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Rigidity of noncompact manifolds with cyclic parallel Ricci curvature

by Yi Hua Deng (Hengyang)

Abstract. We prove that if M is a complete noncompact Riemannian manifold whose Ricci tensor is cyclic parallel and whose scalar curvature is nonpositive, then M is Einstein, provided the Sobolev constant is positive and an integral inequality is satisfied.

1. Introduction. Translating the symmetries of the covariant derivative of the Ricci tensor into an algebraic context, Gray considered the space of 3-tensors with the symmetries of $\nabla\mathrm{Ric}$ (see [G]). This space naturally decomposes under the action of the orthogonal group into three irreducible subspaces $Q \oplus S \oplus A$, where S consists of those manifolds where the Ricci tensor is Codazzi, and A corresponds to the manifolds with cyclic parallel Ricci tensor (see [B] and [FG]). The remaining primitive class Q is given by those manifolds whose Ricci tensor satisfies (see [FG] and [G])

$$\nabla_X \mathrm{Ric}(Y,Z) = \frac{n}{(n+2)(n-1)} X(R)\langle Y,Z\rangle + \frac{n-2}{2(n+2)(n-1)} Y(R)\langle X,Z\rangle + \frac{n-2}{2(n+2)(n-1)} Z(R)\langle X,Y\rangle.$$

A Riemannian manifold (M,g) is said to be of *cyclic parallel Ricci curvature* if the Ricci tensor satisfies

$$\nabla_k R_{ji} + \nabla_j R_{ki} + \nabla_i R_{kj} = 0. \tag{1.1}$$

This definition can be found in [C], [KK] and [M]. If the Ricci tensor satisfies $\nabla_k R_{ji} = \nabla_j R_{ki}$, then (M,g) is called a *manifold with harmonic curvature*. From the definition, we can easily see that if a manifold is both a manifold with harmonic curvature and a manifold with cyclic parallel Ricci curvature, then $\nabla_k R_{ji} = 0$ for each index i, j and k. Therefore, a manifold with

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harmonic curvature is different from a manifold with cyclic parallel Ricci curvature if there exist k and R_{ji} such that $\nabla_k R_{ji} \neq 0$.

Let $p \in M$. It is well known that we have the following expansion:

$$\begin{aligned}(1.2)\qquad \det(g_{ij}) &= 1 - \frac{1}{3}R_{ij}(p)x^i x^j - \frac{1}{6}R_{ij,k}(p)x^i x^j x^k \\ &\quad - \left(\frac{1}{20}R_{ij,kl} + \frac{1}{90}R_{hijm}R_{hklm} - \frac{1}{18}R_{ij}R_{kl}\right)(p)x^i x^j x^k x^l + O(r^5)\end{aligned}$$

in a normal coordinate system $\{x^i\}$ around p for the metric g, where $r = |x|$. Therefore, (1.1) is equivalent to the vanishing of the third item on the right hand of (1.2) for every $p \in M$ (see [YS, pp. 226 and 228], or see [SY2] for more details). Thus, it is necessary to study Riemannian manifolds with cyclic parallel Ricci curvature.

Theorem 4 in [M] shows that a generalized Ricci-recurrent manifold with cyclic parallel Ricci tensor is a quasi-Einstein manifold. Proposition 2 in [KK] shows that if M is a complete and noncompact Riemannian manifold whose Ricci tensor is cyclic parallel and whose sectional curvature is nonpositive, then M is Ricci flat, provided that the square of the length of the Ricci curvature is L_2-summable and the volume of M is infinite.

As far as we know, there exist very few references in the literature on the rigidity of manifolds with cyclic parallel Ricci curvature. In this paper, we will prove that if M is a complete noncompact Riemannian manifold whose Ricci tensor is cyclic parallel and whose scalar curvature is nonpositive, then M is Einstein, provided the Sobolev constant is positive and an integral inequality is satisfied.

From [K], we know that there is no noncompact complete Einstein metric with positive scalar curvature. Therefore, it is pointless to discuss the rigidity of noncompact complete manifolds with positive scalar curvature.

