Correspondence between diffeomorphism groups and singular foliations

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Abstract. It is well-known that any isotopically connected diffeomorphism group $G$ of a manifold determines a unique singular foliation $\mathcal{F}_G$. A one-to-one correspondence between the class of singular foliations and a subclass of diffeomorphism groups is established. As an illustration of this correspondence it is shown that the commutator subgroup $[G, G]$ of an isotopically connected, factorizable and non-fixing $C^r$ diffeomorphism group $G$ is simple iff the foliation $\mathcal{F}_{[G, G]}$ defined by $[G, G]$ admits no proper minimal sets. In particular, the compactly supported $e$-component of the leaf preserving $C^\infty$ diffeomorphism group of a regular foliation $\mathcal{F}$ is simple iff $\mathcal{F}$ has no proper minimal sets.

1. Introduction. Throughout by a foliation we mean a singular foliation (Sussmann [17], Stefan [15]), and by a regular foliation we mean a foliation whose leaves have the same dimension. Introducing the notion of foliations, Sussmann and Stefan emphasized that they play a role of collections of “accessible” sets. Alternatively, they regarded foliations as integrable smooth distributions. Another point of view is to treat foliations as by-products of non-transitive geometric structures (cf. [2], [20] and examples in [10]). In Molino’s approach some types of singular foliations constitute collections of closures of leaves of certain regular foliations ([7], [21]). In this note we regard foliations as a special type of diffeomorphism groups.

Given a $C^\infty$ smooth paracompact boundaryless manifold $M$, $\text{Diff}^r(M)_0$ (resp. $\text{Diff}^r_c(M)_0$), where $1 \leq r \leq \infty$, is the subgroup of the group of all $C^r$ diffeomorphisms $\text{Diff}^r(M)$ on $M$ consisting of diffeomorphisms that can be joined to the identity through a $C^r$ isotopy (resp. compactly supported $C^r$ isotopy) on $M$. A diffeomorphism group $G \leq \text{Diff}^r(M)$ is called isotopically connected if any element $f$ of $G$ can be joined to id$_M$ through a $C^r$ isotopy in $G$. That is, there is a mapping $\mathbb{R} \times M \ni (t, x) \mapsto f_t(x) \in M$ of class $C^r$
with \( f_t \in G \) for all \( t \) and such that \( f_0 = \text{id} \) and \( f_1 = f \). It is well-known that any isotopically connected group \( G \leq \text{Diff}^r(M) \) defines a unique foliation of class \( C^r \), designated by \( F_G \) (see Sect. 2).

Our first aim is to establish a correspondence between the class \( F^r(M) \) of all \( C^r \) foliations on \( M \) and a subclass of diffeomorphism groups on \( M \), and, by using it, to interpret some results and some open problems concerning non-transitive diffeomorphism groups. The second aim is to prove new results (Theorems 1.1 and 1.2) illustrating this correspondence.

A group \( G \leq \text{Diff}^r(M) \) is called factorizable if for every open cover \( U \) and every \( g \in G \) there are \( g_1, \ldots, g_r \in G \) with \( g = g_1 \ldots g_r \) and such that \( g_i \in G_{U_i}, \ i = 1, \ldots, r \), for some \( U_1, \ldots, U_r \in U \). Here for \( U \subset M \) and \( G \leq \text{Diff}^r(M) \), \( G_U \) stands for the identity component of the group of all diffeomorphisms from \( G \) compactly supported in \( U \). Next, \( G \) is said to be non-fixing if \( G(x) \neq \{x\} \) for every \( x \in X \).

**Theorem 1.1.** Assume that \( G \leq \text{Diff}^r_c(M)_0 \), \( 1 \leq r \leq \infty \), is an isotopically connected, non-fixing and factorizable group of diffeomorphisms of a smooth manifold \( M \). Then the commutator group \([G,G] \) is simple if and only if the corresponding foliation \( F_{[G,G]} \) admits no proper (i.e. not equal to \( M \)) minimal set.

