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Complete pluripolar graphs in $\mathbb{C}^N$

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Abstract. Let F be the Cartesian product of N closed sets in $\mathbb{C}$. We prove that there exists a function g which is continuous on F and holomorphic on the interior of F such that $\Gamma_g(F) := \{(z, g(z)) : z \in F\}$ is complete pluripolar in $\mathbb{C}^{N+1}$. Using this result, we show that if D is an analytic polyhedron then there exists a bounded holomorphic function g such that $\Gamma_g(D)$ is complete pluripolar in $\mathbb{C}^{N+1}$. These results are high-dimensional analogs of the previous ones due to Edlund [*Complete pluripolar curves and graphs*, Ann. Polon. Math. 84 (2004), 75–86] and Levenberg, Martin and Poletsky [*Analytic disks and pluripolar sets*, Indiana Univ. Math. J. 41 (1992), 515–532].

1. Introduction. One of traditional problems in complex analysis is the question of holomorphic propagation, like for instance finding the maximal holomorphic object containing a given one. For example, let f be a holomorphic function defined on a domain D in $\mathbb{C}^N$; we search for its holomorphic continuation on a larger domain. A natural counterpart of holomorphic propagation in pluripotential theory is the theory of pluripolar hull. We now recall briefly some elements of pluripotential theory leading to the concept of pluripolar hull.

An upper semicontinuous function u, $u \not\equiv -\infty$, defined on a domain $D \subset \mathbb{C}^N$ is said to be *plurisubharmonic* if, for every complex line l, the restriction of u to each connected component of $D \cap l$ is either a subharmonic function or identically equal to $-\infty$. A subset E of D is said to be *pluripolar* if locally E is included in the $-\infty$ locus of plurisubharmonic functions. According to a well-known result of Josefson, every pluripolar set E is contained in the singular locus of some global plurisubharmonic function u on $\mathbb{C}^N$.

Given a pluripolar subset E of D, following Poletsky and Levenberg [8] we define the *pluripolar hull* of E relative to D as follows:

$$E^*_D := \{z \in D : \forall u \in \mathrm{PSH}(D),\ u|_E \equiv -\infty \Rightarrow u(z) = -\infty\}.$$

Here $\mathrm{PSH}(D)$ denotes the cone of plurisubharmonic functions on D. It is

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clear that E^*_D is pluripolar. It is also obvious that if E is *complete pluripolar* in D, i.e. E coincides with the $-\infty$ locus of an element $u \in \mathrm{PSH}(D)$, then $E^*_D = E$. Conversely, Zeriahi proved in [11] that if $E^*_D = E$ with E being F_σ and G_δ and D being pseudoconvex, then E must be complete pluripolar in D. Recall that a domain $D \subset \mathbb{C}^N$ is said to be *pseudoconvex* if D admits a plurisubharmonic exhaustion function.

This article focuses on the problem of constructing functions defined on a given set such that their graphs are complete pluripolar. In the univariate case, let Δ be the unit disk in $\mathbb{C}$; Levenberg, Martin and Poletsky [7], continuing the work of Sadullaev [10], showed that if $f(z) = \sum_{k\ge 0} a_{n(k)} z^{n(k)}$ is a gap series with radius of convergence 1 and with gaps satisfying $\lim_{k\to\infty} n(k)/n(k+1) = 0$, then $\Gamma_f(\Delta)$ is complete pluripolar in $\mathbb{C}^2$. In this paper, for a subset X of $\mathbb{C}^N$ and a function f defined on X, the graph of f over X is defined by

$$\Gamma_f(X) := \{(z, f(z)) : z \in X\}.$$

Moreover, in the case where the gap series satisfy some additional conditions, $\Gamma_f(\overline{\Delta})$ is also complete pluripolar. Edlund [5] generalized the latter result. More precisely, he showed that if F is a nonempty closed set in the complex plane, then there exists a continuous function f on F such that $\Gamma_f(F)$ is complete pluripolar in $\mathbb{C}^2$. Using the method of Edlund, we construct a continuous function g defined on the Cartesian product F of N univariate closed sets such that $\Gamma_F(g)$ is complete pluripolar. Moreover, the function can be chosen to be holomorphic in the interior of F. It is our first main result, Theorem 2.1.

The second result, Theorem 2.2, is a generalization of the above-mentioned theorem of Levenberg, Martin and Poletsky. Namely, for any connected analytic polyhedron $D \subset \mathbb{C}^N$ we construct a holomorphic function g on D such that $\Gamma_g(D)$ is complete pluripolar in $\mathbb{C}^{N+1}$. For the proof, we first use Theorem 2.1 to get a holomorphic function h on Δ^k continuous up to the boundary such that $\Gamma_h(\overline{\Delta}^k)$ is complete pluripolar in $\mathbb{C}^{k+1}$. Next we let g be the composition of h with the analytic mapping that defines D as an analytic polyhedron. Then by a theorem of Colţoiu on the equivalence between closed locally complete pluripolar sets and closed globally complete pluripolar sets, we conclude that $\Gamma_g(\overline{D})$ is complete pluripolar in $\mathbb{C}^{N+1}$. Finally, we invoke a version of [4, Theorem 4.6] to conclude that the pluripolar hull of $\Gamma_g(D)$ is disjoint from the cylinder $\partial D \times \mathbb{C}$. So by Zeriahi's theorem we get complete pluripolarity of $\Gamma_g(D)$ in $\mathbb{C}^{N+1}$.

