w))(1-|\psi(w)|^2)^{\alpha+n}|\\ \leq C\|l_w\|_{\mathcal{A}^{-\alpha}}|M_v^{\psi}(w)|\rho(\varphi(w),\psi(w))<\infty,\end{aligned}$$

which by (3.8) implies that $|M_u^{\varphi}(w)-M_v^{\psi}(w)|<\infty$ for all $w\in\mathbb{D}$ with $|\varphi(w)|\geq r$.

If $|\varphi(w)|<r$ and $|\psi(w)|\geq\frac{1+r}{2}$, then $\rho(\varphi(w),\psi(w))\geq\frac{1-r}{2(1+r)}$. From (3.1) and (3.2), we can deduce directly that $|M_u^{\varphi}(w)-M_v^{\psi}(w)|<\infty$ in this case.

For $|\varphi(w)| < r$ and $|\psi(w)| < \frac{1+r}{2}$, using the test function $h(z) = z^n/n!$, we see that

$$
\begin{aligned}
(3.9)\qquad \infty &> \|(D^n_{\varphi,u} - D^n_{\psi,v})h\|_{\mathcal{A}^{-\beta}} \\
&\geq (1-|w|^2)^\beta |h^{(n)}(\varphi(w))u(w) - h^{(n)}(\psi(w))v(w)| \\
&= (1-|w|^2)^\beta |u(w) - v(w)| \\
&= \big|(M^\varphi_u(w) - M^\psi_v(w))(1-|\varphi(w)|^2)^{\alpha+n} \\
&\qquad + M^\psi_v(w)[(1-|\varphi(w)|^2)^{\alpha+n} - (1-|\psi(w)|^2)^{\alpha+n}]\big| \\
&\geq |M^\varphi_u(w) - M^\psi_v(w)|(1-|\varphi(w)|^2)^{\alpha+n} \\
&\quad - |M^\psi_v(w)[(1-|\varphi(w)|^2)^{\alpha+n} - (1-|\psi(w)|^2)^{\alpha+n}]|.
\end{aligned}
$$

In view of the boundedness of the derivative of the real function $g(x) = (1-x^2)^{\alpha+n}$ in $[0,1]$, we have

$$
\begin{aligned}
&|(1-|\varphi(w)|^2)^{\alpha+n} - (1-|\psi(w)|^2)^{\alpha+n}| \\
&\qquad\qquad \leq C\big||\varphi(w)| - |\psi(w)|\big| \leq C\rho(\varphi(w), \psi(w)),
\end{aligned}
$$

and hence

$$
|M^\psi_v(w)[(1-|\varphi(w)|^2)^{\alpha+n} - (1-|\psi(w)|^2)^{\alpha+n}]| \leq C|M^\psi_v(w)|\rho(\varphi(w),\psi(w)),
$$

from which, together with (3.2) and (3.9), we obtain $|M^\varphi_g(w) - M^\psi_h(w)| < \infty$ in this case.

Thus we conclude that $|M^\varphi_u(w) - M^\psi_v(w)| < \infty$ for all $w \in \mathbb{D}$, which implies (3.3).

(b)⇒(c). Noticing that $|M^\psi_v(z)| \leq |M^\varphi_u(z)| + |M^\psi_v(z) - M^\varphi_u(z)|$, we have

$$
\begin{aligned}
&|M^\psi_v(z)|\rho(\varphi(z),\psi(z)) \\
&\qquad \leq |M^\varphi_u(z)\big|\rho(\varphi(z),\psi(z)) + |M^\psi_v(z) - M^\varphi_u(z)|\rho(\varphi(z),\psi(z)),
\end{aligned}
$$

which implies (3.2).