In early 1970's Thurston and Mather proved that the group \( \text{Diff}^r_c(M)_0 \), where \( 1 \leq r \leq \infty \), \( r \neq \dim(M) + 1 \), is perfect and simple (see [18], [6], [1]). Next, similar results were proved for classical diffeomorphism groups of class \( C^\infty \) ([1], [13]). For the significance of these simplicity theorems, see, e.g., [1], [13] and references therein.

Let \((M_i, F_i), \ i = 1, 2\), be foliated manifolds. A map \( f : M_1 \to M_2 \) is called foliation preserving if \( f(L_x) = L_{f(x)} \) for any \( x \in M_1 \), where \( L_x \) is the leaf meeting \( x \). Next, if \((M_1, F_1) = (M_2, F_2)\) then \( f \) is leaf preserving if \( f(L_x) = L_x \) for all \( x \in M_1 \). Throughout, \( \text{Diff}^r(M,F) \) will stand for the group of all leaf preserving \( C^r \) diffeomorphisms of a foliated manifold \((M,F)\). Define \( \text{Diff}^r(M,F)_0 \) and \( \text{Diff}^r_c(M,F)_0 \) analogously. Observe that a perfectness theorem for the compactly supported identity component \( \text{Diff}^\infty_c(M,F)_0 \), which is a non-transitive counterpart of Thurston's theorem, has been proved by the author [9] and by Tsuboi [19]. Next, the author [10], following Mather [6, II], showed that \( \text{Diff}^r_c(M,F)_0 \) is perfect provided \( 1 \leq r \leq \dim F \). Observe that, in general, the group \( \text{Diff}^r_c(M,F)_0 \) is not simple for obvious reasons.

**Theorem 1.2.** Let \((M,F)\) be a foliation on a \( C^\infty \) smooth manifold \( M \) with no leaves of dimension 0. Then the commutator subgroup

\[ \mathcal{D} = [\text{Diff}^r_c(M,F)_0, \text{Diff}^r_c(M,F)_0] \]

is simple if and only if \( F_\mathcal{D} \) does not have any proper minimal set. In par-
ticular, if \( F \) is regular, and \( 1 \leq r \leq \dim F \) or \( r = \infty \), then \( \text{Diff}^r_c(M, F)_0 \) is simple if and only if \( F \) has no proper minimal sets.

In the proof of Theorem 1.1 in Sect. 3 some ideas from Ling \[5\] are in use.

2. Foliations correspond to a subclass of the class of diffeomorphism groups. Let \( 1 \leq r \leq \infty \) and let \( L \) be a subset of a \( C^r \) manifold \( M \) endowed with a \( C^r \) differentiable structure which makes it an immersed submanifold. Then \( L \) is weakly imbedded if for any locally connected topological space \( N \) and a continuous map \( f : N \to M \) satisfying \( f(N) \subset L \), the map \( f : N \to L \) is continuous as well. It follows that in this case such a differentiable structure is unique. A foliation of class \( C^r \) is a partition \( F \) of \( M \) into weakly imbedded submanifolds, called leaves, such that the following condition holds. If \( x \) belongs to a \( k \)-dimensional leaf, then there is a local chart \( (U, \varphi) \) of class \( C^r \) with \( \varphi(x) = 0 \), and \( \varphi(U) = V \times W \), where \( V \) is open in \( \mathbb{R}^k \), and \( W \) is open in \( \mathbb{R}^{n-k} \), such that if \( L \in F \) then \( \varphi(L \cap U) = V \times l \), where \( l = \{w \in W : \varphi^{-1}(0, w) \in L\} \). A foliation is called regular if all leaves have the same dimension.

Sussmann \[17\] and Stefan \[15\], \[16\] regarded foliations as collections of accessible sets in the following sense.