The next main result of the paper concerns the problem of determining the pluripolar hull. For a simple explicit example of pluripolar hull, we can take $E = 0 \times \{|z| = 1\} \subset \mathbb{C}^2$ and $D = \mathbb{C}^2$. Then it is easy to check, using the fact that the circle $|z| = 1$ is non-polar in $\mathbb{C}$, that the pluripolar hull is $E^*_D = \{0\} \times \mathbb{C}$. Nevertheless, given a pluripolar set $E \subset D$, it is quite hard in general to determine E^*_D. A typical object in the study of pluripolar hull is the graph $\Gamma_f(D)$ of a holomorphic function f over a domain $D \subset \mathbb{C}^N$. Using techniques from pluripotential theory, Edigarian and Wiegerinck [4] completely solved the problem of describing the pluripolar hull of $\{(z, f(z)) : z \in D \setminus E\}$ in $D \times \mathbb{C}$ where E is a closed polar subset of a domain D in $\mathbb{C}$ and f is holomorphic on $D \setminus E$. They proved in [4, Theorem 5.10] that, for all $z_0 \in E$, $(\Gamma_f(D \setminus E))^*_{D\times C}$ intersects the line $\{z_0\} \times \mathbb{C}$ in at most one point. Later using the rapidly convergent method, Poletsky and Wiegerinck [9, Theorem 3.6] constructed a Cantor compact set $K \subset \mathbb{C}$ and a holomorphic function f on $D := \mathbb{C} \setminus K$ such that $(\Gamma_f(D))^*_{\mathbb{C}^2}$ is two-sheeted over D. Other examples of holomorphic functions with multiple sheeted pluripolar hulls can be found in [2], [3] and [6]. Motivated by the above-mentioned results, we want to determine the additional points of the pluripolar hull of graphs. In Proposition 3.3, we prove that if f is a holomorphic function on a domain $D \subset \mathbb{C}$ such that the complement of $\Gamma_f(D)$ in $(\Gamma_f(D))^*_{\mathbb{C}^2} \cap (D \times \mathbb{C})$ is contained in $A \times \mathbb{C}$ where A is a *countable* set, then $(\Gamma_f(D))^*_{\mathbb{C}^2} \cap (D \times \mathbb{C}) = \Gamma_f(D)$. We suspect that the above result is true in a more general context where D is a domain in $\mathbb{C}^N$ and A is a *pluripolar* subset of D. However, our methods only give this partial result.

2. Graphs of continuous functions and holomorphic functions. The main results of this section are the following generalizations of the theorems due to Edlund [5] and Levenberg, Martin and Poletsky [7].

THEOREM 2.1. *Let $F_1, \ldots, F_N$ be non-empty closed subsets of $\mathbb{C}$ and $F = F_1 \times \cdots \times F_N$. Then there exists a function g which is continuous on F and holomorphic on the interior of F such that $\Gamma_g(F)$ is complete pluripolar in $\mathbb{C}^{N+1}$.*

THEOREM 2.2. *Let Ω be a domain in $\mathbb{C}^N$ and D be a connected analytic polyhedron relatively compact in Ω, i.e.*

$$D := \{z \in \Omega : |f_1(z)| < 1, \ldots, |f_k(z)| < 1\},$$

where $f_1, \ldots, f_k$ are holomorphic on Ω. Then there exists a function g such that

(i) *g is continuous on $\overline{D}$ and holomorphic on D,*
(ii) *$\Gamma_g(D)$ and $\Gamma_g(\overline{D})$ are complete pluripolar in $\mathbb{C}^{N+1}$.*

Since the interior of the set F described in Theorem 2.1 and the domain D given in Theorem 2.2 are pseudoconvex, it is plausible to make the following

CONJECTURE. *Let D be a bounded open pseudoconvex subset of $\mathbb{C}^N$. Then there exists a continuous function g on $\overline{D}$ which is holomorphic on D and such that $\Gamma_g(\overline{D})$ is complete pluripolar in $\mathbb{C}^{N+1}$.*

For the proof of Theorem 2.1, we will rely on the method given in [5]; in that article the case $N = 1$ was considered. It is convenient to first present some technical lemmas, in which we always assume that F_m is a *proper* closed subset of $\mathbb{C}$ for every $1 \le m \le N$. More precisely, for each $1 \le m \le N$, we choose a sequence of distinct points $\mathcal{T}_m = \{\beta_{mj}\}_{j=1}^{\infty} \subset \mathbb{C} \setminus F_m$ and a sequence $\{r_{mj}\}_{j=1}^{\infty}$ of radii such that

(1) $r_{mj} < 1$ for all $j \ge 1$ and $r_{mj} \le r_{m1}$ for all $j > 1$;
(2) for each compact set $K \subset \mathbb{C} \setminus F_m$, there exist finitely many (open) disks $\Delta(\beta_{mj}, r_{mj})$ intersecting K;
(3) $\bigcup_{j=1}^{\infty} \Delta(\beta_{mj}, r_{mj}) = \mathbb{C} \setminus F_m$.

Let us choose positive constants C_k such that

$$C_k > \max\{|\beta_{mj}| : 1 \le m \le N,\ 1 \le j \le k\} + k \qquad \forall 1 \le m \le N,$$

and choose an increasing sequence $\{n_j\}_{j=1}^{\infty}$ of natural numbers with the following properties:

(4) $n_{j+1} \ge j^3 n_j$ for all $j \ge 1$;
(5) the series $\sum_{k=1}^{\infty} (kn_k/n_{k+1}) \log C_k$ is convergent;
(6) the series $\sum_{k=1}^{\infty} (n_k/n_{k+1}) \log D_{mk}$ is convergent for all $m = 1, \ldots, N$, where $D_{mk} = \prod_{i=1}^{k} \operatorname{dist}(\beta_{mi}, F_m)$ and $\operatorname{dist}(\beta_{mi}, F_m)$ is the Euclidean distance from β_{mi} to F_m.

Put

$$a_{1j}^{(m)} = (r_{m1}(1-1/j))^{n_j}, \qquad a_{ij}^{(m)} = (a_{1i}^{(m)})^j (r_{mi}(1-1/j))^{n_j} \quad \text{for } i > 1. \tag{2.1}$$

It is easy to see that $\log a_{ij}^{(m)} = n_j O(1)$ as $j \to \infty$. Thanks to (4), we obtain

$$\sum_{k=1}^{\infty} \frac{\log a_{ik}^{(m)}}{n_{k+1}} > -\infty, \qquad \forall i \ge 1,\ \forall 1 \le m \le N. \tag{2.2}$$

We define the following polynomials and rational functions:

$$q_{mk}(\xi) = \prod_{i=1}^{k} (\xi - \beta_{mi})^{n_k}, \qquad f_{mk}(\xi) = \sum_{i=1}^{k} \sum_{j=1}^{k} \frac{a_{ij}^{(m)}}{(\xi - \beta_{mi})^{n_j}}, \qquad \xi \in \mathbb{C},\ k \ge 1.$$

Now we present several results of Edlund [5] that are useful for proving our main theorems.