(c)⇒(a). For any $f \in \mathcal{A}^{-\alpha}$, by Lemmas 2.1 and 2.2, we have

$$
\begin{aligned}
&\|(D^n_{\varphi,u} - D^n_{\psi,v})f\|_{\mathcal{A}^{-\beta}} = \sup_{z\in\mathbb{D}} (1-|z|^2)^\beta |(D^n_{\varphi,u} - D^n_{\psi,v})f(z)| \\
&= \sup_{z\in\mathbb{D}} (1-|z|^2)^\beta |f^{(n)}(\varphi(z))u(z) - f^{(n)}(\psi(z))v(z)| \\
&= \sup_{z\in\mathbb{D}} |M^\varphi_u(z) f^{(n)}(\varphi(z))(1-|\varphi(z)|^2)^{\alpha+n} - M^\psi_v(z) f^{(n)}(\psi(z))(1-|\psi(z)|^2)^{\alpha+n}| \\
&\leq \sup_{z\in\mathbb{D}} |M^\varphi_u(z) - M^\psi_v(z)|\,|f^{(n)}(\varphi(z))(1-|\varphi(z)|^2)^{\alpha+n}| \\
&\quad + \sup_{z\in\mathbb{D}} |M^\psi_v(z)|\,|f^{(n)}(\varphi(z))(1-|\varphi(z)|^2)^{\alpha+n} - f^{(n)}(\psi(z))(1-|\psi(z)|^2)^{\alpha+n}| \\
&\leq C\|f\|_{\mathcal{A}^{-\alpha}} \sup_{z\in\mathbb{D}} |M^\varphi_u(z) - M^\psi_v(z)| + C\|f\|_{\mathcal{A}^{-\alpha}} \sup_{z\in\mathbb{D}} |M^\psi_v(z)|\rho(\varphi(z),\psi(z)).
\end{aligned}
$$

Therefore conditions (3.2)–(3.3) imply that $D^n_{\varphi,u} - D^n_{\psi,v} : \mathcal{A}^{-\alpha} \to \mathcal{A}^{-\beta}$ is bounded. ∎

From Theorem 3.1 with $v(z)=0$, we obtain a characterization of boundedness of generalized weighted composition operators between growth spaces.

Corollary 3.2. *Let n be a positive integer, $\alpha, \beta > 0$, $\varphi \in S(\mathbb{D})$ and $u \in H(\mathbb{D})$. Then $D^n_{\varphi,u} : \mathcal{A}^{-\alpha} \to \mathcal{A}^{-\beta}$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}} |M^\varphi_u(z)| < \infty.$$

Corollary 3.3. *Let n be a positive integer, $\alpha, \beta > 0$, $\varphi, \psi \in S(\mathbb{D})$ and $u \in H(\mathbb{D})$. Then the following statements are equivalent:*

(a) $D^n_{\varphi,u} - D^n_{\psi,u} : \mathcal{A}^{-\alpha} \to \mathcal{A}^{-\beta}$ *is bounded;*

(b) $\sup_{z\in\mathbb{D}} |M^\varphi_u(z)|\rho(\varphi(z),\psi(z)) < \infty$ *and* $\sup_{z\in\mathbb{D}} |M^\psi_u(z)|\rho(\varphi(z),\psi(z)) < \infty$;

(c) $\sup_{z\in\mathbb{D}} |M^\varphi_u(z)|\rho(\varphi(z),\psi(z)) < \infty$ *and* $\sup_{z\in\mathbb{D}} |M^\varphi_u(z) - M^\psi_u(z)| < \infty$;

(d) $\sup_{z\in\mathbb{D}} |M^\psi_u(z)|\rho(\varphi(z),\psi(z)) < \infty$ *and* $\sup_{z\in\mathbb{D}} |M^\varphi_u(z) - M^\psi_u(z)| < \infty$.

Proof. From Theorem 3.1 with $u = v$, we can directly see that the conditions (a), (c) and (d) are equivalent. It is enough to prove (b)⇒(c).

So assume that (b) holds. For some positive number $r < 1$, it is easy to see that

$$\sup_{\rho(\varphi(z),\psi(z))\ge r,\, z\in\mathbb{D}} |M^\varphi_u(z) - M^\psi_u(z)| < \infty.$$

If $\rho(\varphi(z),\psi(z)) < r$, since for all $z, w \in \mathbb{D}$ the pseudo-hyperbolic metric obeys the inequality [DO, Lemma 3.1]

$$\frac{1-\rho(w,z)}{1+\rho(w,z)} \le \frac{1-|z|^2}{1-|w|^2} \le \frac{1+\rho(w,z)}{1-\rho(w,z)}, \tag{3.10}$$

utilizing the boundedness of the functions

$$\frac{1-(\frac{1-x}{1+x})^{\alpha+n}}{x} \quad \text{and} \quad \frac{(\frac{1+x}{1-x})^{\alpha+n}-1}{x}$$

in $[0,r]$ (where the values of these functions at zero are defined as the limits from the right), we have