**Definition 2.1.** A smooth mapping \( \varphi \) of an open subset of \( \mathbb{R} \times M \) into \( M \) is said to be a \( C^r \) arrow, \( 1 \leq r \leq \infty \), if

1. \( \varphi(t, \cdot) = \varphi_t \) is a local \( C^r \) diffeomorphism for each \( t \), possibly with empty domain,
2. \( \varphi_0 = \text{id} \) on its domain,
3. \( \text{dom}(\varphi_t) \subset \text{dom}(\varphi_s) \) whenever \( 0 \leq s < t \).

Given an arbitrary set \( A \) of arrows, let \( A^* \) be the family of local diffeomorphisms \( \psi \) such that \( \psi = \varphi(t, \cdot) \) for some \( \varphi \in A \), \( t \in \mathbb{R} \). Next, \( \hat{A} \) denotes the set consisting of all local diffeomorphisms which are finite compositions of elements from \( A^* \) or \( (A^*)^{-1} = \{\psi^{-1} : \psi \in A^*\} \), and of the identity. Then the orbits of \( \hat{A} \) are called accessible sets of \( A \).

For \( x \in M \) let \( A(x), \hat{A}(x) \) be the vector subspaces of \( T_x M \) generated by

\[ \{\dot{\varphi}(t, y) : \varphi \in A, \varphi_t(y) = x\}, \quad \{d_y \psi(v) : \psi \in \hat{A}, \psi(y) = x, v \in A(y)\}, \]

respectively. Then we have \([15]\)

**Theorem 2.2.** Let \( A \) be an arbitrary set of \( C^r \) arrows on \( M \). Then

1. every accessible set of \( A \) admits a (unique) \( C^r \) differentiable structure of a connected weakly imbedded submanifold of \( M \);
2. the collection of accessible sets defines a foliation \( F \); and
3. \( \mathcal{D}(F) := \{\hat{A}(x)\} \) is the tangent distribution of \( F \).
Let \( G \leq \text{Diff}^r(M) \) be an isotopically connected group of diffeomorphisms. Let \( \mathcal{A}_G \) be the set of restrictions of isotopies \( \mathbb{R} \times M \ni (t, x) \mapsto f_t(x) \in M \) in \( G \) to open subsets of \( \mathbb{R} \times M \). Then we denote by \( \mathcal{F}_G \) the foliation defined by the set \( \mathcal{A}_G \) of arrows. Observe that \( \mathcal{A}_G = \mathcal{A}_G \), and consequently \( \mathcal{A}_G(x) = \mathcal{A}_G(x) \).

Remark 2.3. (1) Of course, any subgroup \( G \leq \text{Diff}^r(M) \) determines a unique foliation. Namely, \( G \) has a unique maximal subgroup \( G_0 \) which is isotopically connected.

(2) Denote by \( G_c \) the subgroup of all compactly supported elements of \( G \). Then \( G_c \) need not be isotopically connected even if \( G \) is. In fact, let \( G = \text{Diff}^r(\mathbb{R}^n)_0, 1 \leq r \leq \infty \). Then every \( f \in G_c \) is isotopic to the identity but the isotopy need not be in \( G_c \). That is, \( G_c \) is not isotopically connected.

Observe that the \( C^0 \) case is exceptional: due to Alexander’s trick for \( r = 0 \) (see, e.g., [3, p. 70]), \( G_c \) is isotopically connected.

Likewise, let \( C = \mathbb{R} \times \mathbb{S}^1 \) be the annulus and let \( G = \text{Diff}^r(C)_0 \). Then we have the twisting number epimorphism \( T : G_c \to \mathbb{Z} \). It is easily seen that \( f \in G_c \) can be joined to \( \text{id} \) by a compactly supported isotopy iff \( T(f) = 0 \). Consequently, \( G_c \) is not isotopically connected.

