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Three periodic solutions for an ordinary differential inclusion with two parameters

by Antonio Iannizzotto (Catania)

Abstract. Applying a nonsmooth version of a three critical points theorem of Ricceri, we prove the existence of three periodic solutions for an ordinary differential inclusion depending on two parameters.

1. Introduction. In the present paper we will study a second order ordinary differential inclusion subject to periodic boundary conditions, of the following type:

$$\begin{cases} -u'' + u \in H(x,u) & \text{in } [0,1], \\ u(0) = u(1), \\ u'(0) = u'(1). \end{cases} \tag{1.1}$$

Here H is a multifunction defined in $[0,1] \times \mathbb{R}$ whose values are compact intervals in $\mathbb{R}$, measurable with respect to the first variable and upper semicontinuous (u.s.c.) with respect to the second (a set of assumptions that reduces to the usual Carathéodory condition if H is single-valued).

Ordinary differential inclusions (o.d.i.'s) represent a natural generalization of ordinary differential equations (o.d.e.'s). They can be studied through a variety of methods, most of which lead to *existence* results: for instance, we recall the interesting works of Frigon & Granas [4], Erbe & Krawcewicz [3] and Kourogenis [7]. In [4], the existence of a periodic solution for the general o.d.i. $u'' \in K(x,u,u')$ is proved using the method of sub- and super-solutions. In [3], a solution for a system of o.d.i.'s with nonlinear boundary condition is obtained via a priori bounds. In [7], the existence of a solution for a system of o.d.i.'s with general boundary conditions is proved through fixed point theory (a remarkable feature of that paper is that the author deals with both convex- and nonconvex-valued multifunctions).

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In order to achieve *multiplicity* results, the best way is to provide o.d.i.'s with a convenient variational framework, following the ideas of Chang [1] relating to partial differential inclusions. Problem (1.1) cannot be studied in a variational framework using the classical critical point theory for continuously Gâteaux differentiable functionals. Nevertheless, we may define a locally Lipschitz energy functional φ for (1.1) and apply to that functional the theory of generalized differentiation introduced by Clarke [2]. Critical points of φ turn out to be solutions of (1.1) in a very natural sense.

Multiplicity results can be obtained by using different methods from nonsmooth critical point theory: we recall the work of Kandilakis, Kourogenis & Papageorgiou [6], where the existence of two periodic solutions is established via local linking.

The case of o.d.i.'s deriving from o.d.e.'s with discontinuous nonlinearities deserves special interest: in such a case, the multifunction H is usually introduced to fill in the gaps at the discontinuity points of some single-valued nonlinearity. Problems of this type are studied, for instance, in the paper of Papageorgiou & Papalini [9] via variational methods and the theory of monotone operators. We are interested in a more general problem, involving multifunctions which do not necessarily extend discontinuous single-valued mappings.

In the present paper, we will study problem (1.1) with a general set-valued nonlinearity of the type

$$H(x,u)=\lambda F(u)+\mu G(x,u),$$

where F and G are multifunctions and $\lambda,\mu>0$ are parameters. We will apply a nonsmooth version of a three critical points theorem of Ricceri [12], based on a minimax inequality, in order to prove the existence, for λ and μ lying in convenient intervals, of at least three periodic solutions for the o.d.i., plus a uniform estimate on the norms of such solutions (see Theorem 4.1 below).

The paper has the following structure: in Section 2 we recall some basic features of nonsmooth analysis and the abstract result we are going to apply; in Section 3 we introduce a variational method for problem (1.1); and in Section 4 we state and prove our multiplicity result and give some examples.

2. Some nonsmooth analysis. In the present section we will collect, for the convenience of the reader, some basic notions and results of nonsmooth analysis, namely the calculus for locally Lispchitz functionals developed by Clarke [2]. Our main reference is the monograph of Motreanu & Panagiotopoulos [8].

Let $(X, \|\cdot\|)$ be a Banach space, $(X^*, \|\cdot\|_*)$ be its topological dual, and $\varphi : X \to \mathbb{R}$ be a functional. We recall that φ is *locally Lipschitz* (l.L.) if, for all $u \in X$, there exist a neighborhood U of u and a real $L > 0$ such that

$$|\varphi(v) - \varphi(w)| \leq L\|v - w\| \quad \text{for all } v, w \in U.$$

If φ is l.L. and $u \in X$, the *generalized directional derivative* of φ at u along the direction $v \in X$ is

$$\varphi^\circ(u; v) = \limsup_{w \to u,\, \tau \to 0^+} \frac{\varphi(w + \tau v) - \varphi(w)}{\tau}.$$

The *generalized gradient* of φ at u is the set

$$\partial\varphi(u) = \{u^* \in X^* : \langle u^*, v\rangle \leq \varphi^\circ(u; v) \text{ for all } v \in X\}.$$

So, $\partial\varphi : X \to 2^{X^*}$ is a multifunction. We say that φ has a *compact gradient* if $\partial\varphi$ maps bounded subsets of X into relatively compact subsets of X^*.