$$\begin{aligned} &|M^\varphi_u(z) - M^\psi_u(z)| \\ &\quad = \frac{(1-|z|^2)^\beta|u(z)|}{(1-|\varphi(z)|^2)^{\alpha+n}}\left|1-\left(\frac{1-|\varphi(z)|^2}{1-|\psi(z)|^2}\right)^{\alpha+n}\right| \le \frac{(1-|z|^2)^\beta|u(z)|}{(1-|\varphi(z)|^2)^{\alpha+n}} \\ &\quad\quad \times \max\left\{\left(\frac{1+\rho(\varphi(z),\psi(z))}{1-\rho(\varphi(z),\psi(z))}\right)^{\alpha+n} - 1, 1-\left(\frac{1-\rho(\varphi(z),\psi(z))}{1+\rho(\varphi(z),\psi(z))}\right)^{\alpha+n}\right\} \\ &\quad \le C\frac{(1-|z|^2)^\beta|u(z)|}{(1-|\varphi(z)|^2)^{\alpha+n}}\rho(\varphi(z),\psi(z)) = C|M^\varphi_u(z)|\rho(\varphi(z),\psi(z)) < \infty. \end{aligned}$$

Therefore,

$$\sup_{z\in\mathbb{D}} |M_u^{\varphi}(z) - M_u^{\psi}(z)| < \infty. \ \blacksquare$$

From Corollaries 3.2 and 3.3, we obtain the following result.

COROLLARY 3.4. *Let n be a positive integer, $\alpha, \beta > 0$, $\varphi, \psi \in S(\mathbb{D})$. Then $D_\varphi^n - D_\psi^n : \mathcal{A}^{-\alpha} \to \mathcal{A}^{-\beta}$ is bounded if and only if both $\sigma D_\varphi^n = D_{\varphi,\sigma}^n$ and $\sigma D_\psi^n = D_{\psi,\sigma}^n$ are bounded.*

EXAMPLE 3.5. The bounedness of $D_{\varphi,u}^n - D_{\psi,v}^n : \mathcal{A}^{-\alpha} \to \mathcal{A}^{-\beta}$ does not imply that both $D_{\varphi,u}^n : \mathcal{A}^{-\alpha} \to \mathcal{A}^{-\beta}$ and $D_{\psi,v}^n : \mathcal{A}^{-\alpha} \to \mathcal{A}^{-\beta}$ are bounded. In fact, let

$$\begin{aligned} u(z) = v(z) &= \frac{(2s - s^2(1-z))^{\alpha+n}}{(1-z)^{\beta-\alpha-n+1}}, \\ \varphi(z) &= 1 + s(z-1), \qquad \psi(z) = \varphi(z) + t(z-1)^b, \end{aligned}$$

where $0 < s < 1$, $b \geq 3 + n + \alpha$, t is real and $|t|$ is small enough that $\psi \in S(\mathbb{D})$.

Since for every $z \in \overline{\mathbb{D}}$, $\varphi(z)$ lies in the tangent disk $\{z : |1-z|^2 \leq s(1-|z|^2)/(1-s)\}$ with center at $1-s$ and radius s, we have

$$\frac{s}{1-s}(1 - |\varphi(z)|^2) \geq |1 - \varphi(z)|^2 \tag{3.11}$$

for all $z \in \overline{\mathbb{D}}$. Therefore

$$\begin{aligned} |1 - \overline{\varphi(z)}\psi(z)| &= |1 - |\varphi(z)|^2 - t\overline{\varphi(z)}(z-1)^b| \\ &\geq 1 - |\varphi(z)|^2 - |t|\,|z-1|^b \geq s(1-s)|1-z|^2 - |t|\,|z-1|^b \\ &\geq [s(1-s) - 2^{b-2}|t|]\,|1-z|^2 \geq \delta|1-z|^2, \end{aligned}$$

where δ is a positive constant for $|t|$ small enough, and then

$$\rho(\varphi(z), \psi(z)) = \left| \frac{\varphi(z) - \psi(z)}{1 - \overline{\varphi(z)}\psi(z)} \right| \leq C|1-z|^{b-2}. \tag{3.12}$$

From (3.11) and (3.12), we have

$$\begin{aligned} \sup_{z\in\mathbb{D}} |M_u^{\varphi}(z)| \rho(\varphi(z), \psi(z)) &= \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta |u(z)|}{(1-|\varphi(z)|^2)^{\alpha+n}} \rho(\varphi(z), \psi(z)) \\ &\leq C \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta |u(z)|}{|1-\varphi(z)|^{2(\alpha+n)}} \rho(\varphi(z), \psi(z)) \end{aligned} \tag{3.13}$$