Denote by \( \mathfrak{G}^r(M) \) (resp. \( \mathfrak{G}^r_c(M) \)), \( 1 \leq r \leq \infty \), the collection of isotopically connected (resp. isotopically connected through compactly supported isotopies) groups of \( C^r \) diffeomorphisms of \( M \). Next, \( \mathfrak{F}^r(M) \) will stand for the set of all foliations of class \( C^r \) on \( M \). Then each \( G \in \mathfrak{G}^r(M) \) determines a unique foliation from \( \mathfrak{F}^r(M) \), denoted by \( \mathcal{F}_G \). That is, we have the mapping \( \beta_M : \mathfrak{G}^r(M) \ni G \mapsto \mathcal{F}_G \in \mathfrak{F}^r(M) \). Conversely, to any foliation \( \mathcal{F} \in \mathfrak{F}^r(M) \) we assign \( G_{\mathcal{F}} := \text{Diff}^r_c(M, \mathcal{F})_0 \) and we get the mapping \( \alpha_M : \mathfrak{F}^r(M) \ni \mathcal{F} \mapsto G_{\mathcal{F}} \in \mathfrak{G}^r_c(M) \). The following is obvious.

Proposition 2.4. One has \( \beta_M \circ \alpha_M = \text{id}_{\mathfrak{F}^r(M)} \). In particular

\[
\alpha_M : \mathfrak{F}^r(M) \ni \mathcal{F} \mapsto G_{\mathcal{F}} \in \mathfrak{G}^r_c(M)
\]

is an injection identifying the class of \( C^r \) foliations with a subclass of \( C^r \) diffeomorphism groups.

Observe that usually \( (\alpha_M \circ \beta_M)(G) \in \mathfrak{G}^r_c(M) \) is not a subgroup of \( G \) even if \( G \in \mathfrak{G}^r_c(M) \). For instance, take the group of Hamiltonian diffeomorphisms of a Poisson manifold (see [20]). See also examples in [11].

Remark 2.5. Note that we can also define \( \alpha'_M : \mathfrak{F}^r(M) \ni \mathcal{F} \mapsto G'_{\mathcal{F}} \in \mathfrak{G}^r(M) \), where \( G'_{\mathcal{F}} := \text{Diff}^r(M, \mathcal{F})_0 \in \mathfrak{G}^r(M) \), and we get another identification of the class of \( C^r \) foliations with a subclass of \( C^r \) diffeomorphism groups. However we prefer \( \alpha_M \) to \( \alpha'_M \) because of Proposition 2.11 below.

For \( \mathcal{F}_1, \mathcal{F}_2 \in \mathfrak{F}^r(M) \) we say that \( \mathcal{F}_1 \) is a subfoliation of \( \mathcal{F}_2 \) if each leaf of \( \mathcal{F}_1 \) is contained in a leaf of \( \mathcal{F}_2 \). We then write \( \mathcal{F}_1 \prec \mathcal{F}_2 \). By a flag structure
we mean a finite sequence \( \mathcal{F}_1 \prec \cdots \prec \mathcal{F}_k \) of foliations of \( M \). Next, by the intersection of \( \mathcal{F}_1, \mathcal{F}_2 \) we mean the partition \( \mathcal{F}_1 \cap \mathcal{F}_2 := \{ L_1 \cap L_2 : L_i \in \mathcal{F}_i, i = 1, 2 \} \) of \( M \). Clearly, if \( \mathcal{F}_1 \cap \mathcal{F}_2 \) is a foliation then \( \mathcal{F}_1 \cap \mathcal{F}_2 \prec \mathcal{F}_i, i = 1, 2 \).

It is a rare phenomenon that \( \mathcal{F}_1 \cap \mathcal{F}_2 \) is a regular foliation if \( \mathcal{F}_1, \mathcal{F}_2 \) are regular. In the category of (singular) foliations this may happen more often.