At this point, it is worth pointing out that an important problem in Riemannian geometry is to understand classes of metrics that are, in some sense, close to being Einstein or having constant curvature (see [K]). Recently, many authors have shown their interest in rigid properties of manifolds with various curvature conditions. In [K], Kim discussed the rigidity of noncompact complete manifolds with harmonic curvature. In [W] and [CSW], the rigidity of quasi-Einstein manifolds was studied by Wang and Case–Shu–Wei. In [FG] and [PW], some rigid properties of Ricci solitons were obtained by Fernández-López, García-Río, Petersen and Wylie.

2. Notations and preliminaries. Let (M, g) be a noncompact complete Riemannian manifold of dimension $n \geq 3$ with scalar curvature R. The

Sobolev constant $Q(M,g)$ is defined by

$$(2.1)\qquad Q(M,g) := \inf_{0\neq u\in C_0^\infty(M)} \frac{\int_M \left(|\nabla u|^2 + \frac{n-2}{4(n-1)}Ru^2\right) dV_g}{\left(\int_M |u|^{2n/(n-2)}\, dV_g\right)^{(n-2)/n}}.$$

There exist noncompact complete Riemannian manifolds of negative scalar curvature with positive Sobolev constant. For example, any simply connected complete locally conformally flat manifold has positive Sobolev constant (see [SY1]). It is also known that $Q(M,g) \geq 0$ if the noncompact complete Riemannian manifold has zero scalar curvature (see [D]).

Let R_{ijkl} and R_{ij} denote the components of the Riemannian curvature tensor and the Ricci tensor of (M,g) respectively. Then we have $R_{ij} = g^{kl}R_{ikjl}$, $R_{ijkl} = R_{klij}$ and $R_{ijkl} = -R_{ijlk} = -R_{jikl}$. If ϕ is a 2-tensor with components ϕ_{ij} in an orthonormal frame, then $\operatorname{tr}\phi = g_{ij}\phi_{ij}$, $|\phi|^2 = g_{ik}g_{jl}\phi_{ij}\phi_{kl}$ and

$$(2.2)\qquad \phi_{ijkl} - \phi_{ijlk} = -R_{milk}\phi_{mj} - R_{mjlk}\phi_{im},$$

where ϕ_{ijkl} denote the components of $\nabla^2\phi$. It is well known that the *Weyl tensor* W with components W_{ijkl} is defined by

$$\begin{aligned} W_{ijkl} = R_{ijkl} &+ \frac{R}{(n-1)(n-2)}(g_{ik}g_{jl} - g_{il}g_{jk}) \\ &- \frac{1}{n-2}(R_{ik}g_{jl} - R_{il}g_{jk} + R_{jl}g_{ik} - R_{jk}g_{il}). \end{aligned}$$

Direct calculation shows that

$$(2.3)\qquad \begin{aligned} R_{ijkl} = W_{ijkl} &+ \frac{R}{n(n-1)}(g_{ik}g_{jl} - g_{il}g_{jk}) \\ &+ \frac{1}{n-2}(E_{ik}g_{jl} - E_{il}g_{jk} + E_{jl}g_{ik} - E_{jk}g_{il}), \end{aligned}$$

where E_{ij} are the components of the following *traceless Ricci tensor*:

$$(2.4)\qquad E_{ij} = R_{ij} - \frac{1}{n}Rg_{ij}.$$

If the Ricci tensor satisfies (1.1), then (M,g) is called a *manifold with cyclic parallel Ricci curvature*. If the Ricci tensor satisfies $\nabla_k R_{ji} = \nabla_j R_{ki}$, then (M,g) is called a *manifold with harmonic curvature*.

3. Main results

THEOREM 3.1. *If b is a trace-free symmetric 2-tensor satisfying*

$$(3.1)\qquad \nabla_k b_{ji} + \nabla_j b_{ki} + \nabla_i b_{kj} = 0,$$

then $\left|\nabla|b|\right|^2 \leq \frac{3n}{n+2}|\nabla b|^2$ *at any point where* $|b| \neq 0$.

