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Lyapunov type inequalities for a second order differential equation with a damping term

by S. H. SAKER (Riyadh and Mansoura)

Abstract. For a second order differential equation with a damping term, we establish some new inequalities of Lyapunov type. These inequalities give implicit lower bounds on the distance between zeros of a nontrivial solution and also lower bounds for the spacing between zeros of a solution and/or its derivative. We also obtain a lower bound for the first eigenvalue of a boundary value problem. The main results are proved by applying the Hölder inequality and some generalizations of Opial and Wirtinger type inequalities. The results yield conditions for disfocality and disconjugacy. An example is considered to illustrate the main results.

1. Introduction. In this paper, we will establish some inequalities of Lyapunov type for the second-order differential equation with a damping term

$$(r(t)(x'(t))^{\gamma})' + p(t)(x'(t))^{\gamma} + q(t)(x(t))^{\gamma} = 0, \qquad t \in I, \tag{1.1}$$

where I is a nontrivial interval of reals, $\gamma \geq 1$ is a ratio of odd positive integers and $p, q, r : I \to \mathbb{R}$ are continuous measurable functions with $r(t) > 0$. Lyapunov type inequalities yield implicit lower bounds on the distance between consecutive zeros of a nontrivial solution x and also lower bounds for the distance between zeros of a solution $x(t)$ and/or its derivative x'. The best known result of this type for the special case of (1.1) (when $\gamma = 1$, $r(t) = 1$ and $p(t) = 0$) is due to Lyapunov [16]: If $x(t)$ is a solution of the differential equation

$$x''(t) + q(t)x(t) = 0 \tag{1.2}$$

with $x(a) = x(b) = 0$ $(a < b)$ and $x(t) \neq 0$ for $t \in (a, b)$, then

$$\int_a^b q^+(t)\, dt > \frac{4}{b-a}, \tag{1.3}$$

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where q is a real-valued continuous function on a nontrivial interval of reals and $q^+(t) = \max\{q(t), 0\}$. Since the appearance of this inequality various proofs and generalizations or improvements have appeared in the literature for different types of equations. We refer the reader to the papers [3, 9, 10, 18–21, 23–26, 30, 14] and the references cited therein. We also refer the survey [28] of the most basic results on Lyapunov type inequalities and their generalizations.

In this paper, we are concerned with the following three problems:

(i) obtain lower bounds for the spacing $\beta - \alpha$ where x is a solution of (1.1) satisfying $x(\alpha) = x'(\beta) = 0$, or $x'(\alpha) = x(\beta) = 0$,
(ii) obtain lower bounds for the spacing of zeros of a solution of (1.1),
(iii) obtain a lower bound for the first eigenvalue of the boundary value problem

$$-((x'(t))^\gamma)' + p(t)(x'(t))^\gamma + q(t)x^\gamma(t) = \lambda x^\gamma(t), x(0) = x(\pi) = 0.$$

By a solution of (1.1) on the interval $J \subseteq I$, we mean a nontrivial real-valued function $x \in C^1(J)$ which has the property that $r(t)(x'(t))^\gamma \in C^1(J)$ and satisfies equation (1.1) on J. We assume that (1.1) possesses such a nontrivial solution.

A nontrivial solution $x(t)$ of (1.1) is said to be *oscillatory* if it has arbitrarily large zeros. Equation (1.1) is said to be *disconjugate* on the interval $[a, b]$ if there is no nontrivial solution of (1.1) with two zeros in $[a, b]$, and it is said to be *nonoscillatory* on I if there exists $c \in I$ such that this equation is disconjugate on $[c, d]$ for every $d > c$. We say that (1.1) is *right disfocal* (resp. *left disfocal*) on $[\alpha, \beta]$ if the solution $x(t)$ of (1.1) with $x'(\alpha) = 0$ (resp. $x'(\beta) = 0$) has no zeros in $[\alpha, \beta]$.

For oscillation and nonoscillation results for half-linear differential equations, we refer to the book [27].

Our motivation for this paper comes from the papers of Hartman and Wintner [13], Fink and Mary [11], Ha [12], Pachpatte [22] and Lee et al. [15]. Some of their results are presented in the following:

Hartman and Wintner [13] proved that if x is a solution of the equation

$$x'' + p(t)x'(t) + q(t)x(t) = 0, \qquad a \le t \le b, \tag{1.4}$$

such that $x(a) = x(b) = 0$, where $p, q \in C([0, \infty), \mathbb{R})$, then

$$\int_a^b (t-a)(b-t)q^+(t)\,dt + \max\left\{\int_a^b (t-a)|p(t)|\,dt, \int_a^b (b-t)|p(t)|\,dt\right\} > b-a. \tag{1.5}$$

Fink and Mary [11] proved that if x is a solution of (1.4) with no zeros in (a, b) and such that $x(a) = x(b) = 0$, then

$$(b-a)\int_a^b q^+(s)\,ds - 4\exp\Big(-\frac{1}{2}\int_a^b |p(s)|\,ds\Big) > 0. \tag{1.6}$$

Ha [12] proved that if x is a solution of the equation

$$x'' + \lambda x'(t) + q(t)x(t) = 0, \qquad 0 \le t \le \pi, \tag{1.7}$$

with no zeros in $(0,\pi)$ and such that $x(0) = x(\pi) = 0$, then

$$\int_0^\pi |q(s)|\,ds > \begin{cases} 2\sqrt{\lambda}\cot\frac{\sqrt{\lambda}}{2}\pi & \text{if } 0 < \lambda < 1, \\ \sqrt{\lambda}|\sin\sqrt{\lambda}\pi| & \text{if } \lambda > 1. \end{cases} \tag{1.8}$$

Pachpatte [22] considered the equation

$$(r(t)|x'(t)|^{\alpha-1}x'(t))' + p(t)x'(t) + q(t)x(t) = 0, \qquad a \le t \le b, \tag{1.9}$$

where $\alpha > 1$ and $p(t)$, $q(t)$ and $r(t)$ are real measurable functions with $r(t) > 0$, and proved the following: If $x(a) = x(b) = 0$ and $M = \max\{|x(t)| : a \le t \le b\}$, then

$$\frac{1}{M^{\alpha-1}}\Big(\int_a^b r^{-1/\alpha}(t)\,dt\Big)^\alpha\Big(\int_a^b |q(t) - p'(t)/2|\,dt\Big) \ge 2^{\alpha+1}. \tag{1.10}$$

Lee et al. [15] considered the equation

(1.11)

$$(|x'(t)|^{\alpha-2}x'(t))' + p(t)|x'(t)|^{\alpha-2}x'(t) + q(t)|x(t)|^{\alpha-2}x(t) = 0, \qquad a \le t \le b,$$

and extended the results of Fink and Mary as follows: If x is a solution of (1.11) with no zeros in (a,b) and such that $x(a) = x(b) = 0$, then

$$\left.\begin{aligned} (b-a)^{\alpha/\beta}\int_a^b q^+(s)\,ds - 4\exp\Big(-\int_a^b |p(s)|\,ds\Big) > 0, \\ (b-a)\int_a^b q^+(s)\,ds + 4\int_a^b |p(s)|\,ds > 4, \end{aligned}\right\}\ \text{when } \alpha \ge 2, \tag{1.12}$$

and

(1.13)

$$\left.\begin{aligned} (b-a)^{\alpha/\beta}\int_a^b q^+(s)\,ds - 2^\gamma\exp\Big(-\int_a^b |p(s)|\,ds\Big) > 0 \\ (b-a)\int_a^b q^+(s)\,ds + 2^\alpha\int_a^b |p(s)|\,ds > 2^\alpha \end{aligned}\right\}\ \text{when } 1 < \alpha \le 2,$$

where $1/\alpha + 1/\beta = 1$. In [29] the authors also considered the Emden–Fowler type equation

(1.14)

$$(|x'(t)|^{\alpha-2}x'(t))' + g(t)|x'(t)|^{\alpha-2}x'(t) + f(t)|x(t)|^{\beta-2}x(t) = 0, \quad a \le t \le b,$$

and applied the results obtained for a Hamiltonian system to establish new Lyapunov type inequalities similar to (1.12) and (1.13). The results in the above mentioned papers are established by using elementary analysis, integration of the equation, Hölder's inequality and Jensen's inequality.