**PROPOSITION 2.6.**

1. If the distribution \( \mathcal{D}(\mathcal{F}_1 \cap \mathcal{F}_2) \) is of class \( C^r \) \([15]\) then \( \mathcal{F}_1 \cap \mathcal{F}_2 \) is a foliation.
2. If \( G_1, G_2 \in \mathfrak{S}^r(M) \) have the intersection \( G = G_1 \cap G_2 \) isotopically connected then \( \mathcal{F}_G = \mathcal{F}_{G_1} \cap \mathcal{F}_{G_2} \).
3. For \( \mathcal{F}_1, \mathcal{F}_2 \in \mathfrak{S}^r(M) \), if \( \mathcal{F}_1 \cap \mathcal{F}_2 \) is a foliation then there is \( G \in \mathfrak{S}^r(M) \) such that \( G \leq \mathcal{F}_{G_1} \cap \mathcal{F}_{G_2} \) and \( \mathcal{F}_G = \mathcal{F}_1 \cap \mathcal{F}_2 \).
4. For \( \mathcal{F}_1, \mathcal{F}_2 \in \mathfrak{S}^r(M) \), if \( \mathcal{F}_1 \cap \mathcal{F}_2 \) is connected then \( \mathcal{F}_1 \cap \mathcal{F}_2 \) is a foliation.

**Proof.** (1) In fact, the distribution of \( \mathcal{F}_1 \cap \mathcal{F}_2 \) is then integrable.

(2) Denote by \( \mathcal{I}G \) the set of all isotopies in \( G \). Clearly, \( \mathcal{I}(G_1 \cap G_2) = \mathcal{I}G_1 \cap \mathcal{I}G_2 \) for arbitrary \( G_1, G_2 \in \mathfrak{S}^r(M) \). For \( x \in M \), set \( \mathcal{I}G(x) := \{ y \in M : (\exists f \in \mathcal{I}G)(\exists t \in I) y = f_t(x) \} \). By definition, \( L_x = \mathcal{I}G(x) \), where \( L_x \in \mathcal{F}_G \) is a leaf meeting \( x \). Therefore, since \( G_1, G_2, G \) are isotopically connected we have \( L_x = \mathcal{I}G(x) = \mathcal{I}G_1(x) \cap \mathcal{I}G_2(x) = L_{x_1}^1 \cap L_{x_2}^2 \), where \( L_{x_i}^i \in \mathcal{F}_{G_i}, i = 1, 2 \).

(3) Set \( \mathcal{F} = \mathcal{F}_1 \cap \mathcal{F}_2 \) and \( G = G_\mathcal{F} \). Use Prop. 2.4.

(4) In view of Prop. 2.4 we have \( \mathcal{F}_{G_\mathcal{F}_0} = \mathcal{F}_0 \) for all \( \mathcal{F}_0 \in \mathfrak{S}^r(M) \). Put \( G = G_{\mathcal{F}_1} \cap G_{\mathcal{F}_2} \). Therefore, in view of (2), \( \mathcal{F}_1 \cap \mathcal{F}_2 = \mathcal{F}_{G_{\mathcal{F}_1} \cap \mathcal{F}_{G_{\mathcal{F}_2}}} = \mathcal{F}_G \) is a foliation. \( \blacksquare \)

Let \( \mathcal{F}_1 \prec \cdots \prec \mathcal{F}_k \) be a flag structure on \( M \) and let \( x \in M \). If \( x \in L_i \in \mathcal{F}_i \) we write \( p_i(x) = \text{dim} L_i, \; p_i(x) = p_i(x) - p_{i-1}(x) \) \((i = 2, \ldots, k)\) and \( q_i(x) = m - p_i(x) \).

**DEFINITION 2.7.** A chart \((U, \varphi)\) of \( M \) with \( \varphi(0) = x \) is called a distinguished chart at \( x \) with respect to \( \mathcal{F}_1 \prec \cdots \prec \mathcal{F}_k \) if \( U = V_1 \times \cdots \times V_k \times W \) where \( V_1 \subset \mathbb{R}^{p_1(x)}, V_i \subset \mathbb{R}^{p_i(x)} \) \((i \geq 2)\) and \( W \subset \mathbb{R}^{q_k(x)} \) are open balls and for any \( L_i \in \mathcal{F}_i \) we have

\[
\varphi(U) \cap L_i = \varphi(V_1 \times \cdots \times V_i \times L_i),
\]

where \( L_i = \{ w \in V_{i+1} \times \cdots \times V_k \times W : \varphi(0, w) \in L_i \} \) for \( i = 1, \ldots, k \).