Lemma 2.3. *The function f_m defined by*

$$f_m(\xi) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{a_{ij}^{(m)}}{(\xi - \beta_{mi})^{n_j}} \tag{2.3}$$

is continuous and bounded on F_m*. Moreover the estimate*

$$|f_m(\xi) - f_{mk}(\xi)| \le 8k\Big(1 - \frac{1}{k+1}\Big)^{n_{k+1}} \tag{2.4}$$

holds uniformly for $\xi \in F_m$ *for all sufficiently large natural numbers* k*. In particular,* f_m *is holomorphic on the interior of* F_m*.*

Proof. The above facts are basically contained in Lemma 4 of [5]. It remains to check boundedness of f_m on F_m. We set

$$\tilde a_{1j} = (1 - 1/j)^{n_j}, \qquad \tilde a_{ij} = (\tilde a_{1i})^j \tilde a_{1j} \quad \text{for } i > 1. \tag{2.5}$$

Fix $\xi \in F_m$; then $r_{mi}/|\xi - \beta_{mi}| < 1$. Combining (2.1) and (2.3) we obtain

$$\begin{aligned} |f(\xi)| &\le \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \tilde a_{ij} = \sum_{j=1}^{\infty} \tilde a_{1j} + \sum_{i=2}^{\infty}\sum_{j=1}^{\infty} (\tilde a_{1i})^j \tilde a_{1j} \\ &= \sum_{j=1}^{\infty} \tilde a_{1j}\Big(1 + \sum_{i=2}^{\infty} (\tilde a_{1i})^j\Big). \end{aligned} \tag{2.6}$$

Since $n_i \ge i^3$ for all $i \ge 1$, $\sum_{i=2}^{\infty} \tilde a_{1i}^j$ converges to $\sigma_j > 0$ for all $j \ge 1$. Moreover, the sequence $\{\sigma_j\}$ is decreasing. Hence,

$$|f(\xi)| \le \sum_{j=1}^{\infty} \tilde a_{1j}(1 + \sigma_j) \le (1 + \sigma_1)\sum_{j=1}^{\infty} \tilde a_{1j} < \infty. \ \blacksquare \tag{2.7}$$

The result below is essentially Lemma 5 in [5].

LEMMA 2.4. *If* $\xi \in \mathbb{C} \setminus (\mathcal{T}_m \cup F_m)$*, then* $|f_{mk}(\xi)| \to \infty$ *as* $k \to \infty$*.*

In the next lemma, the first part comes from [5, proof of Lemma 6]. The second one follows directly from the definition of D_{mk} and the observation that $|\xi - \beta_{mi}| \ge \operatorname{dist}(\beta_{mi}, F_m)$ for all $\xi \in F_m$. The last two assertions are shown in [5, Lemma 7]. Note also that (d) follows directly from the inequality $|\xi - \beta_{mi}| > \gamma_\xi > 0$ for all i, given in [5, p. 83].

LEMMA 2.5.

(a) *For* $r > 0$*,* $|\xi| < r$ *and sufficiently large* k*,*

$$|q_{mk}(\xi)| \le (C_k)^{kn_k} \quad \text{and} \quad |q_{mk}(\xi) f_{mk}(\xi)| \le k^2 (C_k)^{kn_k}.$$

(b) *If* $\xi \in F_m$*, then*

$$|q_{mk}(\xi)| \ge (D_{mk})^{n_k}.$$

(c) *If* $\xi = \beta_{mi} \in \mathcal{T}_m$*, then there exists* $\delta_\xi > 0$ *that is independent of* k *and such that*

$$|q_{mk}(\xi) f_{mk}(\xi)| \ge a_{ik}^{(m)} (\delta_\xi)^{(k-1)n_k}$$

for all sufficiently large k*.*

(d) *If* $\xi \in \mathbb{C} \setminus (\mathcal{T}_m \cup F_m)$, *then there exists* $\gamma_\xi > 0$ *that is independent of* k *and such that*

$$|q_{mk}(\xi)| \geq (\gamma_\xi)^{kn_k}.$$

Since the sequence $\{f_{mk}\}$ of continuous functions converges uniformly on F_m to f_m as $k \to \infty$, and f_m is bounded on F_m, we can choose a constant $M > 0$ such that

$$\|f_m\| := \sup_{\xi \in F_m} |f_m(\xi)| \leq M - 1, \qquad \|f_{mk}\| := \sup_{\xi \in F_m} |f_{mk}(\xi)| \leq M - 1, \tag{2.8}$$

for all $m = 1, \ldots, N$ and $k = 1, 2, \ldots$. Let P_k denote the polynomial

$$P_k(z,w) = \prod_{m=1}^{N} q_{mk}(z_m)\Big(\prod_{m=1}^{N}(f_{mk}(z_m) + M) - w\Big), \quad z = (z_1, \ldots, z_N). \tag{2.9}$$

Then

$$u_k(z,w) := \frac{1}{n_{k+1}} \log|P_k(z,w)|$$

is a plurisubharmonic function on $\mathbb{C}^{N+1}$.

Lemma 2.6. *The function* u *defined by*

$$u(z,w) = \sum_{k=1}^{\infty} \max\{u_k(z,w), -1\}, \qquad z \in \mathbb{C}^N,\ w \in \mathbb{C}, \tag{2.10}$$

is a plurisubharmonic function on $\mathbb{C}^{N+1}$. *Moreover, if* $u_k(z,w) \geq \alpha_k$ *for all* $k \geq k_0$ *and* $\sum_{k=k_0}^{\infty} \alpha_k$ *is convergent, then* $u(z,w) > -\infty$.

Proof. We first claim that for all $r > 0$, the series in (2.10) converges on the polydisk $\Delta^{N+1}(0,r)$ to a plurisubharmonic function. For $(z,w) \in \Delta^{N+1}(0,r)$, using Lemma 2.5(a) we have, when k is sufficiently large,

$$\Big|w \prod_{m=1}^{N} q_{mk}(z_m)\Big| \leq r(C_k)^{Nkn_k}$$

and

$$|q_{mk}(z_m)(f_{mk}(z_m) + M)| \leq (M + k^2)(C_k)^{kn_k}, \qquad 1 \leq m \leq N.$$

Hence

$$u_k(z,w) \leq \frac{\log(r + (r + k^2)^N)}{n_{k+1}} + N\frac{kn_k \log C_k}{n_{k+1}}, \qquad k \geq k_0.$$

The two series corresponding to the two terms on the right hand side are convergent in view of the conditions (4) and (5). The claim follows.