In this paper, we will employ a technique that depends on the application of some generalizations of Opial's inequality, Wirtinger's inequality and Hölder's inequality to prove several results related to problems (i)–(iii) above. In particular, we obtain some new results different from the results obtained by Hartman and Wintner [13], Fink and Mary [11], Ha [12], Pachpatte [22], Lee et al. [15] and Tiryaki et al. [29]. Of particular interest is the case when q is oscillatory. We remark that our technique is completely different from the techniques used in the above mentioned papers. An example is considered to illustrate the main results.

2. Main results. The Opial [17] inequality

$$\int_a^b |x(t)|\,|x'(t)|\,dt \le \frac{b-a}{4}\int_a^b |x'(t)|^2\,dt, \quad x(a) = x(b) = 0, \tag{2.1}$$

with the best constant $(b-a)/4$ is one of the most important and fundamental integral inequalities in the analysis of qualitative properties of solutions of differential equations. Since its discovery an enormous amount of work has been done, and many papers which deal with new proofs, various generalizations, extensions and discrete analogues have appeared. For more details we refer the reader to the book [1]. In this section, we will employ some generalizations of Opial's inequality to establish our main results.

First, we will apply an inequality due to Beesack and Das [4] which states that if x is absolutely continuous on $[c,d]$ with $x(c) = 0$ and $x'(t)$ does not change sign in (c,d), then

$$\int_c^d B(t)|x(t)|^m|x'(t)|^n\,dt \le K(m,n)\int_c^d A(t)|x'(t)|^{m+n}\,dt, \tag{2.2}$$

where m, n are real numbers with $mn > 0$ and $m+n > 1$, A and B are nonnegative, measurable functions on (c,d) such that $\int_c^t A^{-1/(m+n-1)}(s)\,ds < \infty$ and

(2.3) $K(m,n)$

$$:= \left(\frac{n}{n+m}\right)^{\frac{n}{n+m}}\left[\int_c^d B^{\frac{n+m}{m}}(t)A^{-\frac{n}{m}}(t)\Big(\int_c^t \Big(A^{\frac{-1}{m+n-1}}(s)\,ds\Big)^{m+n-1}dt\right]^{\frac{m}{m+n}}.$$

If we replace $x(c) = 0$ by $x(d) = 0$, then (2.2) holds with

$$(2.4)\quad K(m,n) := \left(\frac{n}{n+m}\right)^{\frac{n}{n+m}} \left[\int_c^d B^{\frac{n+m}{m}}(t) A^{-\frac{n}{m}}(t) \Big(\int_t^d \Big(A^{\frac{-1}{m+n-1}}(s)\, ds\Big)^{m+n-1} dt\right]^{\frac{m}{m+n}}.$$

In the following, we will assume that there exists a nontrivial subinterval $J \subseteq I$ with endpoints $\alpha < \beta$ such that

$$(2.5)\qquad \int_\alpha^t \left(\frac{1}{r(s)}\right)^{1/\gamma} ds < \infty \quad \text{and} \quad \int_\alpha^\beta |q(t)|\, dt < \infty.$$

For simplicity, we introduce the following notations:

$$(2.6)\qquad K_1(\gamma, p, r) = \left(\frac{\gamma}{\gamma+1}\right)^{\frac{\gamma}{\gamma+1}} \left[\int_\alpha^\beta \frac{p^{\gamma+1}(t)}{r^\gamma(t)} \Big(\int_\alpha^t r^{-1/\gamma}(s)\, ds\Big)^\gamma dt\right]^{\frac{1}{1+\gamma}},$$

$$(2.7)\qquad K_1(\gamma, Q, r) = (\gamma+1)^{\frac{\gamma}{\gamma+1}} \left[\int_\alpha^\beta \frac{Q^{\frac{1+\gamma}{\gamma}}(t)}{r^{\frac{1}{\gamma}}(t)} \Big(\int_\alpha^t r^{-1/\gamma}(s)\, ds\Big)^\gamma dt\right]^{\frac{\gamma}{\gamma+1}},$$

$$(2.8)\qquad K_2(\gamma, p, r) = \left(\frac{\gamma}{\gamma+1}\right)^{\frac{\gamma}{\gamma+1}} \left[\int_\alpha^\beta \frac{p^{\gamma+1}(t)}{r^\gamma(t)} \Big(\int_t^\beta r^{-1/\gamma}(s)\, ds\Big)^\gamma dt\right]^{\frac{1}{1+\gamma}},$$

$$(2.9)\qquad K_2(\gamma, Q, r) = (\gamma+1)^{\frac{\gamma}{\gamma+1}} \left[\int_\alpha^\beta \frac{Q^{\frac{1+\gamma}{\gamma}}(t)}{r^{\frac{1}{\gamma}}(t)} \Big(\int_t^\beta r^{-1/\gamma}(s)\, ds\Big)^\gamma dt\right]^{\frac{\gamma}{\gamma+1}}.$$

Now, we are ready to state and prove the main results.

THEOREM 1. *Suppose that x is a nontrivial solution of* (1.1) *and $x'(t)$ does not change sign in (α, β). If $x(\alpha) = x'(\beta) = 0$, then*

$$(2.10)\qquad K_1(\gamma, p, r) + K_1(\gamma, Q, r) \ge 1,$$

where $Q(t) = \int_t^\beta |q(s)|\, ds$. If $x'(\alpha) = x(\beta) = 0$, then

$$(2.11)\qquad K_2(\gamma, p, r) + K_2(\gamma, Q, r) \ge 1,$$

where $Q(t) = \int_\alpha^t |q(s)|\, ds$.

Proof. We prove (2.10). Multiplying (1.1) by x and integrating by parts, we have

$$\int_\alpha^\beta (r(t)(x'(t))^\gamma)' x(t)\, dt = r(t)(x'(t))^\gamma x(t)|_\alpha^\beta - \int_\alpha^\beta r(t)(x'(t))^{\gamma+1}\, dt$$
$$= -\int_\alpha^\beta |p(t)|(x'(t))^\gamma x(t)\, dt - \int_\alpha^\beta |q(t)|(x(t))^{\gamma+1}\, dt.$$

Using the assumptions that $x(\alpha) = x'(\beta) = 0$ and $Q(t) = \int_t^\beta |q(s)|\,ds$, we get

$$\begin{aligned}\int_\alpha^\beta r(t)(x'(t))^{\gamma+1}\,dt &\le \left|\int_\alpha^\beta p(t)(x'(t))^\gamma x(t)\,dt\right| + \left|\int_\alpha^\beta q(t)(x(t))^{\gamma+1}\,dt\right| \\ &\le \int_\alpha^\beta |p(t)|\,|x'(t)|^\gamma |x(t)|\,dt + \int_\alpha^\beta |q(t)|\,|x(t)|^{\gamma+1}\,dt.\end{aligned}$$

Using the fact that $Q(t) = \int_t^\beta |q(s)|\,ds$, we have

$$\int_\alpha^\beta r(t)(x'(t))^{\gamma+1}\,dt \le \int_\alpha^\beta |p(t)|\,|x'(t)|^\gamma |x(t)|\,dt - \int_\alpha^\beta Q'(t)|x(t)|^{\gamma+1}\,dt.$$