$$
\begin{aligned}
&\le C\sup_{z\in\mathbb{D}}(1-|z|)^{\beta}|1-z|^{b-2}\frac{|u(z)|}{|1-\varphi(z)|^{2(\alpha+n)}}\\
&\le C\sup_{z\in\mathbb{D}}|1-z|^{\beta}|1-z|^{b-2}\frac{1}{|s(1-z)|^{2(\alpha+n)}}\left|\frac{(2s-s^2(1-z))^{\alpha+n}}{(1-z)^{\beta-\alpha-n+1}}\right|\\
&\le C\sup_{z\in\mathbb{D}}|1-z|^{b-\alpha-n-3}<\infty.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
(3.14)\quad &\sup_{z\in\mathbb{D}}|M_v^{\psi}(z)|\rho(\varphi(z),\psi(z))=\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^{\beta}|v(z)|}{(1-|\psi(z)|^2)^{\alpha+n}}\rho(\varphi(z),\psi(z))\\
&\le C\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^{\beta}|v(z)|}{|1-\psi(z)|^{2(\alpha+n)}}\rho(\varphi(z),\psi(z))\\
&\le C\sup_{z\in\mathbb{D}}(1-|z|)^{\beta}|1-z|^{b-2}\frac{|v(z)|}{|1-\psi(z)|^{2(\alpha+n)}}\\
&\le C\sup_{z\in\mathbb{D}}\frac{|1-z|^{\beta}|1-z|^{b-2}}{|s(1-z)(1+(t/s)(z-1)^{b-1})|^{2(\alpha+n)}}\left|\frac{(2s-s^2(1-z))^{\alpha+n}}{(1-z)^{\beta-\alpha-n+1}}\right|\\
&\le C\sup_{z\in\mathbb{D}}|1-z|^{b-\alpha-n-3}<\infty.
\end{aligned}
$$

From Corollary 3.3, and by (3.13)–(3.14) we deduce that $D_{\varphi,u}^n-D_{\psi,v}^n:\mathcal{A}^{-\alpha}\to\mathcal{A}^{-\beta}$ is bounded.

On the other hand,

$$
\begin{aligned}
M_u^{\varphi}(r)&=\frac{(1+r)^{\beta}}{1-r}\to\infty,\\
M_v^{\psi}(r)&=\frac{(1+r)^{\beta}(2s-s^2(1-r))^{\alpha+n}}{(1-r)[2(s+t(r-1)^{b-1})+(r-1)(s+t(r-1)^{b-1})^2]^{\alpha+n}}\to\infty
\end{aligned}
$$

as $r\to1^-$. This implies that, by Corollary 3.2, neither $D_{\varphi,u}^n:\mathcal{A}^{-\alpha}\to\mathcal{A}^{-\beta}$ nor $D_{\psi,v}^n:\mathcal{A}^{-\alpha}\to\mathcal{A}^{-\beta}$ is bounded.

4. The compactness of $D_{\varphi,u}^n-D_{\psi,v}^n$. In order to characterize the compactness of $D_{\varphi,u}^n-D_{\psi,v}^n:\mathcal{A}^{-\alpha}\to\mathcal{A}^{-\beta}$, we will use the following conditions in this section:

$$(4.1)\qquad \lim_{|\varphi(z)|\to1}\left|M_u^{\varphi}(z)\right|\rho(\varphi(z),\psi(z))=0;$$

$$(4.2)\qquad \lim_{|\psi(z)|\to1}|M_v^{\psi}(z)|\rho(\varphi(z),\psi(z))=0;$$

$$(4.3)\qquad \lim_{\min\{|\varphi(z)|,|\psi(z)|\}\to1}|M_u^{\varphi}(z)-M_v^{\psi}(z)|=0.$$

THEOREM 4.1. *Let n be a positive integer, $\alpha, \beta > 0$, $\varphi, \psi \in S(\mathbb{D})$ and $u, v \in H(\mathbb{D})$. Then $D^n_{\varphi,u} - D^n_{\psi,v} : \mathcal{A}^{-\alpha} \to \mathcal{A}^{-\beta}$ is compact if and only if $D^n_{\varphi,u} - D^n_{\psi,v} : \mathcal{A}^{-\alpha} \to \mathcal{A}^{-\beta}$ is bounded and conditions* (4.1)–(4.3) *hold.*