Now, the first assertion of the lemma is a consequence of the definition of u. For the second one, we see that $\lim_{k\to\infty} \alpha_k = 0$. Thus

$$\max\{u_k(z,w), -1\} \geq \max\{\alpha_k, -1\} = \alpha_k$$

for all sufficiently large k. Hence $u(z,w) \geq \text{const} + \sum_{k=k_1}^{\infty} \alpha_k > -\infty$. ∎

Now, we set

$$(2.11)\qquad g_k(z)=\prod_{m=1}^{N}(f_{mk}(z_m)+M),\quad g(z)=\prod_{m=1}^{N}(f_m(z_m)+M),\qquad z\in F.$$

Note that g is continuous on F and holomorphic on the interior of F. We will prove that $\Gamma_g(F)$ is complete pluripolar in $\mathbb{C}^{N+1}$. To this end, we need to estimate the error between g and its partial sum g_k.

Lemma 2.7. *The estimate*

$$(2.12)\qquad |g(z)-g_k(z)|\le 8N(2M)^{N-1}k\Big(1-\frac{1}{k+1}\Big)^{n_{k+1}}$$

holds uniformly for $z\in F$ for all sufficiently large k.

Proof. Using (2.8) we get

$$|f_{mk}(z_m)+M|<2M,\quad |f_m(z_m)+M|<2M,\ \forall m=1,\dots,N,\ k\ge 1,\ z_m\in F_m.$$

Hence, Lemma 2.3 yields

$$\begin{aligned}|g(z)-g_k(z)|&=\Big|\prod_{m=1}^{N}(f_m(z_m)+M)-\prod_{m=1}^{N}(f_{mk}(z_m)+M)\Big|\\&\le\sum_{m=1}^{N}\Big(\Big|\prod_{i=1}^{m-1}(f_i(z_i)+M)\Big|\\&\qquad\times|f_m(z_m)-f_{mk}(z_m)|\prod_{i=m+1}^{N}(f_{ik}(z_i)+M)|\Big)\\&\le(2M)^{N-1}\sum_{m=1}^{N}|f_m(z_m)-f_{mk}(z_m)|\\&=8N(2M)^{N-1}k\Big(1-\frac{1}{k+1}\Big)^{n_{k+1}}.\ \blacksquare\end{aligned}$$

Proof of Theorem 2.1. First, we assume that F_m is a proper subset of $\mathbb{C}$ for all $m=1,\dots,N$. Let g be the function defined in (2.11). We will prove that the graph of g over F is complete pluripolar in $\mathbb{C}^{N+1}$. More precisely, we will show

$$\Gamma_g(F)=\{(z,w)\in\mathbb{C}^{N+1}:u(z,w)=-\infty\}.$$

Here u is the function constructed in Lemma 2.6. The proof is divided into several steps.

Step 1: We prove that $u(z,g(z))=-\infty$ for all $z\in F$. Using Lemmas 2.5(a) and 2.7 we obtain the following estimates for $u_k(z,g(z))$ when k is large enough:

$$
\begin{aligned}
u_k(z,g(z)) &= \frac{1}{n_{k+1}}\log\Big|\prod_{m=1}^{N} q_{mk}(z_m)(g_k(z)-g(z))\Big| \\
&\le \frac{1}{n_{k+1}}\log\bigg((C_k)^{Nkn_k}\bigg[8N(2M)^{N-1}k\Big(1-\frac{1}{k+1}\Big)^{n_{k+1}}\bigg]\bigg) \\
&= \frac{\log(8N(2M)^{N-1}k)}{n_{k+1}} + N\frac{kn_k\log C_k}{n_{k+1}} + \log\Big(1-\frac{1}{k+1}\Big).
\end{aligned}
$$

Therefore, there exists a natural number k_0 such that

$$
\begin{aligned}
u(z,g(z)) &\le \mathrm{const} + \sum_{k=k_0}^{\infty} \max\bigg\{\frac{\log(8N(2M)^{N-1}k)}{n_{k+1}} \\
&\qquad\qquad + N\frac{kn_k\log C_k}{n_{k+1}} + \log\Big(1-\frac{1}{k+1}\Big), -1\bigg\} \\
&\le \mathrm{const} + \sum_{k=k_0}^{\infty}\bigg(\frac{\log(8N(2M)^{N-1}k)}{n_{k+1}} + N\frac{kn_k\log C_k}{n_{k+1}}\bigg) \\
&\quad + \sum_{k=k_0}^{\infty}\max\Big\{\log\Big(1-\frac{1}{k+1}\Big), -1\Big\} \\
&= -\infty.
\end{aligned}
$$

Here we use the inequality $\max\{a+t,-1\} \le a+\max\{t,-1\}$ for $a>0$. The middle series converges by (4) and (5), whereas the last one is divergent to $-\infty$.

Step 2: We show that $u(z,w) > -\infty$ for $z \in F$ and $w \ne g(z)$. Indeed, since

$$|g_k(z)-w| \ge |g(z)-w| - |g_k(z)-g(z)|,$$

we can choose $\delta > 0$ and $k_0 > 0$ such that

$$|g_k(z)-w| \ge \delta, \qquad \forall k \ge k_0. \tag{2.13}$$

Combining (2.13) with Lemma 2.5(b) we obtain

$$
\begin{aligned}
u_k(z,w) &= \frac{1}{n_{k+1}}\log\Big|\prod_{m=1}^{N} q_{mk}(z_m)(g_k(z)-w)\Big| \\
&\ge \frac{1}{n_{k+1}}\log\Big(\delta\prod_{m=1}^{N}(D_{mk})^{n_k}\Big) = \frac{\log\delta}{n_{k+1}} + \sum_{m=1}^{N}\frac{n_k}{n_{k+1}}\log D_{mk}.
\end{aligned}
$$

Thanks to (4) and (6), the two series corresponding to the last two terms above are convergent. Thus Lemma 2.6 implies that $u(z,w) > -\infty$.