Integrating by parts in the last term, we get

$$\begin{aligned}\int_\alpha^\beta r(t)|x'(t)|^{\gamma+1}\,dt \le (\gamma+1)&\int_\alpha^\beta Q(t)|x(t)|^\gamma |x'(t)|\,dt \\ &+ \int_\alpha^\beta |p(t)|\,|x'(t)|^\gamma |x(t)|\,dt.\end{aligned} \tag{2.12}$$

Applying the inequality (2.2) with $B(t) = Q(t)$, $A(t) = r(t)$, $m = \gamma$ and $n = 1$, we have

$$(\gamma+1)\int_\alpha^\beta |Q(t)|\,|x(t)|^\gamma |x'(t)|\,dt \le K_1(\gamma, Q, r)\int_\alpha^\beta r(t)|x'(t)|^{\gamma+1}\,dt, \tag{2.13}$$

where $K_1(\gamma, Q, r)$ is as in (2.7). Again applying (2.2) with $B(t) = |p(t)|$, $A(t) = r(t)$, $n = \gamma$ and $m = 1$, we get

$$\int_\alpha^\beta |p(t)|\,|x(t)|\,|x'(t)|^\gamma\,dt \le K_1(\gamma, p, r)\int_\alpha^\beta r(t)|x'(t)|^{\gamma+1}\,dt, \tag{2.14}$$

where $K_1(\gamma, p, r)$ is as in (2.6). Then, from (2.12)–(2.14) we have

$$\begin{aligned}\int_\alpha^\beta r(t)|x'(t)|^{\gamma+1}\,dt \le K_1(\gamma, Q, r)&\int_\alpha^\beta r(t)|x'(t)|^{\gamma+1}\,dt \\ &+ K_1(\gamma, p, r)\int_\alpha^\beta r(t)|x'(t)|^{\gamma+1}\,dt.\end{aligned}$$

Dividing both sides by $\int_\alpha^\beta r(t)|x'(t)|^{\gamma+1}\,dt$, we obtain

$$K_1(\gamma, p, r) + K_1(\gamma, Q, r) \ge 1,$$

which is the desired inequality (2.10). The proof of (2.11) is similar using integration by parts and (2.2), and (2.4) instead of (2.3). ■

By using the maximum of Q and $|p|$ on $[\alpha,\,\beta]$, we will now derive a new formula for the spacing between a zero of a solution of (1.1) and a zero of its derivative by employing (2.2) and the Calvert inequality (see [8])

$$\int_c^d |x(t)|^m |x'(t)|\,dt \le \frac{1}{m+1}\Big(\int_c^d \lambda^{-1/m}(t)\,dt\Big)^m \int_c^d \lambda(t)|x'(t)|^{m+1}\,dt, \tag{2.15}$$

where $x(c)=0$ or $x(d)=0$ and $\lambda(t)>0$. For simplicity, we set

$$K_1(\gamma,r) := \Big(\frac{\gamma}{\gamma+1}\Big)^{\frac{\gamma}{\gamma+1}} \bigg[\int_\alpha^\beta \frac{1}{r^\gamma(t)}\Big(\int_\alpha^t r^{-1/\gamma}(s)\,ds\Big)^\gamma dt\bigg]^{\frac{1}{1+\gamma}}, \tag{2.16}$$

$$K_2(\gamma,r) := \Big(\frac{\gamma}{\gamma+1}\Big)^{\frac{\gamma}{\gamma+1}} \bigg[\int_\alpha^\beta \frac{1}{r^\gamma(t)}\Big(\int_t^\beta r^{-1/\gamma}(s)\,ds\Big)^\gamma dt\bigg]^{\frac{1}{1+\gamma}}. \tag{2.17}$$

THEOREM 2. *Suppose that x is a nontrivial solution of* (1.1) *and $x'(t)$ does not change sign in (α,β). If $x(\alpha)=x'(\beta)=0$, then*

$$(\gamma+1)\max_{\alpha\le t\le\beta} Q(t)\Big(\int_\alpha^\beta r^{-1/\gamma}(t)\,dt\Big)^\gamma + K_1(\gamma,r)\max_{\alpha\le t\le\beta}|p(t)| \ge 1, \tag{2.18}$$

where $Q(t)=\int_t^\beta |q(s)|\,ds$ and $K_1(\gamma,r)$ is as in (2.16). *If $x'(\alpha)=x(\beta)=0$, then*

$$(\gamma+1)\max_{\alpha\le t\le\beta} Q(t)\Big(\int_\alpha^\beta r^{-1/\gamma}(t)\,dt\Big)^\gamma + K_2(\gamma,r)\max_{\alpha\le t\le\beta}|p(t)| \ge 1, \tag{2.19}$$

where $Q(t)=\int_\alpha^t |q(s)|\,ds$ and $K_2(\gamma,r)$ is as in (2.17).

Proof. We prove (2.18). Multiplying (1.1) by x and integrating by parts and proceeding as in the proof of Theorem 1 we get

$$\begin{aligned}\int_\alpha^\beta r(t)|x'(t)|^{\gamma+1}\,dt &\le \int_\alpha^\beta |p(t)|\,|x'(t)|^\gamma |x(t)|\,dt \\ &\quad + (\gamma+1)\int_\alpha^\beta Q(t)|x(t)|^\gamma |x'(t)|\,dt \\ &\le \max_{\alpha\le t\le\beta}|p(t)|\int_\alpha^\beta |x'(t)|^\gamma |x(t)|\,dt \\ &\quad + (\gamma+1)\max_{\alpha\le t\le\beta} Q(t)\int_\alpha^\beta |x(t)|^\gamma |x'(t)|\,dt.\end{aligned} \tag{2.20}$$

Applying (2.15) with $\lambda(t)=r(t)$ and $m=\gamma$, we have

$$(\gamma+1)\max_{\alpha\le t\le\beta}Q(t)\int_{\alpha}^{\beta}|x(t)|^{\gamma}|x'(t)|\,dt \le (\gamma+1)\max_{\alpha\le t\le\beta}Q(t)\Big(\int_{\alpha}^{\beta}r^{-1/\gamma}(t)\,dt\Big)^{\gamma}\int_{\alpha}^{\beta}r(t)|x'(t)|^{\gamma+1}\,dt. \tag{2.21}$$

Applying (2.2) with $B(t)=1$, $A(t)=r(t)$ and $m=1$ and $n=\gamma$, we get

$$\max_{\alpha\le t\le\beta}|p(t)|\int_{\alpha}^{\beta}|x(t)|\,|x'(t)|^{\gamma}\,dt \le K_1(\gamma,r)\max_{\alpha\le t\le\beta}|p(t)|\int_{\alpha}^{\beta}r(t)|x'(t)|^{\gamma+1}\,dt, \tag{2.22}$$

where $K_1(\gamma,r)$ is as in (2.16). Then from (2.20)–(2.22) we have

$$(\gamma+1)\max_{\alpha\le t\le\beta}|Q(t)|\Big(\int_{\alpha}^{\beta}r^{-1/\gamma}(t)\,dt\Big)^{\gamma}+K_1(\gamma,r)\max_{\alpha\le t\le\beta}|p(t)|\ge 1,$$

which is the desired inequality (2.18). The proof of (2.19) when $x'(\alpha)=x(\beta)=0$ is similar by using integration by parts and (2.15). ∎

