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A short note on Seshadri constants and packing constants

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Abstract. The note is about a connection between Seshadri constants and packing constants and presents another proof of Lazarsfeld's result from [Math. Res. Lett. 3 (1996), 439–447].

1. Introduction. This note concerns certain connections between the local positivity of line bundles on algebraic varieties and the symplectic packing of balls into symplectic manifolds. On the one hand we have a smooth, complex, projective variety X with an ample line bundle L. Local positivity of the bundle may be measured by Seshadri constants (see [13], [20], [8]), which, roughly speaking, say how large the degree of a curve must be if the curve passes through given points with given multiplicities. On the other hand, a variety as above may be treated as a symplectic manifold. The symplectic packing problem concerns the existence of a symplectic embedding of a disjoint union of (Euclidean) balls into X. The amount of the volume of X which may be filled by the symplectic images of balls is measured by symplectic packing constants (see [10], [14], [5]). Analogously we may consider symplectic and holomorphic packing constants (see [12]). It seems that there exists a close connection between Seshadri constants and packing constants. This connection was first observed in [14] and then in [5], [6], [12] and other papers. Lazarsfeld in [12] proved a bound on Seshadri constants by means of symplectic and holomorphic packing constants. The aim of this note is to give another proof of this bound using the fact that holomorphic curves are minimal surfaces.

2. Seshadri constants. Let X be a projective algebraic manifold with an ample line bundle L. Let $P_1, \dots, P_r$ be r different points on X. Seshadri constants are defined as follows.

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Definition 2.1. The *Seshadri constant* of L in $P_1, \dots, P_r$ is defined as the number

$$\varepsilon(X, L, P_1, \dots, P_r) := \inf\left\{ \frac{LC}{\operatorname{mult}_{P_1} C + \dots + \operatorname{mult}_{P_r} C} \right\},$$

where the infimum is taken over the curves C on X passing through at least one P_i. Equivalently

$$\varepsilon(X, L, P_1, \dots, P_r) := \sup\{\varepsilon \mid \pi^* L - \varepsilon(E_1 + \dots + E_r) \text{ is numerically effective}\},$$

where $\pi : \tilde{X} \to X$ is the blow-up of X in $P_1, \dots, P_r$ with the exceptional divisors $E_1, \dots, E_r$.

If the points $P_1, \dots, P_r$ are very general on X we will write $\varepsilon(X, L, r)$ instead of $\varepsilon(X, L, P_1, \dots, P_r)$; if X is clear from the context we will write $\varepsilon(L, P_1, \dots, P_r)$ or respectively $\varepsilon(L, r)$.

Remark 2.2. For an ample line bundle L on X we have (see [13])

$$0 < \varepsilon(L, P_1, \dots, P_r) \le \sqrt[n]{L^n/r}.$$

Finding the exact values of Seshadri constants is in most cases a difficult problem. For $\mathbb{P}^2$ with $L = \mathcal{O}_{\mathbb{P}^2}(1)$ the exact values of $\varepsilon(L, r)$ are known only if $r \le 9$ or $r = k^2$, $k \in \mathbb{N}$. The famous conjecture of Nagata states that $\varepsilon(\mathcal{O}_{\mathbb{P}^2}(1), r) = \sqrt{1/r}$ (so it is maximal possible) for $r \ge 10$ (cf. [11]). The generalized conjecture, called the Nagata–Biran–Szemberg Conjecture, says that for any algebraic surface X with an ample line bundle L there exists a number N such that for all $r \ge N$, $\varepsilon(L, r) = \sqrt{L^2/r}$ (cf. e.g. [19]).

So far, all known values of Seshadri constants are rational. In general, it is hard to find the value of a Seshadri constant even at one point. For example, in the case of surfaces, if we can prove the existence of so-called *submaximal curves*, i.e. curves C on X such that

$$\frac{LC}{\operatorname{mult}_{P_1} C + \dots + \operatorname{mult}_{P_r} C} < \sqrt{\frac{L^2}{r}},$$

then the Seshadri constant is necessarily rational and less than the maximal value $\sqrt{L^2/r}$ (see e.g. [17]).