Observe that actually the above \( \varphi \) is an inverse chart; following \([16]\) we call it a chart for simplicity. Notice as well that in the above definition one need not assume that \( \mathcal{F}_i \) is a foliation but only that it is a partition into weakly imbedded submanifolds; that \( \mathcal{F}_i \) is a foliation then follows by definition.
Theorem 2.8. Let $G_1 \leq \cdots \leq G_k \leq \text{Diff}^r(M)$ be an increasing sequence of diffeomorphism groups of $M$. Then $\mathcal{F}_{G_1} \prec \cdots \prec \mathcal{F}_{G_k}$ admits a distinguished chart at any $x \in M$.

In fact, this is a straightforward consequence of Theorem 2 in [11].

Corollary 2.9. Let $G_1 \leq \cdots \leq G_k \leq \text{Diff}^r(M)$ and let $(L, \sigma)$ be a leaf of $\mathcal{F}_{G_k}$. Then all $G_i$ preserve $L$, and $\mathcal{F}_{G_1|L} \prec \cdots \prec \mathcal{F}_{G_{k-1}|L}$ is a flag structure on $L$. Moreover, a distinguished chart at $x$ for $\mathcal{F}_{G_1|L} \prec \cdots \prec \mathcal{F}_{G_{k-1}|L}$ is the restriction to $L$ of a distinguished chart at $x$ for $\mathcal{F}_{G_1} \prec \cdots \prec \mathcal{F}_{G_k}$.

The following property of paracompact spaces is well-known.

Lemma 2.10. If $X$ is a paracompact space and $U$ is an open cover of $X$, then there exists an open cover $V$ starwise finer than $U$, that is, for all $V \in V$ there is $U \in U$ such that $\text{star}(V) \subset U$. Here $\text{star}(V) := \bigcup \{V' \in V : V' \cap V \neq \emptyset \}$. In particular, for all $V_1, V_2 \in V$ with $V_1 \cap V_2 \neq \emptyset$ there is $U \in U$ such that $V_1 \cup V_2 \subset U$.

Proposition 2.11. If $\mathcal{F} \in \mathfrak{F}^r(M)$ then $G_\mathcal{F} = \alpha(\mathcal{F})$ is factorizable.

Proof. Let $\mathfrak{X}_c(M, \mathcal{F})$ be the Lie algebra of all compactly supported vector fields on $M$ tangent to $\mathcal{F}$. Then there is a one-to-one correspondence between isotopies $f_t$ in $G_\mathcal{F}$ and smooth paths $X_t$ in $\mathfrak{X}_c(M, \mathcal{F})$ given by the equation

$$\frac{df_t}{dt} = X_t \circ f_t \quad \text{with} \quad f_0 = \text{id}.$$ 

Let $f = (f_t) \in \mathcal{I}G_\mathcal{F}$ and let $X_t$ be the corresponding family in $\mathfrak{X}_c(M, \mathcal{F})$. By considering $f_{(p/m)t} f_{(p-1/m)t}^{-1}$, $p = 1, \ldots, m$, instead of $f_t$ we may assume that $f_t$ is close to the identity.