We now apply an inequality due to Boyd [5] and the Hölder inequality to obtain results similar to Theorem 1. The Boyd inequality states that if $x\in C^1[a,\,b]$ with $x(a)=0$ (or $x(b)=0$), then

$$\int_a^b|x(t)|^{\nu}|x'(t)|^{\eta}\,dt\le N(\nu,\eta,s)(b-a)^{\nu}\Big(\int_a^b|x'(t)|^{s}\,dt\Big)^{\frac{\nu+\eta}{s}}, \tag{2.23}$$

where $\nu>0$, $s>1$, $0\le\eta<s$,

$$N(\nu,\eta,s):=\frac{(s-\eta)\nu^{\nu}}{(s-1)(\nu+\eta)(I(\nu,\eta,s))^{\nu}}\sigma^{\nu+\eta-s}, \tag{2.24}$$

$$\sigma:=\Big\{\frac{\nu(s-1)+(s-\eta)}{(s-1)(\nu+\eta)}\Big\}^{1/s},$$

$$I(\nu,\eta,s):=\int_0^1\Big\{1+\frac{s(\eta-1)}{s-\eta}t\Big\}^{-(\nu+\eta+s\nu)/s\nu}[1+(\eta-1)t]t^{1/\nu-1}\,dt.$$

The inequality (2.23) has an immediate application, when $\eta=s$, to the case where $x(a)=0$ (or $x(b)=0$). In this case (2.23) becomes

$$\int_a^b|x(t)|^{\nu}|x'(t)|^{\eta}\,dt\le L(\nu,\eta)(b-a)^{\nu}\Big(\int_a^b|x'(t)|^{\eta}\,dt\Big)^{\frac{\nu+\eta}{\eta}}, \tag{2.25}$$

where

$$L(\nu,\eta) := \frac{\eta\nu^\eta}{\nu+\eta}\Big(\frac{\nu}{\nu+\eta}\Big)^{\nu/\eta}\bigg(\frac{\Gamma\big(\frac{\eta+1}{\eta}+\frac{1}{\nu}\big)}{\Gamma\big(\frac{\eta+1}{\eta}\big)\Gamma\big(\frac{1}{\nu}\big)}\bigg)^\nu, \tag{2.26}$$

and Γ is the Gamma function. For simplicity, we define

$$M(\gamma) := (\gamma+1)(\beta-\alpha)^\gamma N^{\frac{\gamma}{\gamma+1}}\Big(\gamma+1, \frac{\gamma+1}{\gamma}, \gamma+1\Big),$$
$$M_1(\gamma) := (\beta-\alpha)L^{\frac{\gamma}{\gamma+1}}\Big(\frac{\gamma+1}{\gamma}, \gamma+1\Big).$$

Theorem 3. *Assume that $r(t)$ is a nonincreasing function on $[\alpha,\beta]$. Suppose that x is a nontrivial solution of (1.1) and $x'(t)$ does not change sign in (α,β). If $x(\alpha)=x'(\beta)=0$, then*

$$M(\gamma)\Big(\int_\alpha^\beta Q^{\gamma+1}(t)\,dt\Big)^{\frac{1}{\gamma+1}} + M_1(\gamma)\Big(\int_\alpha^\beta |p(t)|^{\gamma+1}\,dt\Big)^{\frac{1}{\gamma+1}} \ge r(\beta), \tag{2.27}$$

where $Q(t)=\int_t^\beta |q(s)|\,ds$. If $x'(\alpha)=x(\beta)=0$, then

$$M(\gamma)\Big(\int_\alpha^\beta Q^{\gamma+1}(t)\,dt\Big)^{\frac{1}{\gamma+1}} + M_1(\gamma)\Big(\int_\alpha^\beta |p(t)|^{\gamma+1}\,dt\Big)^{\frac{1}{\gamma+1}} \ge r(\beta), \tag{2.28}$$

where $Q(t)=\int_\alpha^t |q(s)|\,ds$.

Proof. Proceeding as in the proof of Theorem 1 and using $x(\alpha)=x'(\beta)=0$, we get

$$\int_\alpha^\beta r(t)|x'(t)|^{\gamma+1}\,dt \le \int_\alpha^\beta |p(t)|\,|x'(t)|^\gamma|x(t)|\,dt + (\gamma+1)\int_\alpha^\beta Q(t)|x(t)|^\gamma|x'(t)|\,dt. \tag{2.29}$$

Applying the Hölder inequality with exponents $\gamma+1$ and $(\gamma+1)/\gamma$, we see that

$$\int_\alpha^\beta Q(t)|x(t)|^\gamma|x'(t)|\,dt \le \Big(\int_\alpha^\beta Q^{\gamma+1}(t)\,dt\Big)^{\frac{1}{\gamma+1}} \times \Big(\int_\alpha^\beta |x(t)|^{\gamma+1}|x'(t)|^{\frac{\gamma+1}{\gamma}}\,dt\Big)^{\frac{\gamma}{\gamma+1}}. \tag{2.30}$$

Applying the Boyd inequality (2.23) with $\nu=(\gamma+1)$, $\eta=(\gamma+1)/\gamma$ and $s=\gamma+1$, we obtain

$$\int_{\alpha}^{\beta} |x(t)|^{\gamma+1}|x'(t)|^{\frac{\gamma+1}{\gamma}}\,dt \le N\Big(\gamma+1,\frac{\gamma+1}{\gamma},\gamma+1\Big)(\beta-\alpha)^{\gamma+1}\Big(\int_{a}^{b}|x'(t)|^{\gamma+1}\,dt\Big)^{\frac{\gamma+1}{\gamma}}. \tag{2.31}$$

Substituting (2.31) into (2.30) and using the fact that $r(t)$ is nonincreasing, we have

$$(\gamma+1)\int_{\alpha}^{\beta} Q(t)|x(t)|^{\gamma}|x'(t)|\,dt \le \frac{M(\gamma)}{r(\beta)}\Big(\int_{\alpha}^{\beta} Q^{\gamma+1}(t)\,dt\Big)^{\frac{1}{\gamma+1}}\Big(\int_{\alpha}^{\beta} r(t)|x'(t)|^{\gamma+1}\,dt\Big). \tag{2.32}$$

Applying the Hölder inequality to the term $\int_{\alpha}^{\beta}|p(t)|\,|x'(t)|^{\gamma}|x(t)|\,dt$, we see that

$$\int_{\alpha}^{\beta}|p(t)|\,|x'(t)|^{\gamma}|x(t)|\,dt \le \Big(\int_{\alpha}^{\beta}|p(t)|^{\gamma+1}\,dt\Big)^{\frac{1}{\gamma+1}} \times\Big(\int_{\alpha}^{\beta}|x(t)|^{\frac{\gamma+1}{\gamma}}|x'(t)|^{\gamma+1}\,dt\Big)^{\frac{\gamma}{\gamma+1}}. \tag{2.33}$$

Applying the Boyd inequality (2.25) with $\nu=(\gamma+1)/\gamma$ and $\eta=\gamma+1$, we get

$$\int_{\alpha}^{\beta}|x(t)|^{\frac{\gamma+1}{\gamma}}|x'(t)|^{\gamma+1}\,dt \le L\Big(\frac{\gamma+1}{\gamma},\gamma+1\Big) \times(\beta-\alpha)^{\frac{\gamma+1}{\gamma}}\Big(\int_{\alpha}^{\beta}|x'(t)|^{\gamma+1}\,dt\Big)^{\frac{\gamma+1}{\gamma}}. \tag{2.34}$$

Substituting (2.34) into (2.33) and using the fact that $r(t)$ is nonincreasing, we have

$$\int_{\alpha}^{\beta}|p(t)|\,|x'(t)|^{\gamma}|x(t)|\,dt \le \frac{M_1(\gamma)}{r(\beta)}\Big(\int_{\alpha}^{\beta}|p(t)|^{\gamma+1}\,dt\Big)^{\frac{1}{\gamma+1}}\int_{\alpha}^{\beta} r(t)|x'(t)|^{\gamma+1}\,dt. \tag{2.35}$$

Substituting (2.32) and (2.35) into (2.29) and cancelling $\int_{\alpha}^{\beta} r(t)|x'(t)|^{\gamma+1}\,dt$ yields

$$\frac{M(\gamma)}{r(\beta)}\Big(\int_{\alpha}^{\beta} Q^{\gamma+1}(t)\,dt\Big)^{\frac{1}{\gamma+1}}+\frac{M_1(\gamma)}{r(\beta)}\Big(\int_{\alpha}^{\beta}|p(t)|^{\gamma+1}\,dt\Big)^{\frac{1}{\gamma+1}}\ge 1,$$

which is the desired inequality (2.27). A similar argument yields (2.28) when $x'(\alpha) = x(\beta) = 0$. ∎

Remark 1. Theorems 1–3 yield sufficient conditions for disfocality of (1.1), i.e., sufficient conditions so that there does not exist a nontrivial solution x satisfying either $x(\alpha) = x'(\beta) = 0$ or $x'(\alpha) = x(\beta) = 0$.