On the other hand, there are (so far) not many ways of proving the nonexistence of submaximal curves. This makes it difficult to prove that the Seshadri constants are maximal. One way to attack the problem is to give lower bounds on the Seshadri constants.

3. Packing constants. Let us recall that a *symplectic manifold* is a smooth real manifold, of dimension $2n$, with a closed nondegenerate differential 2-form ω. The volume of X is given by $(1/n!)\int_X \omega^{\wedge n}$. The classical example of a symplectic manifold is $\mathbb{R}^{2n}$ with the 2-form $\omega_0 := dx_1 \wedge dy_1 + \dots + dx_n \wedge dy_n$.

Another example is given by a complex algebraic variety X (of dimension n), with an ample line bundle L. This variety may be treated as a real ($2n$-dimensional) manifold with the closed nondegenerate differential 2-form given by the first Chern class of L, $\omega_L = c_1(L)$. Thus, X is a symplectic manifold, with the volume given as $\mathrm{vol}(X) = (1/n!)L^n$.

If (X_1, ω_1) and (X_2, ω_2) are two symplectic manifolds, we define a symplectic embedding of X_1 to X_2 as follows.

DEFINITION 3.1. We say that $f : X_1 \to X_2$ is a *symplectic embedding* if f is a $\mathcal{C}^\infty$-diffeomorphism onto its image and

$$f^*\omega_2 = \omega_1.$$

We will use the notation

$$f : (X_1, \omega_1) \xrightarrow{s} (X_2, \omega_2).$$

Let (X, ω) be a symplectic manifold of dimension $2n$ and let $(B^{2n}(R), \omega_0)$ be a ball of radius R in $\mathbb{R}^{2n}$ with the standard symplectic form $\omega_0 = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$. We may consider the so-called *symplectic packing problem*: find a maximal radius R such that there exists a symplectic embedding of the disjoint union of r balls of radius R into a given symplectic manifold (X, ω),

$$f : \coprod_{i=1}^{r} (B^{2n}(R), \omega_0) \xrightarrow{s} (X, \omega).$$

If the volume of X is finite, then there is an obvious upper bound on R:

$$r \cdot \mathrm{vol}(B^{2n}(R)) \leq \mathrm{vol}(X).$$

However, even if the volume of X is infinite, there may be obstructions for packing balls into X. For example the famous Gromov Nonsqueezing Theorem (see [10]) says that if there exists a symplectic embedding of a ball $B^{2n}(R)$ into $(B^2(\epsilon) \times \mathbb{R}^{2n-2}, \omega_0)$, then $R \leq \epsilon$.

From now on assume that the volume of a symplectic manifold X is finite. To measure how much of the volume of (X, ω) we may pack with the symplectic images of balls we define so-called packing constants (or packing numbers) (cf. [5], [14], [10]).

DEFINITION 3.2. Let (X, ω) be a symplectic manifold and let r be a natural number. The *symplectic packing constant* is defined as

$$v_r := \sup\left\{ \frac{r\,\mathrm{vol}(B^{2n}(R))}{\mathrm{vol}(X)} \right\},$$

where the supremum is taken over all R such that there exists a symplectic packing $f : \coprod_{i=1}^{r} (B^{2n}(R), \omega_0) \xrightarrow{s} (X, \omega)$.

If $v_r = 1$ we say that *a full packing exists.*

We may, following Lazarsfeld [12], define similar constants for embeddings being both symplectic and holomorphic:

Definition 3.3. Let (X,ω) be a symplectic and holomorphic manifold and let r be a natural number. The *symplectic and holomorphic packing constant* is defined as

$$v_r^h := \sup\left\{\frac{r\,\mathrm{vol}(B^{2n}(R))}{\mathrm{vol}(X)}\right\},$$

where the supremum is taken over all R such that there exists a symplectic and holomorphic packing $f : \coprod_{i=1}^{r}(B^{2n}(R),\omega_0) \xrightarrow{s,\,\mathrm{hol}} (X,\omega)$.