Proof. First, we prove sufficiency. Assume $D^n_{\varphi,u} - D^n_{\psi,v} : \mathcal{A}^{-\alpha} \to \mathcal{A}^{-\beta}$ is bounded. Then conditions (3.1)–(3.3) hold. Since (4.1)–(4.3) hold by assumption, for each $\varepsilon > 0$ there exists $0 < r < 1$ such that

$$|M^\varphi_u(z)|\rho(\varphi(z), \psi(z)) < \varepsilon \quad \text{when } |\varphi(z)| > r; \tag{4.4}$$

$$|M^\psi_v(z)|\rho(\varphi(z), \psi(z)) < \varepsilon \quad \text{when } |\psi(z)| > r; \tag{4.5}$$

$$|M^\varphi_u(z) - M^\psi_v(z)| < \varepsilon \quad \text{when } |\varphi(z)|, |\psi(z)| > r. \tag{4.6}$$

Let $(f_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{A}^{-\alpha}$ such that $\|f_k\|_{\mathcal{A}^{-\alpha}} \le 1$ and which converges to zero uniformly on compact subsets of $\mathbb{D}$. In order to prove that $D^n_{\varphi,u} - D^n_{\psi,v}$ is compact, by recalling Lemma 2.4, we only need to show that $\|(D^n_{\varphi,u} - D^n_{\psi,v})f_k\|_{\mathcal{A}^{-\beta}} \to 0$ as $k \to \infty$.

It is easy to see that

$$\begin{aligned} \|(D^n_{\varphi,u} - D^n_{\psi,v})f_k\|_{\mathcal{A}^{-\beta}} &= \sup_{z\in\mathbb{D}} (1-|z|^2)^\beta |(D^n_{\varphi,u} - D^n_{\psi,v})f_k(z)| \\ &= \sup_{z\in\mathbb{D}} (1-|z|^2)^\beta |f_k^{(n)}(\varphi(z))u(z) - f_k^{(n)}(\psi(z))v(z)| \\ &= \sup_{z\in\mathbb{D}} |M^\varphi_u(z) f_k^{(n)}(\varphi(z))(1-|\varphi(z)|^2)^{\alpha+n} - M^\psi_v(z) f_k^{(n)}(\psi(z))(1-|\psi(z)|^2)^{\alpha+n}|. \end{aligned} \tag{4.7}$$

We set

$$M^\varphi_u(z) f_k^{(n)}(\varphi(z))(1-|\varphi(z)|^2)^{\alpha+n} - M^\psi_v(z) f_k^{(n)}(\psi(z))(1-|\psi(z)|^2)^{\alpha+n} = A_k(z) + B_k(z),$$

where

$$A_k(z) = (M^\varphi_u(z) - M^\psi_v(z)) f_k^{(n)}(\varphi(z))(1-|\varphi(z)|^2)^{\alpha+n},$$
$$B_k(z) = M^\psi_v(z)[f_k^{(n)}(\varphi(z))(1-|\varphi(z)|^2)^{\alpha+n} - f_k^{(n)}(\psi(z))(1-|\psi(z)|^2)^{\alpha+n}].$$

(i) If $|\varphi(z)| \le r$ and $|\psi(z)| \le r$, by (3.3), we have $|A_k(z)| < C|f_k^{(n)}(\varphi(z))|$. From Remark 2.3 and (3.2), we get

$$|B_k(z)| \le C|M^\psi_v(z)|\rho(\varphi(z), \psi(z))\mathcal{A}^{-\alpha}_r(f_k) \le C\mathcal{A}^{-\alpha}_r(f_k).$$

(ii) If $|\varphi(z)| \le r$ and $|\psi(z)| > r$, with the same argument as in case (i), we obtain $|A_k(z)| < C|f_k^{(n)}(\varphi(z))|$. Applying Lemma 2.2 and (4.5) leads to

$$|B_k(z)| \le C\|f_k\|_{\mathcal{A}^{-\alpha}} |M^\psi_v(z)|\rho(\varphi(z), \psi(z)) \le C\varepsilon.$$

(iii) If $|\varphi(z)| > r$ and $|\psi(z)| > r$, by Lemma 2.2 and (4.6), we have

$$|A_k(z)| < C|M^\varphi_u(z) - M^\psi_v(z)|\, \|f_k\|_{\mathcal{A}^{-\alpha}} < C\varepsilon.$$

With the same argument as in case (ii), we get

$$|B_k(z)| \le C\|f_k\|_{\mathcal{A}^{-\alpha}}|M_v^{\psi}(z)|\rho(\varphi(z),\psi(z)) \le C\varepsilon.$$