Proof. In an orthonormal frame, we have $|\nabla b|^2 = \sum_{ijk}(\nabla_k b_{ij})^2$. It follows from (3.1) that $\nabla_k b_{kk} = 0$ and $\nabla_k b_{ii} = -2\nabla_i b_{ki}$. Thus

$$|\nabla b|^2 \geq \sum_{i,k}\Big(-\frac{1}{2}\nabla_k b_{ii}\Big)^2 = \frac{1}{12}\Big(\sum_k (\nabla_k b_{kk})^2 + 3\sum_{i\neq k}(\nabla_k b_{ii})^2\Big). \tag{3.2}$$

Since $|b|^2 = \sum_i b_{ii}^2$, $\sum_i b_{ii} = 0$ and $\sum_i(\nabla_k b_{ii}) = 0$, according to (3.2) and using the methods in [HV] we have $|\nabla(|b|^2)|^2 \leq \frac{12n}{n+2}|b|^2|\nabla b|^2$. Therefore, we get $\big|\nabla|b|\big|^2 \leq \frac{3n}{n+2}|\nabla b|^2$. ∎

THEOREM 3.2. *If (M,g) is a Riemannian manifold with cyclic parallel Ricci curvature, then the Laplacian of the traceless Ricci tensor is*

$$\Delta E_{ij} = -R_{im}E_{mj} - R_{kijm}E_{km} - R_{jm}E_{im} - R_{kjim}E_{km}.$$

Proof. To simplify notations, similarly to [K] we will work in an orthonormal frame. Since (M,g) is a manifold with cyclic parallel Ricci curvature, by (1.1) and the Bianchi identity we get

$$\nabla_k R = 2\nabla_j R_{kj} = -\nabla_k R_{jj} = -\nabla_k R.$$

Therefore, the scalar curvature R is a constant and consequently the traceless Ricci tensor satisfies

$$\nabla_k E_{ji} + \nabla_j E_{ki} + \nabla_i E_{kj} = 0,$$

which means that

$$\nabla_k E_{ji} = -\nabla_j E_{ki} - \nabla_i E_{kj}, \qquad 2\nabla_k E_{kj} = -\nabla_j E_{kk} = 0. \tag{3.3}$$

By (2.2), (3.3) and the properties of E_{ij} and R_{ijkl}, we have

$$\begin{aligned}\Delta E_{ij} &= \nabla_k\nabla_k E_{ij} = -\nabla_k\nabla_i E_{kj} - \nabla_k\nabla_j E_{ki}\\ &= -\nabla_i\nabla_k E_{kj} - R_{kikm}E_{mj} - R_{kijm}E_{km}\\ &\quad - \nabla_j\nabla_k E_{ki} - R_{kjkm}E_{mi} - R_{kjim}E_{km}\\ &= -R_{im}E_{mj} - R_{kijm}E_{km} - R_{jm}E_{im} - R_{kjim}E_{km}.\end{aligned}$$ ∎

THEOREM 3.3. *Let (M,g) be a noncompact complete n-dimensional Riemannian manifold with cyclic parallel Ricci curvature. If $R \leq 0$, then the traceless Ricci tensor satisfies*

$$|E|\Delta|E| \geq \frac{2-2n}{3n}\big|\nabla|E|\big|^2 - 2\sqrt{\frac{n-2}{2(n-1)}}\,|W|\,|E|^2 - 2\sqrt{\frac{n}{n-1}}\,|E|^3, \tag{3.4}$$

where W is the Weyl tensor of (M,g).

Proof. Since E_{ij} is traceless, by (2.3) and (2.4) we obtain

$$(3.5)\quad \begin{aligned} &E_{ij}(R_{jm}E_{im} + R_{kjim}E_{km}) \\ &\quad = \Big(E_{jm} + \frac{1}{n}Rg_{mj}\Big)E_{ji}E_{im} + W_{kjim}E_{ij}E_{km} \\ &\qquad + \frac{R}{n(n-1)}g_{ki}E_{ij}g_{jm}E_{mk} \\ &\qquad + \frac{1}{n-2}(g_{jm}E_{mk}E_{ki}E_{ij} + g_{ki}E_{ij}E_{jm}E_{mk}) \\ &\quad = W_{kjim}E_{ij}E_{km} + \frac{R}{n-1}|E|^2 + \frac{n}{n-2}\operatorname{tr} E^3. \end{aligned}$$

On the other hand, according to the proof of Theorem 1 in [K] we have

$$(3.6)\quad E_{ij}(R_{im}E_{mj}+R_{kijm}E_{km}) = W_{kijm}E_{ij}E_{km}+\frac{R}{n-1}|E|^2+\frac{n}{n-2}\operatorname{tr} E^3.$$