STEP 3: For $w \in \mathbb{C}$ and $z = (z_1, \dots, z_N) \notin F$ such that $z_m \in \mathcal{T}_m$ for some m, we will verify that $u(z,w) > -\infty$. Let us partition the set $\{1, \dots, N\}$ into three parts I_1, I_2, I_3 by setting

$$I_1 = \{m : z_m \in \mathcal{T}_m\}, \quad I_2 = \{m : z_m \in F_m\}, \quad I_3 = \{m : z_m \in \mathbb{C} \setminus (F_m \cup \mathcal{T}_m)\}.$$

Note that I_1 is non-empty, but I_2 and I_3 may be empty. Next, we will find a lower bound of the polynomial

$$|q_{mk}(z_m)(f_{mk}(z_m) + M)|$$

separately for m belonging to I_1, I_2, I_3 .

For $m \in I_1$, there exists $i_m \geq 1$ such that $z_m = \beta_{mi_m}$. Hence $q_{mk}(z_m) = 0$ for sufficiently large k, say $k \geq k_0$. By Lemma 2.5(c) we can choose $\delta_{z_m} > 0$ so that

$$\begin{aligned} |q_{mk}(z_m)(f_{mk}(z_m) + M)| &= |q_{mk}(z_m) f_{mk}(z_m)| \\ &\geq a^{(m)}_{i_m k} (\delta_{z_m})^{(k-1)n_k}, \qquad k \geq k_0. \end{aligned} \tag{2.14}$$

Since $I_1 \neq \emptyset$, the product $w \prod_{m=1}^N q_{mk}(z_m)$ contains at least one vanishing factor. Thus

$$w \prod_{m=1}^{N} q_{mk}(z_m) = 0, \qquad k \geq k_0. \tag{2.15}$$

Next we treat the case $m \in I_2$. Since $\|f_{mk}\| := \sup\{|f_{mk}(z_m)| : z_m \in F_m\} \leq M - 1$, we have $|f_{mk}(z_m) + M| \geq 1$ for all $k \geq 1$. Now using Lemma 2.5(b) again, we obtain

$$|q_{mk}(z_m)(f_{mk}(z_m) + M)| \geq |q_{mk}(z_m)| \geq (D_{mk})^{n_k}. \tag{2.16}$$

Finally, if $m \in I_3$, then Lemma 2.4 gives $|f_{mk}(z_m)| \to \infty$ as $k \to \infty$. Thus $|f_{mk}(z_m) + M| \geq 1$ for all $k \geq k_1$. Now, Lemma 2.5(d) implies that there exists $\gamma_{z_m} > 0$ such that

$$|q_{mk}(z_m)(f_{mk}(z_m) + M)| \geq |q_{mk}(z_m)| \geq (\gamma_{z_m})^{kn_k}, \qquad \forall k \geq k_1. \tag{2.17}$$

Collecting the above estimates, we get, for all $k \geq \max(k_0, k_1)$,

$$\begin{aligned} u_k(z,w) &= \frac{1}{n_{k+1}} \log\Big| \prod_{m=1}^{N} q_{mk}(z_m)(g_k(z) - w) \Big| \\ &= \frac{1}{n_{k+1}} \log\Big| \prod_{m=1}^{N} (q_{mk}(z_m)(f_{mk}(z_m) + M)) \Big| \\ &\geq \frac{1}{n_{k+1}} \log\Big(\prod_{m \in I_1} a^{(m)}_{i_m k} (\delta_{z_m})^{(k-1)n_k} \cdot \prod_{m \in I_2} (D_{mk})^{n_k} \cdot \prod_{m \in I_3} (\gamma_{z_m})^{kn_k} \Big) \end{aligned}$$

$$= \sum_{m\in I_1}\Big(\frac{a^{(m)}_{i_m k}}{n_{k+1}} + \frac{(k-1)n_k}{n_{k+1}}\log\delta_{z_m}\Big)$$
$$+ \sum_{m\in I_2}\frac{n_k}{n_{k+1}}\log D_{mk} + \sum_{m\in I_3}\frac{kn_k}{n_{k+1}}\log\gamma_{z_m}$$
$$:= \alpha_k,$$

where we use (2.15) to get the second equality, and (2.14), (2.16) and (2.17) to get the inequality. Moreover, using (4), (6) and (2.2) we deduce that $\sum_{k=\max(k_0,k_1)}^{\infty}\alpha_k < \infty$. Therefore, Lemma 2.10 implies that $u(z,w) > -\infty$.

Step 4: We will prove that if $w \in \mathbb{C}$ and $z \notin F$ is such that $z_m \notin \mathcal{T}_m$ for all $m = 1,\ldots,N$, then $u(z,w) > -\infty$. Indeed, proceeding as in Step 3, we divide the set $\{1,\ldots,N\}$ into

$$J_1 = \{m : z_m \in F_m\}, \qquad J_2 = \{m : z_m \in \mathbb{C}\setminus(F_m \cup \mathcal{T}_m)\}.$$

Notice that $J_2 \neq \emptyset$. Using the same arguments as in Step 3, we get $|f_{mk}(z_m) + M| \ge 1$ for all $m \in I_1$, but $|f_{mk}(z_m) + M| \to \infty$ for all $m \in I_2$. Therefore we can take k_0 such that

$$|g_k(z) - w| \ge \prod_{m=1}^{N}|f_{mk}(z_m) + M| - |w|$$
$$= \prod_{m\in J_1}|f_{mk}(z_m) + M|\prod_{m\in J_2}|f_{mk}(z_m) + M| - |w| \ge 1, \quad \forall k \ge k_0.$$

On the other hand, if $m \in J_1$, then

$$|q_{mk}(z_m)| \ge (D_{mk})^{n_k},$$

and if $m \in J_2$, then we can choose $\gamma_{z_m} > 0$ so that

$$|q_{mk}(z_m)| \ge (\gamma_{z_m})^{kn_k}.$$

Hence, we obtain for all $k \ge k_0$ the following estimates:

$$u_k(z,w) = \frac{1}{n_{k+1}}\log\Big(\prod_{m=1}^{k}|q_{mk}(z_m)|\,|g_k(z) - w|\Big)$$
$$\ge \frac{1}{n_{k+1}}\log\Big(\prod_{m\in J_1}(D_{mk})^{n_k}\cdot\prod_{m\in J_2}(\gamma_{z_m})^{kn_k}\Big)$$
$$= \sum_{m\in J_1}\frac{n_k}{n_{k+1}}\log D_{mk} + \sum_{m\in J_2}(\log\gamma_{z_m})\frac{kn_k}{n_{k+1}}.$$

Again by the same reasoning as in Step 3, we have $u(z,w) > -\infty$. This step finishes the proof of the special case when F_m is a proper subset of $\mathbb{C}$ for all $m = 1,\ldots,N$.