Let $\mathcal{U}$ be an open cover of $M$. We choose a family of open sets, $(V_j)_{j=1}^s$, which is starwise finer than $\mathcal{U}$, and satisfies $\text{supp}(f_t) \subset V_1 \cup \cdots \cup V_s$ for each $t$. Let $(\lambda_j)_{j=1}^s$ be a partition of unity subordinate to $(V_j)$, and let $Y_t^j = \lambda_j X_t$. We set

$$X_t^j = Y_t^1 + \cdots + Y_t^j, \quad j = 1, \ldots, s,$$

and $X_t^0 = 0$. Each of the smooth families $X_t^j$ integrates to an isotopy $g_t^j$ with support in $V_1 \cup \cdots \cup V_j$. We get the fragmentation

$$f_t = g_t^s \circ \cdots \circ g_t^1,$$

where $f_t^j = g_t^j \circ (g_t^{-1})^{-1}$, with the required inclusions

$$\text{supp}(f_t^j) = \text{supp}(g_t^j \circ (g_t^{-1})^{-1}) \subset \text{star}(V_j) \subset U_{i(j)}$$

which hold if $f_t$ is sufficiently small. Thus the group of isotopies of $G_\mathcal{F}$ is factorizable. Consequently, $G_\mathcal{F}$ itself is factorizable. ■
Remark 2.12. The identification $\alpha_M$ enables us to consider several new properties of foliations from $\mathfrak{F}(M)$. For instance, one can say that a foliation $\mathcal{F}$ is perfect if so is the corresponding diffeomorphism group $G_\mathcal{F} = \alpha_M(\mathcal{F})$. As mentioned before, it is known that $G_\mathcal{F} = \text{Diff}^r_c(M,\mathcal{F})_0$ is perfect provided $\mathcal{F}$ is regular and $1 \leq r \leq \dim \mathcal{F}$ or $r = \infty$ (see [9], [19], [10]). It is not known whether $G_\mathcal{F}$ is perfect for singular foliations and a possible proof seems to be very difficult. In turn, possible perfectness of $G_\mathcal{F} = \text{Diff}^r_c(M,\mathcal{F})_0$ with $r$ large is closely related to the simplicity of $\text{Diff}^{n+1}_c(M^n)_0$ (see [4]).

Likewise, one can consider uniformly perfect or bounded foliations by using the corresponding notions for groups (see [4] and references therein).

Finally consider the following important feature of subclasses of the class $\mathfrak{F}(M)$, depending also on $M$ and $r$. A subclass $\mathcal{R}$ of $\mathfrak{F}(M)$ is called faithful if the following holds: For all $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{R}$ and for any group isomorphism $\Phi : \alpha_M(\mathcal{F}_1) \cong \alpha_M(\mathcal{F}_2)$ there is a $C^r$ foliated diffeomorphism $\varphi : (M,\mathcal{F}_1) \cong (M,\mathcal{F}_2)$ such that for all $f \in \alpha_M(\mathcal{F}_1)$, $\Phi(f) = \varphi \circ f \circ \varphi^{-1}$. From the reconstruction results of Rybicki [12] and Rubin [8] it is known that the class of regular foliations of class $C^\infty$, $\mathfrak{F}^\infty_{\text{reg}}(M)$, is faithful.

3. Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1. First observe that the fact that a foliation $\mathcal{F}$ has no proper minimal set is equivalent to the statement that all leaves of $\mathcal{F}$ are dense.

($\Rightarrow$) Assume that $\emptyset \neq L \subset M$ is a proper closed saturated subset of $M$. Choose $x \in M \setminus L$. We will prove the following statement:

(*) there are a ball $U \subset M \setminus L$ with $x \in U$ and $g \in [G_U, G_U]$ such that $g(x) \neq x$.

Then we are done by setting $H := \{g \in [G, G] : g|_L = \text{id}_L\}$. To prove (*), choose balls $U$ and $V$ in $M$ such that $x \in V \subset \overline{V} \subset U$. Take $f \in G$ such that $f(x) \neq x$. By assumption, for $\mathcal{U} = \{U, \overline{V}\}$ we may write $g = g_r \ldots g_1$, where all $g_i$ are supported in elements of $\mathcal{U}$. Let $s := \min\{i \in \{1, \ldots, r\} : \text{supp}(g_i) \subset U \text{ and } g_i(x) \neq x\}$. Then $g_s \in G_U$ satisfies $g_s(x) \neq x$.