Since $R \le 0$, by (3.5), (3.6) and Theorem 3.2 we obtain

$$(3.7)\quad \begin{aligned} E_{ij}\Delta E_{ij} &= -W_{kijm}E_{ij}E_{km} - W_{kjim}E_{ij}E_{km} - \frac{2R}{n-1}|E|^2 - \frac{2n}{n-2}\operatorname{tr} E^3 \\ &\ge -W_{kijm}E_{ij}E_{km} - W_{kjim}E_{ij}E_{km} - \frac{2n}{n-2}\operatorname{tr} E^3. \end{aligned}$$

According to [H], we have

$$(3.8)\quad |W_{kijm}E_{ij}E_{km}| \le \sqrt{\frac{n-2}{2(n-1)}}\,|W|\,|E|^2, \qquad |\operatorname{tr} E^3| \le \frac{n-2}{\sqrt{n(n-1)}}|E^3|.$$

By (3.7), (3.8) and Theorem 3.1 we conclude that

$$\begin{aligned} |E|\Delta|E| &= |\nabla E|^2 - \big|\nabla|E|\big|^2 + E_{ij}\Delta E_{ij} \\ &\ge \frac{2-2n}{3n}\big|\nabla|E|\big|^2 - 2\sqrt{\frac{n-2}{2(n-1)}}\,|W|\,|E|^2 - 2\sqrt{\frac{n}{n-1}}\,|E|^3. \ \blacksquare \end{aligned}$$

MAIN THEOREM 3.4. *Let (M,g) be a noncompact complete n-dimensional Riemannian manifold with cyclic parallel Ricci curvature. If $R \le 0$ and $Q(M,g) > 0$, then there exists a positive real number c_0 depending only on the dimension n such that if*

$$\int_M |E|^{n/2}\,dV_g + \int_M |W|^{n/2}\,dV_g \le c_0 Q(M,g),$$

then (M,g) is an Einstein manifold.

Proof. Let $u = |E|$. Suppose that $\phi \neq 0$ and $\phi \in C_0^\infty(M)$. Multiplying (3.4) by $\phi^2 u^{n/2-2}$ and then integrating on M, we get

$$\int_M \phi^2 u^{n/2-1} \Delta u \, dV_g \geq \frac{2-2n}{3n} \int_M \phi^2 u^{n/2-2} |\nabla u|^2 \, dV_g - 2\sqrt{\frac{n-2}{2(n-1)}} \int_M \phi^2 u^{n/2} |W| \, dV_g - 2\sqrt{\frac{n}{n-1}} \int_M \phi^2 u^{n/2+1} \, dV_g. \tag{3.9}$$

It follows from integration by parts that

$$\begin{aligned} &\int_M \phi^2 u^{n/2-1} \Delta u \, dV_g \\ &= -\int_M 2\phi u^{n/2-1} \nabla\phi \nabla u \, dV_g - \left(\frac{8}{n} - \frac{16}{n^2}\right) \int_M \phi^2 |\nabla u^{n/4}|^2 \, dV_g \\ &\leq \frac{1}{3n^2} \int_M \phi^2 |\nabla u^{n/4}|^2 \, dV_g + 48 \int_M u^{n/2} |\nabla \phi|^2 \, dV_g - \left(\frac{8}{n} - \frac{16}{n^2}\right) \int_M \phi^2 |\nabla u^{n/4}|^2 \, dV_g \\ &\leq \left(\frac{49}{3n^2} - \frac{8}{n}\right) \int_M \phi^2 |\nabla u^{n/4}|^2 \, dV_g + 48 \int_M u^{n/2} |\nabla \phi|^2 \, dV_g. \end{aligned} \tag{3.10}$$

Since $|\nabla u^{n/4}|^2 = \frac{n^2}{16} u^{n/2-2} |\nabla u|^2$, by (3.9), (3.10) and the Hölder inequality we have

$$\begin{aligned} &\frac{24n^2 - 81n + 32}{3n^3} \int_M \phi^2 |\nabla u^{n/4}|^2 \, dV_g - 48 \int_M u^{n/2} |\nabla \phi|^2 \, dV_g \\ &\leq 2\sqrt{\frac{n-2}{2(n-1)}} \Big(\int_M |W|^{n/2} \, dV_g \Big)^{2/n} \Big(\int_M |\phi u^{n/4}|^{2n/(n-2)} \, dV_g \Big)^{(n-2)/n} \\ &\quad + 2\sqrt{\frac{n}{n-1}} \Big(\int_M u^{n/2} \, dV_g \Big)^{2/n} \Big(\int_M |\phi u^{n/4}|^{2n/(n-2)} \, dV_g \Big)^{(n-2)/n}. \end{aligned} \tag{3.11}$$