Next, we treat the general case. The assertion is trivial if $F_m = \mathbb{C}$ for all $m = 1, \dots, N$. Assume that $F_1, \dots, F_m$ are proper closed subsets of $\mathbb{C}$ and $F_{m+1} = \dots = F_N = \mathbb{C}$. By the special case considered above, we can find a function h defined on $F_1 \times \dots \times F_m$ such that $\Gamma_{F_1\times\dots\times F_m}(h)$ is complete pluripolar in $\mathbb{C}^{m+1}$. Thus there exists $v \in \mathrm{PSH}(\mathbb{C}^{m+1})$ such that

$$\Gamma_{F_1\times\dots\times F_m}(h) = \{(z_1, \dots, z_m, w) \in \mathbb{C}^{m+1} : v(z_1, \dots, z_m, w) = -\infty\}.$$

The function g defined on F by $g(z_1, \dots, z_N) = h(z_1, \dots, z_m)$ is continuous on F and holomorphic on the interior of F. On the other hand, the function $\tilde{v}(z_1, \dots, z_N, w) = v(z_1, \dots, z_m, w)$ is in $\mathrm{PSH}(\mathbb{C}^{N+1})$. We easily check that

$$\Gamma_F(g) = \{(z, w) \in \mathbb{C}^{N+1} : \tilde{v}(z, w) = -\infty\}.$$

Thus $\Gamma_F(g)$ is complete pluripolar in $\mathbb{C}^{N+1}$. This completes the proof of Theorem 2.1. ■

For the proof of Theorem 2.2, we need the following fact whose proof will be postponed until the end of this paper.

LEMMA 2.8. *Let E be an F_σ pluripolar subset of $\mathbb{C}^n$ and let F be a holomorphic mapping from an open neighborhood U of $E^*_{\mathbb{C}^n}$ to $\mathbb{C}$ such that $F(E) \subset \Delta$ and $F(E^*_{\mathbb{C}^n}) \subset \overline{\Delta}$. Then $F(E^*_{\mathbb{C}^n}) \subset \Delta$.*

Proof of Theorem 2.2. We have

$$D = \{z \in \Omega : |f_1(z)| < 1, \dots, |f_k(z)| < 1\},$$

where $f_1, \dots, f_k$ are holomorphic on Ω. According to Theorem 2.1, we can find a continuous function h on $\overline{\Delta^k}$, holomorphic on Δ^k, such that $\Gamma_h(\overline{\Delta}^k)$ is complete pluripolar in $\mathbb{C}^{k+1}$. Let $F := (f_1, \dots, f_k)$ and $g := h\circ F$. Obviously, g is continuous on $\overline{D}$ and holomorphic on D. We will show that $X := \Gamma_g(D)$ is complete pluripolar in $\mathbb{C}^{N+1}$.

To see this, we first claim that $\overline{X} = \Gamma_g(\overline{D})$ is complete pluripolar in $\mathbb{C}^{N+1}$. Indeed, let u be a plurisubharmonic function on $\mathbb{C}^{k+1}$ such that $u = -\infty$ precisely on $\Gamma_h(\overline{\Delta}^k)$. Consider $\varphi(z, w) := u(F(z), w)$. Then φ is plurisubharmonic on $\Omega \times \mathbb{C}$ and $\varphi = -\infty$ exactly on $\overline{X}$. Since $\overline{X}$ is closed in $\mathbb{C}^{N+1}$, by a result of Colţoiu [1] about equivalence between locally complete pluripolarity and globally complete pluripolarity, we see that $\overline{X}$ is complete pluripolar in $\mathbb{C}^{N+1}$. This proves the claim.

Next, we show $X^*_{\mathbb{C}^{N+1}} = X$. By the above reasoning $X^*_{\mathbb{C}^{N+1}} \subset \overline{X}$. For every $1 \le j \le k$, consider the map $F_j : \Omega \times \mathbb{C} \to \mathbb{C}$ defined by

$$F_j(z_1, \dots, z_N, w) = f_j(z_1, \dots, z_N).$$

Then F_j is holomorphic on an open neighborhood of $X^*_{\mathbb{C}^{N+1}}$ and

$$F_j(X^*_{\mathbb{C}^{N+1}}) \subset F_j(\overline{X}) \subset \overline{\Delta}.$$

Observe that $F_j(X) \subset \Delta$, so we may apply Lemma 2.8 to conclude that

$$F_j(X^*_{\mathbb{C}^{N+1}}) \subset \Delta.$$

Since this is true for every $1 \le j \le k$, we infer $X^*_{\mathbb{C}^{N+1}} = X$.

Since $X = \overline{X} \setminus \Gamma_g(\partial D)$ and $\Gamma_g(\partial D)$ is closed we deduce that X is a G_δ set. Finally, since X is also F_σ, we apply Zeriahi's theorem to conclude that X is complete pluripolar in $\mathbb{C}^{N+1}$. The proof of Theorem 2.2 is complete. ∎

Remark. If D is a simply connected, proper subdomain in $\mathbb{C}$ with real analytic boundary, then by Riemann's mapping theorem we can find a bijective holomorphic map f from D onto Δ. Since D has real analytic boundary, the map f extends to a larger neighborhood Ω of $\overline{D}$. Thus D is a connected analytic polyhedron in Ω, and so by Theorem 2.2 we can find a continuous function g on $\overline{D}$ which is holomorphic on D and such that $\Gamma_g(D)$ and $\Gamma_g(\overline{D})$ are complete pluripolar in $\mathbb{C}^2$.