(iv) If $|\varphi(z)| > r$ and $|\psi(z)| \le r$, we reset

$$M_u^{\varphi}(z)f_k^{(n)}(\varphi(z))(1-|\varphi(z)|^2)^{\alpha+n} - M_v^{\psi}(z)f_k^{(n)}(\psi(z))(1-|\psi(z)|^2)^{\alpha+n} = E_k(z)+F_k(z),$$

where

$$E_k(z) = -(M_v^{\psi}(z) - M_u^{\varphi}(z))f_k^{(n)}(\psi(z))(1-|\psi(z)|^2)^{\alpha+n},$$
$$F_k(z) = -M_u^{\varphi}(z)\big[f_k^{(n)}(\psi(z))(1-|\psi(z)|^2)^{\alpha+n} - f_k^{(n)}(\varphi(z))(1-|\varphi(z)|^2)^{\alpha+n}\big].$$

Using (3.3) again, we have $|E_k(z)| < C|f_k^{(n)}(\psi(z))|$. Applying Lemma 2.2 and (4.4), we obtain

$$|F_k(z)| \le C\|f_k\|_{\mathcal{A}^{-\alpha}}|M_u^{\varphi}(z)|\rho(\varphi(z),\psi(z)) \le C\varepsilon.$$

Therefore, from (4.7), we see that

$$\text{(4.8)}\quad \|(D_{\varphi,u}^n - D_{\psi,v}^n)f_k\|_{\mathcal{A}^{-\beta}} \le C\mathcal{A}_r^{-\alpha}(f_k) + C\sup_{|\varphi(z)|\le r}|f_k^{(n)}(\varphi(z))| + C\varepsilon + C\sup_{|\psi(z)|\le r}|f_k^{(n)}(\psi(z))|.$$

Since $\{z\in\mathbb{D} : |z|\le r\}$ is compact, and since

$$\mathcal{A}_r^{-\alpha}(f_k) = \max\Big\{\sup_{z\in\mathbb{D}_r}(1-|z|^2)^{\alpha+n}|f_k^{(n)}(z)|, \sup_{z\in\mathbb{D}_r}(1-|z|^2)^{\alpha+n+1}|f_k^{(n+1)}(z)|\Big\} < \max\Big\{\sup_{z\in\mathbb{D}_r}|f_k^{(n)}(z)|, \sup_{z\in\mathbb{D}_r}|f_k^{(n+1)}(z)|\Big\},$$

inequality (4.8) implies that $\|(D_{\varphi,u}^n - D_{\psi,v}^n)f_k\|_{\mathcal{A}^{-\beta}} \to 0$. Consequently, $D_{\varphi,u}^n - D_{\psi,v}^n$ is compact by Lemma 2.4.

Next we assume that $D_{\varphi,u}^n - D_{\psi,v}^n : \mathcal{A}^{-\alpha}\to\mathcal{A}^{-\beta}$ is compact. In that case $D_{\varphi,u}^n - D_{\psi,v}^n : \mathcal{A}^{-\alpha}\to\mathcal{A}^{-\beta}$ is bounded. Let $\{z_k\}$ be a sequence of points in $\mathbb{D}$ such that $|\varphi(z_k)|\to 1$ as $k\to\infty$. Define

$$l_k(z) = \frac{1}{\tau(\alpha+n)\overline{\varphi(z_k)}^n}\,\frac{1-|\varphi(z_k)|^2}{(1-\overline{\varphi(z_k)}z)^{\alpha+1}},$$
$$f_k(z) = \frac{l_w(z)}{\lambda_n+1}(\sigma_{\varphi(z_k)}(z) + \lambda_n/\overline{\varphi(z_k)}),$$

where λ_n and $\tau(\alpha+n)$ are defined as in the proof of Theorem 3.1. From

(3.4) and (3.5), we see that

$$\begin{aligned}(4.9)\quad &\|(D^n_{\varphi,u}-D^n_{\psi,v})l_k\|_{\mathcal{A}^{-\beta}}\\ &\geq \left|M^{\varphi}_u(z_k)\rho(\varphi(z_k),\psi(z_k))-\frac{M^{\psi}_v(z_k)\rho(\varphi(z_k),\psi(z_k))(1-|\psi(z_k)|^2)^{\alpha+n}}{(1-\overline{\varphi(z_k)}\psi(z_k))^{\alpha+n+1}(1-|\varphi(z_k)|^2)^{-1}}\right|,\end{aligned}$$

$$(4.10)\quad \|(D^n_{\varphi,u}-D^n_{\psi,v})f_k\|_{\mathcal{A}^{-\beta}}\geq\left|\frac{M^{\psi}_v(z_k)\rho(\varphi(z_k),\psi(z_k))(1-|\psi(z_k)|^2)^{\alpha+n}}{(1-\overline{\varphi(z_k)}\psi(z_k))^{\alpha+n+1}(1-|\varphi(z_k)|^2)^{-1}}\right|.$$

Since $D^n_{\varphi,u}-D^n_{\psi,v}$ is compact, by Lemma 2.4, $\|(D^n_{\varphi,u}-D^n_{\psi,v})l_k\|_{\mathcal{A}^{-\beta}}\to 0$ and $\|(D^n_{\varphi,u}-D^n_{\psi,v})f_k\|_{\mathcal{A}^{-\beta}}\to 0$ as $k\to\infty$. From (4.9) and (4.10), we conclude that (4.1) holds. Changing the test functions $l_k(z)$ and $f_k(z)$ by exchanging φ and ψ, we can prove (4.2).