There are many interesting results about the constants v_r; see e.g. [5], [6], [14]. In his famous paper [6], Biran proved the following theorem (here restricted to algebraic surfaces with the symplectic form ω_L):

Theorem 3.4. *Let (X,L) be a projective algebraic surface, treated as a four-dimensional symplectic manifold with the symplectic form ω_L. Then there exists a number N_0 such that for any $r \ge N_0$ there exists a full packing, i.e. $v_r = 1$. Moreover, this N_0 can be taken equal to $k_0^2L^2$ where k_0 is such that the linear system $|k_0L|$ contains a curve C of genus at least one.*

It seems that there exists a close connection between Seshadri constants and packing numbers. This connection was first observed in [14] and then in [5, 6, 12] and other papers.

Consider the following example. Let $X = \mathbb{P}^2$ with $L = \mathcal{O}_{\mathbb{P}^2}(1)$. For $r = 1,\dots,9$ we have $\varepsilon(L,r) = 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{2}{5}, \frac{2}{5}, \frac{3}{8}, \frac{6}{17}, \frac{1}{3}$ respectively. In the same range of r, we have (cf. [5]) $v_r = 1, \frac{1}{2}, \frac{3}{4}, 1, \frac{20}{25}, \frac{24}{25}, \frac{63}{64}, \frac{288}{289}, 1$, so $\varepsilon(L,r) = \sqrt{L^2v_r/r}$ here. For $r \ge 10$ we know by the results of Biran [6, 5] that $v_r = 1$, whereas $\varepsilon(L,r)$ is still unknown (unless r is the square of a natural number, in which case $\varepsilon(L,r) = 1/\sqrt{r}$, cf. e.g. [11]). As mentioned above, Nagata's conjecture says that $\varepsilon(L,r) = 1/\sqrt{r}$ for all $r > 9$ (cf. e.g. [16], [11], [19]), so conjecturally $\varepsilon(L,r) = \sqrt{v_rL^2/r}$ for $\mathbb{P}^2$ with $L = \mathcal{O}_{\mathbb{P}^2}(1)$ and for any r.

In [7, Theorem G] Biran and Cieliebak noted the following upper bound on Seshadri constants:

Theorem 3.5. *For a projective manifold X with an ample line bundle L,*

$$\sqrt[n]{v_rL^n/r} \ge \varepsilon(L,r).$$

On the other hand, holomorphic and symplectic packing constants give a lower bound. Lazarsfeld [12] proved the following theorem:

Theorem 3.6. *Let X be a projective manifold with an ample line bundle L and the symplectic form ω_L. Let v_r^h be the symplectic and holomorphic packing constant, and let $\varepsilon(L,r)$ be the Seshadri constant of L in r general points* *of X. Then*

$$\varepsilon(L,r) \geq \sqrt[n]{v_r^h L^n / r}.$$

REMARK 3.7. Lazarsfeld's proof of this result is based on the construction of a symplectic blowing up (cf. [14]). The theorem in [12] is actually stated for $r = 1$, but it can be generalized to $r \geq 1$.

It is in general not true that for any projective variety M with an ample line bundle L and symplectic form given by $c_1(L)$ the following equality holds:

$$\varepsilon(L,r) = \sqrt[n]{v_r L^n / r},$$

but anyway, it would be useful to understand when and why it does (or does not) hold.

REMARK 3.8. Note that the above formula also holds for $\mathbb{P}^1 \times \mathbb{P}^1$ with the line bundle of type $(1,1)$, for any $r \leq 8$. Then $\varepsilon(L,r) = 1, 1, \frac{2}{3}, \frac{2}{3}, \frac{3}{5}, \frac{4}{7}, \frac{8}{15}, \frac{1}{2}$ and $v_r = \frac{1}{2}, 1, \frac{2}{3}, \frac{8}{9}, \frac{9}{10}, \frac{48}{49}, \frac{224}{225}, 1$ for $r = 1, \ldots, 8$ respectively (see for example [5], [19]).