3. The pluripolar hulls of graphs. We recall some major tools that will be used in the study of pluripolar hulls. According to Levenberg and Poletsky [8], the negative pluripolar hull E^-_D of a pluripolar set $E \subset D$ is defined by

$$E^-_D = \bigcap \{z \in D : u(z) = -\infty \text{ if } u \in \mathrm{PSH}(D),\ u < 0,\ u|_E = -\infty\}.$$

The following result (Theorem 2.4 in [8]) gives a relation between the pluripolar hull and the negative pluripolar hull.

Theorem 3.1. *Let D be a pseudoconvex domain in $\mathbb{C}^n$. Let $\{D_j\}$ be an increasing sequence of relatively compact subdomains with $\bigcup_{j=1}^{\infty} D_j = D$. Let $E \subset D$ be pluripolar. Then*

$$E^*_D = \bigcup_{j=1}^{\infty} (E \cap D_j)^-_{D_j}.$$

Now for every subset E of $\overline{D}$ we define the *pluriharmonic measure of E relative to D* by

$$\omega(z, E, D) = -\sup\Big\{u(z) : u \in \mathrm{PSH}(D),\ u \le 0 \text{ on } D \text{ and } \limsup_{D \ni \xi \to w} u(\xi) \le -1 \text{ for } w \in E\Big\}, \quad z \in D.$$

The following connection between the pluriharmonic measure and negative pluripolar hull is again due to Levenberg and Poletsky [8]:

$$E^-_D = \{z \in D : \omega(z, E, D) > 0\}. \tag{3.1}$$

The estimate of pluriharmonic measures given below, due to Levenberg and Poletsky (Lemma 3.4 in [8]), is useful in studying pluripolar hulls.

LEMMA 3.2. *Let $D \subset\subset G$ be a domain in $\mathbb{C}^n$. Let $E \subset D$ be compact, and let V be a domain in G that contains a point $z \in D$ and does not intersect E. Let $K = \overline{\partial V \cap D}$. Then there exists a point $w \in K$ such that*

$$\omega(z, E, D) \leq \omega(w, E, G).$$

The following proposition is the main result of this section.

PROPOSITION 3.3. *Let D be a domain in $\mathbb{C}$ and f be holomorphic on D. Assume that there exists a countable subset A of D satisfying*

$$(\Gamma_f(D))^*_{\mathbb{C}^2} \cap (D \times \mathbb{C}) \subset \Gamma_f(D) \cup (A \times \mathbb{C}).$$

Then $(\Gamma_f(D))^*_{\mathbb{C}^2} \cap (D \times \mathbb{C}) = \Gamma_f(D)$.

For the proof, we first need the following

LEMMA 3.4. *Let f be a holomorphic function on a bounded domain D in $\mathbb{C}$. Assume that there exists a polar subset E of D such that*

$$(\Gamma_f(D))^*_{\mathbb{C}^2} \cap (D \times \mathbb{C}) \subset \Gamma_f(D) \cup (E \times \mathbb{C}).$$

Let $(z_0, w_0) \in (\Gamma_f(D))^*_{\mathbb{C}^2} \setminus \Gamma_f(D)$ with $z_0 \in E$, $w_0 \neq f(z_0)$. Then for every neighborhood U of z_0 in $\mathbb{C}$ we can find $z' \in E \cap U$, $z'' \in U \setminus \{z'\}$ such that $(z', f(z'')) \in (\Gamma_f(D))^*_{\mathbb{C}^2}$ and $f(z') \neq f(z'')$.

Proof. Choose a closed disk S in D such that $z_0 \notin S$. For $j \geq 1$ we set

$$\Delta^2_j := \{(z, w) \in \mathbb{C}^2 : |z| < j,\ |w| < j\}.$$

Then from Theorem 3.1 we get the relation

$$(z_0, w_0) \in (\Gamma_f(D))^*_{\mathbb{C}^2} = (\Gamma_f(S))^*_{\mathbb{C}^2} = \bigcup_{j \geq 1} (\Gamma_f(S) \cap \Delta^2_j)^-_{\Delta^2_j}.$$

So we can choose j_0 so large that the following properties hold:

(1) $\Gamma_f(S) \subset \Delta^2_{j_0}$, $D \Subset \{|z| < j_0\}$;
(2) $(z_0, w_0) \in (\Gamma_S(f))^-_{\Delta^2_{j_0}}$.

Since $z_0 \notin S$, we see that E is polar, and since f is non-constant, we can choose an open disk D' around z_0 satisfying the following conditions:

(3) $D' \cap S = \emptyset$, $w_0 \notin f(\overline{D'})$;
(4) $\partial D' \cap E = \emptyset$;
(5) $f(z_0) \notin f(\partial D')$.

In view of (5) we can choose another disk $D'' \Subset D'$ such that $z_0 \in D''$, $f(\partial D') \cap f(\overline{D''}) = \emptyset$ and $\partial D'' \cap E = \emptyset$.

Now we consider the open set

$$V := D'' \times \{\{w \in \mathbb{C} : |w| < j_0 + 1\} \setminus f(\overline{D'})\}.$$

It follows from (3) that $(z_0, w_0) \in V$ and $V \cap \Gamma_f(S) = \emptyset$. So we may apply Lemma 3.2 to obtain a point $\xi = (\xi_1, \xi_2) \in \overline{(\partial V) \cap \Delta^2_{j_0}} \subset (\partial V) \cap \overline{\Delta^2_{j_0}}$ such that

$$0 < \omega((z_0, w_0), \Gamma_S(f), \Delta^2_{j_0}) \le \omega(\xi, \Gamma_S(f), \Delta^2_{j_0+1});$$

here we use (2) and (3.1) to get the first inequality. Hence, we may use Theorem 3.1 and get

$$\xi \in (\Gamma_f(S))^-_{\Delta^2_{j_0+1}} \subset (\Gamma_f(S))^*_{\mathbb{C}^2} = (\Gamma_f(D))^*_{\mathbb{C}^2}.$$