From (3.6), we have

$$\begin{aligned}&\|(D^n_{\varphi,u}-D^n_{\psi,v})l_k\|_{\mathcal{A}^{-\beta}}\geq|M^{\varphi}_u(z_k)-M^{\psi}_v(z_k)|\\ &\quad-\left|M^{\psi}_v(z_k)\left[l_k^{(n)}(\varphi(z_k))(1-|\varphi(z_k)|^2)^{\alpha+n}-l_k^{(n)}(\psi(z_k))(1-|\psi(z_k)|^2)^{\alpha+n}\right]\right|.\end{aligned}$$

Since $\|(D^n_{\varphi,u}-D^n_{\psi,v})l_k\|_{\mathcal{A}^{-\beta}}\to 0$ as $|\varphi(z_k)|\to 1$, and since

$$\begin{aligned}&\left|M^{\psi}_v(z_k)\left[l_k^{(n)}(\varphi(z_k))(1-|\varphi(z_k)|^2)^{\alpha}-l_k^{(n)}(\psi(z_k))(1-|\psi(z_k)|^2)^{\alpha}\right]\right|\\ &\qquad\leq C\|l_k\|_{\mathcal{A}^{-\alpha}}|M^{\psi}_v(z_k)|\rho(\varphi(z_k),\psi(z_k))\to 0\end{aligned}$$

as $|\psi(z_k)|\to 1$ from Lemma 2.2 and (4.2), we get $|M^{\varphi}_u(z_k)-M^{\psi}_v(z_k)|\to 0$ as $|\varphi(z_k)|\to 1$ and $|\psi(z_k)|\to 1$. This implies (4.3). ∎

From Theorem 4.1 with $v(z)=0$, we obtain a characterization of compactness of generalized weighted composition operators between growth spaces.

Corollary 4.2. *Let $\varphi\in S(\mathbb{D})$, $\alpha,\beta>0$, $u\in H(\mathbb{D})$. Then $D^n_{\varphi,u}:\mathcal{A}^{-\alpha}\to\mathcal{A}^{-\beta}$ is compact if and only if $D^n_{\varphi,u}:\mathcal{A}^{-\alpha}\to\mathcal{A}^{-\beta}$ is bounded and*

$$\lim_{|\varphi(z)|\to 1}|M^{\psi}_u(z)|=0.$$

Corollary 4.3. *Let n be a positive integer, $\alpha,\beta>0$, $\varphi,\psi\in S(\mathbb{D})$, $u\in H(\mathbb{D})$. Then $D^n_{\varphi,u}-D^n_{\psi,u}:\mathcal{A}^{-\alpha}\to\mathcal{A}^{-\beta}$ is compact if and only if $D^n_{\varphi,u}-D^n_{\psi,u}:\mathcal{A}^{-\alpha}\to\mathcal{A}^{-\beta}$ is bounded and the following conditions hold:*

$$\lim_{|\varphi(z)|\to 1}|M^{\varphi}_u(z)|\rho(\varphi(z),\psi(z))=0,\qquad \lim_{|\psi(z)|\to 1}|M^{\psi}_u(z)|\rho(\varphi(z),\psi(z))=0.$$

Proof. From Theorem 4.1 with $u=v$, necessity is obvious.

For the converse, from Theorem 4.1, in order to prove that $D^n_{\varphi,u}-D^n_{\psi,u}:\mathcal{A}^{-\alpha}\to\mathcal{A}^{-\beta}$ is compact, it is enough to show that

$$(4.11)\qquad \lim_{\min\{|\varphi(z)|,|\psi(z)|\}\to 1}|M^{\varphi}_u(z)-M^{\psi}_u(z)|=0.$$

Suppose (4.11) does not hold; then there exist $\varepsilon_0 > 0$ and a sequence of points $\{z_k\}$ in $\mathbb{D}$ such that $|\varphi(z_k)| \to 1$ and $|\psi(z_k)| \to 1$ as $k \to \infty$ and

$$|M_u^{\varphi}(z_k) - M_u^{\psi}(z_k)| \geq \varepsilon_0. \tag{4.12}$$

We claim that $\rho(\varphi(z_k), \psi(z_k)) \to 0$ as $k \to \infty$.