In the following, we employ new Opial and Wirtinger type inequalities to determine the lower bound for the distance between consecutive zeros of solutions of (1.1). Note that the inequality (2.23) has an immediate application to the case where $x(a) = x(b) = 0$. Choose $c = (a+b)/2$ and apply (2.23) to $[a,c]$ and $[c,b]$ and then add to obtain

$$\int_a^b |x(t)|^{\nu}|x'(t)|^{\eta}\,dt \le N(\nu,\eta,s)\Big(\frac{b-a}{2}\Big)^{\nu}\Big(\int_a^b |x'(t)|^{s}\,dt\Big)^{\frac{\nu+\eta}{s}}, \tag{2.36}$$

where $N(\nu,\eta,s)$ is defined as in (2.24). Note that application of (2.36) allows the use of an arbitrary anti-derivative Q in the above arguments. We define

$$M^*(\gamma) := (\gamma+1)\Big(\frac{b-a}{2}\Big)^{\gamma} N^{\frac{\gamma}{\gamma+1}}\Big(\gamma+1, \frac{\gamma+1}{\gamma}, \gamma+1\Big),$$
$$M_1^*(\gamma) := \frac{b-a}{2} N^{\frac{\gamma}{\gamma+1}}\Big(\frac{\gamma+1}{\gamma}, \gamma+1, \gamma+1\Big).$$

The following theorems give the lower bound for the distance between zeros of the solution of (1.1).

Theorem 4. *Assume that $r(t)$ is a nonincreasing function and $Q'(t) = |q(t)|$ on $[a,b]$. Suppose that x is a nontrivial solution of* (1.1) *and $x'(t)$ does not change sign in (a,b). If $x(a) = x(b) = 0$, then*

$$M^*(\gamma)\Big(\int_a^b Q^{\frac{\gamma+1}{\gamma}}(t)\,dt\Big)^{\frac{\gamma}{\gamma+1}} + M_1^*(\gamma)\Big(\int_a^b |p(t)|^{\gamma+1}\,dt\Big)^{\frac{1}{\gamma+1}} \ge r(b). \tag{2.37}$$

Proof. As in the proof of Theorem 1, by multiplying (1.1) by $x(t)$ and integrating by parts, and using $x(a) = x(b) = 0$, we have

$$\int_a^b r(t)|x'(t)|^{\gamma+1}\,dt \le \int_a^b |p(t)|\,|x'(t)|^{\gamma}|x(t)|\,dt + (\gamma+1)\int_a^b Q(t)|x(t)|^{\gamma}|x'(t)|\,dt. \tag{2.38}$$

Applying the Hölder inequality and (2.36) to (2.38) and proceeding as in the proof of Theorem 3, we have

$$\int_a^b r(t)|x'(t)|^{\gamma+1}\,dt \le \frac{M^*(\gamma)}{r(b)}\Big(\int_a^b Q^{\frac{\gamma+1}{\gamma}}(t)\,dt\Big)^{\frac{\gamma}{\gamma+1}}\int_a^b r(t)|x'(t)|^{\gamma+1}\,dt + \frac{M_1^*(\gamma)}{r(b)}\Big(\int_a^b |p(t)|^{\gamma+1}\,dt\Big)^{\frac{1}{\gamma+1}}\int_a^b r(t)|x'(t)|^{\gamma+1}\,dt.$$

From this inequality, after cancelling $\int_a^b r(t)|x'(t)|^{\gamma+1}\,dt$, we get (2.37). ∎

We note that the inequality (2.36) can be applied only when $s > \eta \ge 0$. So Theorem 4 cannot be applied when $\gamma = 1$. In the following, we make use of the inequality (2.25) which can be applied when $\eta = s$. The inequality (2.25) has an immediate application to the case when $x(a) = x(b) = 0$. In this case (2.25) becomes

$$\int_a^b |x(t)|^{\nu}|x'(t)|^{\eta}\,dt \le L(\nu,\eta)\Big(\frac{b-a}{2}\Big)^{\nu}\Big(\int_a^b |x'(t)|^{\eta}\,dt\Big)^{\frac{\nu+\eta}{\eta}}, \tag{2.39}$$

where $L(\nu,\eta)$ is as in (2.26).

THEOREM 5. *Assume that $r(t)$ is a nonincreasing function and $Q'(t) = |q(t)|$ on $[a,b]$. Suppose that x is a nontrivial solution of* (1.1) *and $x'(t)$ does not change sign in (a,b). If $x(a) = x(b) = 0$, then*

$$(\gamma+1)N(\gamma)\Big(\int_a^b Q^{\frac{\gamma+1}{\gamma}}(t)\,dt\Big)^{\frac{\gamma}{\gamma+1}} + N^*(\gamma)\Big(\int_a^b |p(t)|^{\frac{1}{\gamma+1}}\,dt\Big)^{\gamma+1} \ge r(b), \tag{2.40}$$

where

$$N(\gamma) := \Big(\frac{b-a}{2}\Big)^{\gamma} L^{\frac{1}{\gamma+1}}(\gamma(\gamma+1),\gamma+1),$$
$$N^*(\gamma) := \frac{b-a}{2} L^{\frac{\gamma}{\gamma+1}}\Big(\frac{\gamma+1}{\gamma},\gamma+1\Big).$$

Proof. As in the proof of Theorem 1, we have

$$\int_a^b r(t)|x'(t)|^{\gamma+1}\,dt \le (\gamma+1)\int_a^b Q(t)|x(t)|^{\gamma}|x'(t)|\,dt + \int_\alpha^\beta |p(t)|\,|x'(t)|^{\gamma}|x(t)|\,dt. \tag{2.41}$$

Applying the Hölder inequality, we get

$$\int_a^b Q(t)|x(t)|^{\gamma}|x'(t)|\,dt \le \Big(\int_a^b Q^{\frac{\gamma+1}{\gamma}}(t)\,dt\Big)^{\frac{\gamma}{\gamma+1}} \times \Big(\int_a^b |x(t)|^{\gamma(\gamma+1)}|x'(t)|^{\gamma+1}\,dt\Big)^{\frac{1}{\gamma+1}}. \tag{2.42}$$

Applying the inequality (2.39) to the term

$$\left(\int_a^b |x(t)|^{\gamma(\gamma+1)}|x'(t)|^{\gamma+1}\,dt\right)^{\frac{1}{\gamma+1}} \tag{2.43}$$

with $\nu = \gamma(\gamma+1)$ and $\eta = \gamma+1$, we see that

$$\left(\int_a^b |x(t)|^{\gamma(\gamma+1)}|x'(t)|^{\gamma+1}\,dt\right)^{\frac{\gamma}{\gamma+1}} \le L(\gamma(\gamma+1),\gamma+1)\left(\frac{b-a}{2}\right)^{\gamma(\gamma+1)}\left(\int_a^b |x'(t)|^{\gamma+1}\,dt\right)^{\gamma+1}. \tag{2.44}$$