Since $\xi_1 \in D$, in view of our assumption we have

$$\xi \in ((\partial V) \cap \overline{\Delta^2_{j_0}}) \cap (\Gamma_f(D) \cup (E \times \mathbb{C})).$$

It is easy to see that $\partial V = A_1 \cup A_2 \cup A_3$ where

$$A_1 = \partial D'' \times (\{w \in \mathbb{C} : |w| < j_0 + 1\} \setminus f(\overline{D'})),$$
$$A_2 = \partial D'' \times (\{w \in \mathbb{C} : |w| = j_0 + 1\} \cup f(\partial D')),$$
$$A_3 = D'' \times (\{w \in \mathbb{C} : |w| = j_0 + 1\} \cup f(\partial D')).$$

Since $\partial D'' \cap E = \emptyset$, we find that $A_1 \cap (E \times \mathbb{C}) = \emptyset$. This implies

$$A_1 \cap (\Gamma_f(D) \cup (E \times \mathbb{C})) = \emptyset.$$

In the same way, we have

$$A_2 \cap \overline{\Delta^2_{j_0}} = (\partial D'' \times f(\partial D')) \cap \overline{\Delta^2_{j_0}}.$$

Using the facts that $f(\partial D') \cap f(\overline{D''}) = \emptyset$ and $\partial D'' \cap E = \emptyset$ we obtain

$$A_2 \cap (\Gamma_f(D) \cup (E \times \mathbb{C})) = \emptyset.$$

Now we observe that

$$A_3 \cap \overline{\Delta^2_{j_0}} = (D'' \times f(\partial D')) \cap \overline{\Delta^2_{j_0}}.$$

Since $f(\partial D') \cap f(\overline{D''}) = \emptyset$, we have $(D'' \times f(\partial D')) \cap \Gamma_f(D) = \emptyset$.

Putting all this together we deduce that

$$\xi \in (D'' \times f(\partial D')) \cap (E \times \mathbb{C}).$$

This means that $\xi_1 \in D'' \cap E$ and $\xi_2 = f(\eta)$, where $\eta \in \partial D'$. By letting D' shrink towards z_0 we complete the proof. ∎

We also need the next result (Theorem 3.4 in [2]) in the proof of Proposition 3.3.

PROPOSITION 3.5. *Let Ω be a domain in $\mathbb{C}^n$ and E be a pluripolar subset of Ω such that $E \cap H = \emptyset$, where H is the hyperplane $z_1 = 0$. Then $E^*_\Omega \cap H$ is pluripolar* (*relative to H*).

Proof of Proposition 3.3. Suppose that there exists a point $(a, b) \in (\Gamma_f(D))^*_{\mathbb{C}^2}$ with $b \ne f(a)$. It follows that f is non-constant and $a \in A$. Let S be a closed ball contained in D, $A \cap S = \emptyset$ and $G := \Gamma_f(S)$. Then $G \cap (A \times \mathbb{C}) = \emptyset$ and $G^*_{\mathbb{C}^2} = (\Gamma_f(D))^*_{\mathbb{C}^2}$. Since A is countable, using Proposition 3.5, we deduce that the set $T := \{w \in \mathbb{C} : (z, w) \in G^*_{\mathbb{C}^2},\ z \in A\}$ is pluripolar. Since f is non-constant we deduce $f^{-1}(T)$ is polar in D. So we can choose a disk $D' \subset D$ around a such that

$$D' \cap S = \emptyset, \quad b \notin f(\overline{D'}), \quad f(a) \notin f(\partial D'), \quad \partial D' \cap f^{-1}(T) = \emptyset.$$

We also take a smaller disk D'' relatively compact in D' such that $a \in D''$ and $f(\partial D') \cap f(\overline{D''}) = \emptyset$. Then, following the proof of Lemma 3.4, we can find a point $(a', b') \in G^*_{\mathbb{C}^2}$ where $a' \in D'' \cap A$ and $b' \in f(\partial D')$. In particular $b' \in T$. This contradicts our choice of the set D'. ∎

We end up this paper by giving the announced proof of Lemma 2.8.

Proof of Lemma 2.8. This result can be deduced from the proof of Theorem 4.6 in [3]. For the reader's convenience, we give some details. Assume that there exists a point $z_0 \in E^*_{\mathbb{C}^n}$ such that $F(z_0) \in \partial\Delta$. Fix $0 < r < 1$ and $R > 1$. For $\delta > 0$ we let $\Delta_\delta := \{z \in \mathbb{C} : |z| < \delta\}$ and $U_\delta := \{z \in U : F(z) \in \Delta_{1+\delta}\}$. The key observation is that if $\varepsilon > 0$, then U_ε is an open neighborhood of $E^*_{\mathbb{C}^n}$. In particular $\partial U_\varepsilon \cap E^*_{\mathbb{C}^n} = \emptyset$. Set $\mathbb{B}_R := \{z \in \mathbb{C}^n : |z| < R\}$. By applying the localization principle of Edigarian and Wiegerinck [3, Theorem 4.1], for every $\varepsilon > 0$ we have

$$\begin{aligned} 0 &\le \omega(z_0, E \cap F^{-1}(\Delta_r) \cap \mathbb{B}_R, \mathbb{B}_R) = \omega(z_0, E \cap F^{-1}(\Delta_r) \cap \mathbb{B}_R, \mathbb{B}_R \cap U_\varepsilon) \\ &\le \omega(F(z_0), \Delta_r, \Delta_{1+\varepsilon}). \end{aligned}$$

By letting $\varepsilon \to 0$ we get

$$\omega(z_0, E \cap F^{-1}(\Delta_r) \cap \mathbb{B}_R, \mathbb{B}_R) = 0.$$

So

$$z_0 \notin (E \cap F^{-1}(\Delta_r) \cap \mathbb{B}_R, \mathbb{B}_R)^-_{\mathbb{B}_R}.$$

Since $F(E) \subset \Delta$, by letting $r \to 1$ we infer that $z_0 \notin (E \cap \mathbb{B}_R, \mathbb{B}_R)^-_{\mathbb{B}_R}$. Finally, since $R > 1$ is arbitrarily large, by Theorem 3.1 we get $z_0 \notin E^*_{\mathbb{C}^n}$, a contradiction. ∎

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