In fact, if this is not the case, then there exists a subsequence $\{z_{n_k}\}$ of $\{z_k\}$ such that $\rho(\varphi(z_{n_k}), \psi(z_{n_k})) \to s > 0$. On the other hand,

$$\lim_{k\to\infty} |M_u^{\varphi}(z_{n_k})|\rho(\varphi(z_{n_k}), \psi(z_{n_k})) = 0,$$
$$\lim_{k\to\infty} |M_u^{\psi}(z_{n_k})|\rho(\varphi(z_{n_k}), \psi(z_{n_k})) = 0.$$

Therefore

$$\lim_{k\to\infty} |M_u^{\varphi}(z_{n_k})| = 0, \qquad \lim_{k\to\infty} |M_u^{\psi}(z_{n_k})| = 0,$$

which contradicts (4.12).

Therefore we may assume that $\rho(\varphi(z_k), \psi(z_k)) < r < 1$ for all n. Using similar arguments to those in the proof of Corollary 3.3, we have

$$|M_u^{\varphi}(z_k) - M_u^{\psi}(z_k)| \leq C|M_u^{\varphi}(z_k)|\rho(\varphi(z_k), \psi(z_k)) \to 0,$$

which contradicts (4.12) again. So we conclude that (4.11) holds. ■

From Corollaries 3.4, 4.2 and 4.3, we obtain the following result.

COROLLARY 4.4. *Let n be a positive integer, $\alpha, \beta > 0$, $\varphi, \psi \in S(\mathbb{D})$. Then $D_{\varphi}^n - D_{\psi}^n : \mathcal{A}^{-\alpha} \to \mathcal{A}^{-\beta}$ is compact if and only if both $\sigma D_{\varphi}^n = D_{\varphi,\sigma}^n$ and $\sigma D_{\psi}^n = D_{\psi,\sigma}^n$ are compact.*

EXAMPLE 4.5. The compactness of $D_{\varphi,u}^n - D_{\psi,v}^n : \mathcal{A}^{-\alpha} \to \mathcal{A}^{-\beta}$ does not imply that both $D_{\varphi,u}^n : \mathcal{A}^{-\alpha} \to \mathcal{A}^{-\beta}$ and $D_{\psi,v}^n : \mathcal{A}^{-\alpha} \to \mathcal{A}^{-\beta}$ are compact. In fact, define $\varphi(z)$ and $\psi(z)$ as in Example 3.5, and

$$u(z) = v(z) = \frac{(2s - s^2(1-z))^{\alpha+n}}{(1-z)^{\beta-\alpha-n}}.$$

Then

$$M_u^{\varphi}(r) = (1+r)^{\beta} \to 2^{\beta},$$
$$M_v^{\psi}(r) = \frac{(1+r)^{\beta}(2s - s^2(1-r))^{\alpha+n}}{[2(s+t(r-1)^{b-1}) + (r-1)(s+t(r-1)^{b-1})^2]^{\alpha+n}} \to 2^{\beta}$$

as $r \to 1^-$, i.e. $\varphi(r) \to 1$. In view of $\rho < 1$, from this, we conclude by Corollary 3.3 that $D_{\varphi,u}^n - D_{\psi,v}^n : \mathcal{A}^{-\alpha} \to \mathcal{A}^{-\beta}$ is bounded, and that neither $D_{\varphi,u}^n : \mathcal{A}^{-\alpha} \to \mathcal{A}^{-\beta}$ nor $D_{\psi,v}^n : \mathcal{A}^{-\alpha} \to \mathcal{A}^{-\beta}$ is compact by Corollary 4.2.

On the other hand, since $\rho(\varphi(z), \psi(z)) \to 0$ as $z \to 1$, we have

$$|M_u^{\varphi}(z)|\rho(\varphi(z), \psi(z)), |M_v^{\psi}(z)|\rho(\varphi(z), \psi(z)) \leq C\rho(\varphi(z), \psi(z)) \to 0$$

as $|\varphi(z)|, |\psi(z)| \to 1$, which implies that $D^n_{\varphi,u} - D^n_{\psi,v} : \mathcal{A}^{-\alpha} \to \mathcal{A}^{-\beta}$ is compact from Corollary 4.3.

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