Substituting (2.44) into (2.42) and using the fact that $r(t)$ is nonincreasing, we have

$$\int_a^b |Q(t)|\,|x(t)|^{\gamma}|x'(t)|\,dt \le \frac{L^{\frac{1}{\gamma+1}}(\gamma(\gamma+1),\gamma+1)}{r(b)}\left(\frac{b-a}{2}\right)^{\gamma}\left(\int_a^b Q^{\frac{\gamma+1}{\gamma}}(t)\,dt\right)^{\frac{\gamma}{\gamma+1}} \times\left(\int_a^b r(t)|x'(t)|^{\gamma+1}\,dt\right). \tag{2.45}$$

Applying the Hölder inequality, we also get

$$\int_\alpha^\beta |p(t)|\,|x'(t)|^{\gamma}|x(t)|\,dt \le \left(\int_a^b |p(t)|^{\frac{1}{\gamma+1}}\,dt\right)^{\gamma+1} \times\left(\int_a^b |x'(t)|^{\gamma+1}|x(t)|^{\frac{\gamma+1}{\gamma}}\,dt\right)^{\frac{\gamma}{\gamma+1}}. \tag{2.46}$$

Applying the inequality (2.39) to the term

$$\left(\int_a^b |x'(t)|^{\gamma+1}|x(t)|^{\frac{\gamma+1}{\gamma}}\,dt\right)^{\frac{\gamma}{\gamma+1}} \tag{2.47}$$

with $\eta = \gamma+1$ and $\nu = (\gamma+1)/\gamma$, we see that

$$\left(\int_a^b |x(t)|^{\frac{\gamma+1}{\gamma}}|x'(t)|^{\gamma+1}\,dt\right)^{\frac{\gamma}{\gamma+1}} \le L\left(\frac{\gamma+1}{\gamma},\gamma+1\right)\left(\frac{b-a}{2}\right)^{\frac{\gamma+1}{\gamma}}\left(\int_a^b |x'(t)|^{\gamma+1}\,dt\right)^{\frac{\gamma+1}{\gamma}}. \tag{2.48}$$

Substituting (2.48) into (2.46) and using the fact that $r(t)$ is nonincreasing, we have

$$\int_a^b p(t)|x(t)|^\gamma |x'(t)|\,dt \le \frac{L^{\frac{\gamma}{\gamma+1}}(\frac{\gamma+1}{\gamma},\gamma+1)}{r(b)}\,\frac{b-a}{2}\Big(\int_a^b |p(t)|^{\frac{1}{\gamma+1}}\,dt\Big)^{\gamma+1} \times \Big(\int_a^b r(t)|x'(t)|^{\gamma+1}\,dt\Big). \tag{2.49}$$

Substituting (2.45) and (2.49) into (2.41) and cancelling $\int_a^b r(t)|x'(t)|^{\gamma+1}\,dt$, we get

$$1 \le (\gamma+1)L^{\frac{1}{\gamma+1}}(\gamma(\gamma+1),\gamma+1)\Big(\frac{b-a}{2}\Big)^\gamma\Big(\int_a^b Q^{\frac{\gamma+1}{\gamma}}(t)\,dt\Big)^{\frac{\gamma}{\gamma+1}} + L^{\frac{\gamma}{\gamma+1}}\Big(\frac{(\gamma+1)}{\gamma},\gamma+1\Big)\frac{b-a}{2}\Big(\int_a^b |p(t)|^{\frac{1}{\gamma+1}}\,dt\Big)^{\gamma+1},$$

which is the desired inequality (2.40). ∎

As a special case when $\gamma = 1$, we have the following result for the equation

$$x''(t) + p(t)x'(t) + q(t)x(t) = 0, \qquad a \le t \le b. \tag{2.50}$$

Corollary 6. *Assume that $Q'(t) = |q(t)|$ on $[a,b]$. Suppose that x is a nontrivial solution of* (2.50) *and $x'(t)$ does not change sign in (a,b). If $x(a) = x(b) = 0$, then*

$$\Big(\int_a^b Q^2(t)\,dt\Big)^{1/2} + \frac{1}{2}\Big(\int_a^b \sqrt{p(t)}\,dt\Big)^2 \ge \frac{\pi}{2(b-a)}. \tag{2.51}$$

In the following, we establish a new formula for the spacing between zeros of (1.1) by applying the Wirtinger type inequality (see [1])

$$\int_c^d \lambda(t)|x(t)|^{\gamma+1}\,dt \le \frac{1}{2}\Big(\int_c^d (t(d-t))^{\gamma/2}\lambda(t)\,dt\Big)\int_c^d |x'(t)|^{\gamma+1}\,dt, \tag{2.52}$$

where $x(c) = x(d) = 0$, $\lambda(t) > 0$ is a continuous function on $[c,d]$, and $x(t)$ is an absolutely continuous function on $[c,d]$.

Theorem 7. *Assume that $r(t)$ is a nonincreasing function and $Q'(t) = |q(t)|$ on $[a,b]$. Suppose that x is a nontrivial solution of* (1.1) *and $x'(t)$ does* *not change sign in* (α,β). *If* $x(a)=x(b)=0$, *then*

$$\frac{1}{2}(\gamma+1)^{\frac{\gamma+1}{\gamma}}\int_a^b (t(b-t))^{\frac{\gamma}{2}} Q^{\frac{\gamma+1}{\gamma}}(t)\,dt + \left(\frac{1}{2}\right)^{\frac{1}{\gamma+1}}\left(\int_\alpha^\beta (t(b-t))^{\frac{\gamma}{2}}|p(t)|^{\gamma+1}\,dt\right)^{\frac{1}{\gamma+1}} \geq r(b). \tag{2.53}$$

Proof. Proceeding as in the proof of Theorem 1 and using $x(a)=x(b)=0$, we get

$$\int_a^b r(t)|x'(t)|^{\gamma+1}\,dt \leq (\gamma+1)\int_a^b Q(t)|x(t)|^{\gamma}|x'(t)|\,dt + \int_a^b |p(t)|\,|x'(t)|^{\gamma}|x(t)|\,dt. \tag{2.54}$$

Applying the Hölder inequality to the two terms on the right hand side, we see that

$$\begin{aligned}\int_a^b r(t)|x'(t)|^{\gamma+1}\,dt &\leq (\gamma+1)\Big(\int_a^b |x'(t)|^{\gamma+1}\,dt\Big)^{\frac{1}{\gamma+1}}\Big(\int_a^b Q^{\frac{\gamma+1}{\gamma}}(t)|x(t)|^{\gamma+1}\,dt\Big)^{\frac{\gamma}{\gamma+1}} \\ &\quad + \Big(\int_a^b |x'(t)|^{\gamma+1}\,dt\Big)^{\frac{\gamma}{\gamma+1}}\Big(\int_a^b |p(t)|^{\gamma+1}|x(t)|^{\gamma+1}\,dt\Big)^{\frac{1}{\gamma+1}}.\end{aligned} \tag{2.55}$$