Other examples of manifolds (X, L), for which the equality $\varepsilon(L,r) = \sqrt[n]{v_r L^n / r}$ holds may be given by principally polarized abelian surfaces in case $r = 2k^2$. We know from [6] that for these surfaces $v_r = 1$ if $r > 1$. On the other hand, there is the following result of Roé and Ross [18] (here we give a slightly restricted version):

THEOREM 3.9. *Let X be a projective variety of dimension n with an ample line bundle L. Let s, N be integers. Then*

$$\varepsilon(X, L, sN) \geq \varepsilon(X, L, s)\varepsilon(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1), N).$$

If X is an abelian surface with L of type $(1,1)$, then $\varepsilon(X, L, 2) = 1$, moreover $\varepsilon(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1), k^2) = 1/k$. The above theorem implies that then

$$\varepsilon(X, L, 2k^2) \geq \varepsilon(X, L, 2)\varepsilon(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1), k^2) = 1/k.$$

From this it follows that $\varepsilon(X, L, 2k^2) = 1/k$ and the conjectural equality holds here.

From [2], [3], [4] we know the Seshadri constant (at one point) for some abelian or $K3$ surfaces, but v_1 remains to be computed for them.

4. A proof of Theorem 3.6. In this section we present the proof of Theorem 3.6, using some facts from Geometric Measure Theory (cf. [15], [9], [1]).

DEFINITION 4.1. Let S be a surface in $\mathbb{R}^{2n}$. We say that S is *minimal* if the mean curvature of S is zero.

REMARK 4.2. Any analytic curve in $\mathbb{C}^n$ is a minimal surface.

Let now S be an analytic curve in $\mathbb{C}^n$. Let S pass through a point $P \in \mathbb{C}^n$. As S is analytic, the multiplicity of S in P is defined. Assume that $\operatorname{mult}_P S = m$. Take then a ball $B^{2n}(R)$ with center P. By the volume of $S \cap B^{2n}(R)$ we mean the area of S (in the Euclidean metric) in $B^{2n}(R)$. Wirtinger's Theorem says that in this situation, the volume of a surface equals the integral of the symplectic form on S:

Remark 4.3 (Wirtinger's Theorem, see [9]). 1. In the situation as above, $\operatorname{vol}(S \cap B^{2n}(R)) = \int_{S\cap B^{2n}(R)} \omega_0$.

2. If C is an analytic curve in a polarized variety (X, L), then we have $\operatorname{vol}(C) = LC$.

The following fact will be crucial for us.

Theorem 4.4 (Monotonicity Lemma, see [15, Theorem 9.3]). *In the situation described above,*

$$\operatorname{vol}(S \cap B^{2n}(R)) \geq m\pi R^2.$$

Let now (X, L) be a smooth projective variety, with an ample line bundle L and a symplectic form ω_L. Take R such that there exists a symplectic and holomorphic embedding f of r disjoint balls of radius R into X.

Let $f(Q_1), \ldots, f(Q_r)$ be the images of the centers of these balls. Take an algebraic curve C on X, passing through $f(Q_1), \ldots, f(Q_r)$ with multiplicities $m_1, \ldots, m_r$ respectively. Let $S_i := f^{-1}(C \cap f(B(Q_i, R)))$, where $B(Q_i, R)$ denotes the ball of radius R with center Q_i. As f is symplectic and holomorphic, S_i is an analytic curve in $B(Q_i, R)$. Moreover, $\operatorname{mult}_{Q_i} S_i = m_i$. From the Monotonicity Lemma it follows that $\operatorname{vol}(S_i) \geq m_i\pi R^2$.

Thus,

$$LC = \operatorname{vol}(C) \geq \sum_{i=1}^{r} \operatorname{vol}(C \cap f(B(Q_i, R))) \overset{f\text{ sympl.}}{=} \sum_{i=1}^{r} \operatorname{vol}(S_i) \overset{\text{Monot. Lemma}}{\geq} \sum_{i=1}^{r} m_i\pi R^2.$$

From this,

$$\frac{LC}{\sum_{i=1}^{r} m_i} \geq \pi R^2$$

for any R such that a symplectic and holomorphic embedding exists. Thus

$$\varepsilon(L, r) \geq \pi R^2$$

and from the definition of v_r^h, and the fact that the volume of the ball of radius R in $(\mathbb{R}^{2n}, \omega_0)$ is $(\pi R^2)^n/n!$, we get the required inequality

$$\varepsilon(L, r) \geq \sqrt[n]{L^n v_r^h / r}.$$

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