Now take an open $W$ such that $x \in W \subset U$ and $g_s(x) \notin W$. Choose $f \in G_W$ with $f(x) \neq x$ by an argument similar to the above. It follows that $f(g_s(x)) = g_s(x) \neq g_s(f(x))$, and therefore $[f, g_s](x) \neq x$. Thus $g = [f, g_s]$ satisfies the claim.

($\Leftarrow$) First observe the following commutator formulae for all $f, g, h \in G$:

\begin{equation}
[f, g, h] = f[g, h]f^{-1}[f, h], \quad [f, gh] = [f, g][f, h]g^{-1}.
\end{equation}

Next, in view of a theorem of Ling [5] we know that $[G, G]$ is a perfect group,
that is,
\[(3.2) \quad [G, G] = [[G, G], [G, G]].\]

Suppose that \(H\) is a non-trivial normal subgroup of \([G, G]\). Let \(x \in M\) satisfy \(h(x) \neq x\) for some \(h \in H\). Fix a ball \(U_0\) such that \(h(U_0) \cap U_0 = \emptyset\). By the definition of \(\mathcal{F}_{[G, G]}\) and the assumption that each leaf \(L \in \mathcal{F}_{[G, G]}\) is dense, for every \(y \in M\) there are a ball \(U_y\) with \(y \in U_y\) and \(f_y \in [G, G]\) such that \(f_y(U_y) \subset U\). Let \(U = \{U_y\}_{y \in M}\).

By Lemma 2.10 we can find an open cover \(\mathcal{V}\) starwise finer than \(U\). We denote \(U^G = \{g(U) : U \in \mathcal{U}, g \in [G, G]\}\) and \(G^U = \prod_{U \in U^G} [G_U, G_U]\).

By assumption \(G\) is factorizable with respect to \(\mathcal{V}\). First we show that \([G, G] \subset G^U\), i.e. any \([g_1, g_2] \in [G, G]\) can be expressed as a product of elements of the form \([h_1, h_2]\), where \(h_1, h_2 \in G_U\) for some \(U \in U^G\). In view of (3.1) and (3.2) we may assume that \(g_1, g_2 \in [G, G]\). Now the relation \([G, G] \subset G^U\) is an immediate consequence of (3.1) and the fact that \(\mathcal{V}\) is starwise finer than \(U\).

Next we have to show that \(G^U \subset H\). It suffices to check that for every \(f, g \in G_U\) with \(U \in \mathcal{U}\) the bracket \([f, g]\) belongs to \(H\). This implies that for every \(f, g \in G_U\) with \(U \in U^G\) one has \([f, g] \in H\), since \(H\) is a normal subgroup in \([G, G]\).

We have fixed \(h \in H\) and \(U_0\) such that \(h(U_0) \cap U_0 = \emptyset\). If \(U \in \mathcal{U}\) and \(f, g \in G_U\), take \(k \in [G, G]\) such that \(k(U) \subset U_0\), and put \(\bar{f} = kfk^{-1}, \quad \bar{g} = kgk^{-1}\). It follows that \([hf^{-1}, g] = id\). Therefore, \([\bar{f}, \bar{g}] = [h, \bar{f}], \bar{g}] \in H\), and we also have \([f, g] \in H\). Thus \(G^U \subset H\), and consequently \([G, G] \leq H\), as required.

**Proof of Theorem 1.2.** By assumption and Prop. 2.11, \(\text{Diff}^r(M, \mathcal{F})_0\) is factorizable and non-fixing. Since \(\text{Diff}^r(M, \mathcal{F})_0\) is isotopically connected, the first assertion follows from Theorem 1.1. The second assertion is a consequence of \(\text{Diff}^r(M, \mathcal{F})_0\) being perfect (\([9]\) and \([19]\) for \(r = \infty\), and \([6]\) and \([10]\) for \(1 \leq r \leq \dim \mathcal{F}\)).

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**References**


Diffeomorphism groups and foliations


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