Applying the Wirtinger inequality (2.52) to $(\int_a^b Q^{\frac{\gamma+1}{\gamma}}(t)|x(t)|^{\gamma+1}\,dt)^{\frac{\gamma}{\gamma+1}}$, we have

$$\begin{aligned}\Big(\int_a^b |Q(t)|^{\frac{\gamma+1}{\gamma}}|x(t)|^{\gamma+1}\,dt\Big)^{\frac{\gamma}{\gamma+1}} &\leq \left(\frac{1}{2}\Big(\int_a^b (t(b-t))^{\frac{\gamma}{2}} Q^{\frac{\gamma+1}{\gamma}}(t)\,dt\Big)\int_a^b |x'(t)|^{\gamma+1}\,dt\right)^{\frac{\gamma}{\gamma+1}} \\ &= \left(\frac{1}{2}\right)^{\frac{\gamma}{\gamma+1}}\Big(\int_a^b (t(b-t))^{\frac{\gamma}{2}} Q^{\frac{\gamma+1}{\gamma}}(t)\,dt\Big)^{\frac{\gamma}{\gamma+1}}\Big(\int_a^b |x'(t)|^{\gamma+1}\,dt\Big)^{\frac{\gamma}{\gamma+1}}.\end{aligned} \tag{2.56}$$

Again applying the Wirtinger inequality (2.52) to $(\int_a^b |p(t)|^{\gamma+1}|x(t)|^{\gamma+1}\,dt)^{\frac{1}{\gamma+1}}$, we have

(2.57)
$$\Big(\int_a^b |p(t)|^{\gamma+1}|x(t)|^{\gamma+1}\,dt\Big)^{\frac{1}{\gamma+1}} \le \Big(\frac{1}{2}\Big)^{\frac{1}{\gamma+1}}\Big(\int_a^b (t(b-t))^{\frac{\gamma}{2}}|p(t)|^{\gamma+1}\,dt\Big)^{\frac{1}{\gamma+1}} \times \Big(\int_a^b |x'(t)|^{\gamma+1}\,dt\Big)^{\frac{1}{\gamma+1}}.$$

Substituting (2.56) and (2.57) into (2.55), we obtain

$$\begin{aligned}\int_a^b r(t)|x'(t)|^{\gamma+1}\,dt &\le \Big(\frac{1}{2}\Big)^{\frac{\gamma}{\gamma+1}}(\gamma+1)\Big(\int_a^b |x'(t)|^{\gamma+1}\Big)^{\frac{1}{\gamma+1}}\\ &\quad\times\Big(\int_a^b (t(b-t))^{\frac{\gamma}{2}}Q^{\frac{\gamma+1}{\gamma}}(t)\,dt\Big)^{\frac{\gamma}{\gamma+1}}\Big(\int_a^b r(t)|x'(t)|^{\gamma+1}\Big)^{\frac{\gamma}{\gamma+1}}\\ &\quad+\Big(\int_\alpha^\beta |x'(t)|^{\gamma+1}\,dt\Big)^{\frac{\gamma}{\gamma+1}}\Big(\int_\alpha^\beta |x'(t)|^{\gamma+1}\,dt\Big)^{\frac{1}{\gamma+1}}\\ &\quad\times\Big(\frac{1}{2}\Big)^{\frac{1}{\gamma+1}}\Big(\int_a^b (t(b-t))^{\frac{\gamma}{2}}|p(t)|^{\gamma+1}\,dt\Big)^{\frac{1}{\gamma+1}}\\ &\le \frac{1}{r(b)}\Big(\frac{1}{2}\Big)^{\frac{\gamma}{\gamma+1}}(\gamma+1)\Big(\int_a^b (t(b-t))^{\frac{\gamma}{2}}Q^{\frac{\gamma+1}{\gamma}}(t)\,dt\Big)^{\frac{\gamma}{\gamma+1}}\cdot\int_a^b r(t)|x'(t)|^{\gamma+1}\,dt\\ &\quad+\frac{1}{r(b)}\Big(\frac{1}{2}\Big)^{\frac{1}{\gamma+1}}\Big(\int_a^b (t(b-t))^{\frac{\gamma}{2}}|p(t)|^{\gamma+1}\,dt\Big)^{\frac{1}{\gamma+1}}\cdot\int_a^b r(t)|x'(t)|^{\gamma+1}\,dt.\end{aligned}$$

Cancelling $\int_a^b r(t)|x'(t)|^{\gamma+1}\,dt$, we get

$$\begin{aligned}r(b) &\le \Big(\frac{1}{2}\Big)^{\frac{\gamma}{\gamma+1}}(\gamma+1)\Big(\int_a^b (t(b-t))^{\frac{\gamma}{2}}Q^{\frac{\gamma+1}{\gamma}}(t)\,dt\Big)^{\frac{\gamma}{\gamma+1}}\\ &\quad+\Big(\frac{1}{2}\Big)^{\frac{1}{\gamma+1}}\Big(\int_a^b (t(b-t))^{\frac{\gamma}{2}}|p(t)|^{\gamma+1}\,dt\Big)^{\frac{1}{\gamma+1}},\end{aligned}$$

which is the desired inequality (2.53). ∎

Remark 2. One can apply the Wirtinger inequality

$$\int_a^b \lambda(t)|x(t)|^{\gamma+1}\,dt \le \Big(\int_a^b (t^{-\gamma}+(1-t)^{-\gamma})^{-1}\lambda(t)\,dt\Big)\int_a^b |x'(t)|^{\gamma+1}\,dt, \qquad (2.58)$$

due to Brnetić and Pečarić [7] where $x(t)\in C^1[a,b]$, $\lambda(t)>0$ is a continuous function on $[\alpha,\beta]$, and $x(a)=x(b)=0$, and follow the proof of Theorem 7 to establish a new formula for the spacing between zeros of (1.1). The details are left to the reader.

REMARK 3. Theorems 4–6 yield sufficient conditions for disconjugacy of (1.1), i.e., sufficient conditions so that there does not exist a nontrivial solution x satisfying $x(\alpha) = x(\beta) = 0$.

As an application, we will show how Opial and Wirtinger type inequalities may be used to find a lower bound for the first eigenvalue of a boundary value problem. In particular, we will apply the Wirtinger inequality

$$\int_0^\pi (x'(t))^{k+1}\,dt \ge \frac{2\Gamma(k+2)}{\pi^{k+1}\Gamma^2((k+2)/2)} \int_0^\pi x^{k+1}(t)\,dt \quad \text{for } k \ge 1, \tag{2.59}$$

where $x \in C^1[0,\pi]$ and $x(0) = x(\pi) = 0$, due to Agarwal and Pang [2], to establish a new explicit lower bound of the first eigenvalue λ_0 of the eigenvalue problem

$$-((x'(t))^\gamma)' - p(t)(x'(t))^\gamma + q(t)x^\gamma(t) = \lambda x^\gamma(t), \quad x(0) = x(\pi) = 0, \tag{2.60}$$

where $\gamma \ge 1$ is an odd positive integer.

THEOREM 8. *Assume that λ_0 is the first positive eigenvalue of* (2.60) *and $Q'(t) = q(t) + \mu$, where $0 < \mu < \lambda_0$. Then*

$$\lambda_0 \ge \mu + \frac{1}{\Psi(\gamma)}\bigg[1 - \Big(\frac{1}{2}\Big)^{\frac{\gamma}{\gamma+1}}(\gamma+1)\Big(\int_0^\pi (t(\pi-t))^{\frac{\gamma}{2}} Q^{\frac{\gamma+1}{\gamma}}(t)\,dt\Big)^{\frac{\gamma}{\gamma+1}} - \Big(\frac{1}{2}\Big)^{\frac{1}{\gamma+1}}\Big(\int_0^\pi (t(\pi-t))^{\frac{\gamma}{2}}|p(t)|^{\gamma+1}\,dt\Big)^{\frac{1}{\gamma+1}}\bigg], \tag{2.61}$$

where $\Psi(\gamma) = \pi^{\gamma+1}\Gamma^2((\gamma+2)/2)/2\Gamma(\gamma+2)$.