Since $R \leq 0$, by (2.1) and (3.11) we get

$$\begin{aligned} &\frac{24n^2 - 81n + 32}{3n^3} Q(M,g) \Big(\int_M |\phi u^{n/4}|^{2n/(n-2)} \, dV_g \Big)^{(n-2)/n} \\ &\leq \frac{24n^2 - 81n + 32}{3n^3} \int_M |\nabla(\phi u^{n/4})|^2 \, dV_g \\ &= \frac{24n^2 - 81n + 32}{3n^3} \Big[\int_M \phi^2 |\nabla u^{n/4}|^2 \, dV_g \\ &\quad + \int_M u^{n/2} |\nabla \phi|^2 \, dV_g + \int_M 2\phi u^{n/4} \nabla \phi \nabla u^{n/4} \, dV_g \Big] \end{aligned} \tag{3.12}$$

$$\le \frac{48n^2-162n+64}{3n^3}\int\limits_M \phi^2|\nabla u^{n/4}|^2\,dV_g + \frac{48n^2-162n+64}{3n^3}\int\limits_M u^{n/2}|\nabla\phi|^2\,dV_g$$
$$\le \left(96+\frac{48n^2-162n+64}{3n^3}\right)\int\limits_M u^{n/2}|\nabla\phi|^2\,dV_g$$
$$+4\sqrt{\frac{n-2}{2(n-1)}}\Big(\int\limits_M |W|^{n/2}\,dV_g\Big)^{2/n}\Big(\int\limits_M |\phi u^{n/4}|^{2n/(n-2)}\,dV_g\Big)^{(n-2)/n}$$
$$+4\sqrt{\frac{n}{n-1}}\Big(\int\limits_M u^{n/2}\,dV_g\Big)^{2/n}\Big(\int\limits_M |\phi u^{n/4}|^{2n/(n-2)}\,dV_g\Big)^{(n-2)/n}.$$

From (3.12), we find that there exists a constant c small enough such that if $\int_M u^{n/2}\,dV_g + \int_M |W|^{n/2}\,dV_g \le cQ(M,g)$, then there exists a constant c' such that

$$c'\Big(\int\limits_M |\phi u^{n/4}|^{2n/(n-2)}\,dV_g\Big)^{(n-2)/n} \le \left(96+\frac{48n^2-162n+64}{3n^3}\right)\int\limits_M u^{n/2}|\nabla\phi|^2\,dV_g. \tag{3.13}$$

Similarly to [K], we let ϕ be a cut-off function such that $\phi=0$ on the ball $B(r)$, $\phi=1$ on $M\setminus B(2r)$ and $|\nabla\phi|\le C/r$ on $B(2r)\setminus B(r)$ for some positive constant C. By (3.13), we have

$$c'\Big(\int\limits_M |\phi u^{n/4}|^{2n/(n-2)}\,dV_g\Big)^{(n-2)/n} \le \frac{C^2}{r^2}\left(96+\frac{48n^2-162n+64}{3n^3}\right)\int\limits_{B(2r)\setminus B(r)} u^{n/2}\,dV_g.$$

Taking $r\to\infty$, we get $u=0$. Thus (M,g) is an Einstein manifold. ∎

From Theorem 3.4, we can easily get the following result:

THEOREM 3.5. *Let (M,g) be a noncompact complete n-dimensional Riemannian manifold with cyclic parallel Ricci curvature satisfies and $Q(M,g) > 0$. Assume that the scalar curvature R is nonpositive. If (M,g) is not Einstein, then there exists a positive real number c depending only on the dimension n such that $\int_M |E|^{n/2}\,dV_g + \int_M |W|^{n/2}\,dV_g \ge cQ(M,g)$.*

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Yi Hua Deng
Department of Mathematics and Computational Science
Hengyang Normal University
421002 Hengyang, China
E-mail: dengchen4032@126.com

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