Proof. Let $x(t)$ be the eigenfunction of (2.60) corresponding to λ_0. Multiplying (2.60) by $x(t)$ and proceeding as in the proof of Theorem 3 we get

$$-\int_0^\pi p(t)(x'(t))^\gamma x(t)\,dt + \int_0^\pi q(t)x^{\gamma+1}(t)\,dt = \lambda_0 \int_0^\pi x^{\gamma+1}(t)\,dt + \int_0^\pi ((x'(t))^\gamma)'x(t)\,dt.$$

This implies, after integrating by parts and using the fact that $x(0) = x(\pi)$

$= 0$, that

$$
\begin{aligned}
(\lambda_0 - \mu) \int_0^\pi x^{\gamma+1}(t)\, dt
&= \int_0^\pi (x'(t))^{\gamma+1}\, dt + \int_0^\pi Q'(t) x^{\gamma+1}(t)\, dt - \int_0^\pi p(t)(x'(t))^\gamma x(t)\, dt \\
&= \int_0^\pi (x'(t))^{\gamma+1}\, dt - (\gamma+1) \int_0^\pi Q(t) x^\gamma(t) x'(t)\, dt - \int_0^\pi p(t)(x'(t))^\gamma x(t)\, dt \\
&\geq \int_0^\pi |x'(t)|^{\gamma+1}\, dt - (\gamma+1) \int_0^\pi Q(t)|x(t)|^\gamma |x'(t)|\, dt - \int_0^\pi |p(t)|\, |x'(t)|^\gamma x(t)\, dt.
\end{aligned}
$$

Proceeding as in the proof of Theorem 6 by applying (2.52) to

$$
\int_0^\pi Q(t)|x(t)|^\gamma |x'(t)|\, dt \quad \text{and} \quad \int_0^\pi |p(t)|\, |x'(t)|^\gamma x(t)\, dt,
$$

we obtain

$$
\begin{aligned}
(\lambda_0 - \mu) \int_0^\pi |x(t)|^{\gamma+1}\, dt \geq \int_0^\pi (|x'(t)|)^{\gamma+1}\, dt& \\
- \left(\frac{1}{2}\right)^{\frac{\gamma}{\gamma+1}} (\gamma+1) \Big(\int_0^\pi (t(\pi - t))^{\frac{\gamma}{2}} Q^{\frac{\gamma+1}{\gamma}}(t)\, dt \Big)^{\frac{\gamma}{\gamma+1}} \cdot \int_0^\pi |x'(t)|^{\gamma+1}\, dt& \\
- \left(\frac{1}{2}\right)^{\frac{1}{\gamma+1}} \Big(\int_0^\pi (t(\pi - t))^{\frac{\gamma}{2}} |p(t)|^{\gamma+1}\, dt \Big)^{\frac{1}{\gamma+1}} \cdot \int_0^\pi |x'(t)|^{\gamma+1}\, dt.&
\end{aligned}
$$

Now, applying the Wirtinger inequality (2.59), we have

$$
\begin{aligned}
(\lambda_0 - \mu)\Psi(\gamma) \int_0^\pi (x'(t))^{\gamma+1}\, dt& \\
\geq \int_0^\pi (x'(t))^{\gamma+1}\, dt - \left(\frac{1}{2}\right)^{\frac{\gamma}{\gamma+1}} (\gamma+1) \Big(\int_0^\pi (t(\pi - t))^{\frac{\gamma}{2}} Q^{\frac{\gamma+1}{\gamma}}(t)\, dt \Big)^{\frac{\gamma}{\gamma+1}}& \\
\times \int_0^\pi (x'(t))^{\gamma+1}\, dt& \\
- \left(\frac{1}{2}\right)^{\frac{1}{\gamma+1}} \Big(\int_0^\pi (t(\pi - t))^{\frac{\gamma}{2}} |p(t)|^{\gamma+1}\, dt \Big)^{\frac{1}{\gamma+1}} \cdot \int_0^\pi |x'(t)|^{\gamma+1}\, dt.&
\end{aligned}
$$

This implies that

$$(\lambda_0-\mu)\Psi(\gamma)\ge 1-\Big(\frac{1}{2}\Big)^{\frac{\gamma}{\gamma+1}}(\gamma+1)\Big(\int_0^{\pi}(t(\pi-t))^{\frac{\gamma}{2}}Q^{\frac{\gamma+1}{\gamma}}(t)\,dt\Big)^{\frac{\gamma}{\gamma+1}}$$
$$-\Big(\frac{1}{2}\Big)^{\frac{1}{\gamma+1}}\Big(\int_0^{\pi}(t(\pi-t))^{\frac{\gamma}{2}}|p(t)|^{\gamma+1}\,dt\Big)^{\frac{1}{\gamma+1}}.$$

From this, (2.61) follows. ■

The following example illustrates the results.

Example 1. Consider the equation

$$x''(t)+(\mu\sin^2(kt))x'(t)+(\lambda\cos(kt))x(t)=0,\quad t\in I, \tag{2.62}$$

where $p(t)=\mu\sin^2(kt)$, $q(t)=\lambda\cos(kt)$ and λ, μ, k are positive constants. Let $x(t)$ be a solution of (2.62) with $x(a)=x(b)=0$ where $[a,b]\subseteq I$. By (2.51) we see that

$$\frac{\pi}{2(b-a)}\le\frac{\sqrt{\lambda}}{k}\Big(\int_a^b\sin^2(kt)\,dt\Big)^{1/2}+\frac{\mu}{2}\Big(\int_a^b\sin(kt)\,dt\Big)^2. \tag{2.63}$$

Applying the Cauchy–Schwarz inequality we see that

$$\Big(\int_a^b\sin(kt)\,dt\Big)^2\le(b-a)\int_a^b\sin^2(kt)\,dt.$$

Substituting into (2.63), we see that

$$\begin{aligned}\frac{\pi}{2(b-a)}&\le\frac{\sqrt{\lambda}}{k}\Big(\int_a^b\sin^2(kt)\,dt\Big)^{1/2}+\frac{\mu}{2}(b-a)\int_a^b\sin^2(kt)\,dt\\&\le\frac{\sqrt{\lambda}}{k}\sqrt{b-a}+\frac{\mu}{2}(b-a)^2\\&=\Big(\frac{\sqrt{\lambda}}{k}+\frac{\mu}{2}(b-a)^{3/2}\Big)\sqrt{b-a}.\end{aligned}$$

This implies that

$$(b-a)^3\Big(\frac{\sqrt{\lambda}}{k}+\frac{\mu}{2}(b-a)^{3/2}\Big)^2\ge\frac{\pi^2}{4}. \tag{2.64}$$

The inequality (2.64) gives a lower bound for the spacing of zeros of (2.62). If $\mu=0$, then (2.64) reduces to

$$b-a\ge\Big(\frac{\pi^2}{4\lambda}\Big)^{1/3}k^{2/3}, \tag{2.65}$$

which gives a lower bound for the spacing of zeros of the equation

$$x''(t)+(\lambda\cos(kt))x(t)=0. \tag{2.66}$$

We note that the estimate for (2.66) that has been obtained by the usual Lyapunov inequality for $\lambda = 1$ and $a = 0$ is

$$\int_0^b (\cos(kt))_+ \, dt > \frac{4}{b}, \tag{2.67}$$

where + denotes the positive part of a function. This inequality gives a lower bound of order $\sqrt{k}$ for the spacing of zeros, which is different from the order in (2.65). It is clear that the estimate (2.65) improves (2.67).

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S. H. Saker
Department of Mathematics Skills, PYD
King Saud University
Riyadh 11451, Saudi Arabia
and
Department of Mathematics
Mansoura University
Mansoura 35516, Egypt
E-mail: mathcoo@py.ksu.edu.sa, shsaker@mans.edu.eg

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