The following lemmata yield some useful properties of the above defined tools:

Lemma 2.1 ([8, Proposition 1.1]). *Let $\varphi \in C^1(X)$ be a functional. Then φ is l.L. and*

(i) $\varphi^\circ(u; v) = \langle\varphi'(u), v\rangle$ *for all $u, v \in X$;*
(ii) $\partial\varphi(u) = \{\varphi'(u)\}$ *for all $u \in X$.*

Lemma 2.2 ([8, Proposition 1.3]). *Let $\varphi : X \to \mathbb{R}$ be a l.L. functional. Then*

(i) $\varphi^\circ(u; \cdot)$ *is subadditive and positively homogeneous for all $u \in X$;*
(ii) $\varphi^\circ(u; v) \leq L\|v\|$ *for all $u, v \in X$, where $L > 0$ is a Lipschitz constant for φ around u.*

Lemma 2.3 ([8, Proposition 1.6]). *Let $\varphi, \psi : X \to \mathbb{R}$ be l.L. functionals. Then*

(i) $\partial(\lambda\varphi)(u) = \lambda\partial\varphi(u)$ *for all $u \in X$, $\lambda \in \mathbb{R}$;*
(ii) $\partial(\varphi + \psi)(u) \subseteq \partial\varphi(u) + \partial\psi(u)$ *for all $u \in X$.*

Lemma 2.4 ([5, Lemma 6]). *Let $\varphi : X \to \mathbb{R}$ be a l.L. functional with a compact gradient. Then φ is sequentially weakly continuous.*

We say that $u \in X$ is a *critical point* of a l.L. functional φ if $0 \in \partial\varphi(u)$ (of course, this relation is the link between nonsmooth analysis and differential inclusions).

With appropriate technical devices, most results from the classical critical point theory can be adapted to the nonsmooth framework. The following result, which we are going to apply, is a nonsmooth extension of a three critical points theorem of Ricceri [12], based on a minimax inequality. Before stating it, we recall that an operator $A : X \to X^*$ is of *type* $(S)_+$ if, for any sequence (u_n) in X, whenever $u_n \rightharpoonup u$ and

$$\limsup_n \langle A(u_n), u_n - u\rangle \le 0,$$

then $u_n \to u$.

THEOREM 2.5 (A particular case of [5, Theorem 14]). *Let $(X, \|\cdot\|)$ be a reflexive Banach space, $I \subseteq \mathbb{R}$ be an interval, $\mathcal{N} \in C^1(X)$ be a sequentially weakly l.s.c. functional whose derivative is of type $(S)_+$, $\mathcal{F} : X \to \mathbb{R}$ be a l.L. functional with a compact gradient, and $\rho \in \mathbb{R}$. Assume that*

(i) $\lim_{\|u\|\to\infty} [\mathcal{N}(u) - \lambda\mathcal{F}(u)] = +\infty$ *for all* $\lambda \in I$;

(ii) $\sup_{\lambda\in I} \inf_{u\in X} [\mathcal{N}(u) + \lambda(\rho - \mathcal{F}(u))] < \inf_{u\in X} \sup_{\lambda\in I} [\mathcal{N}(u) + \lambda(\rho - \mathcal{F}(u))]$.

Then there exist $\alpha, \beta \in I$ $(\alpha < \beta)$ and $r > 0$ with the following property: for any $\lambda \in [\alpha, \beta]$ and any l.L. functional $\mathcal{G} : X \to \mathbb{R}$ with a compact gradient, there exists $\delta > 0$ such that, for all $\mu \in [0, \delta]$, the functional

$$\varphi_{\lambda,\mu} = \mathcal{N} - \lambda\mathcal{F} - \mu\mathcal{G}$$

admits at least three critical points in X with norms less than r.

The main hypothesis of Theorem 2.5 above is the minimax inequality (ii). An easy way to have it satisfied is illustrated by the following result due again to Ricceri [11]:

LEMMA 2.6 ([11, Proposition 3.1]). *Let X be a nonempty set, $\mathcal{N}, \mathcal{F} : X \to \mathbb{R}$ be functions, $\check{u}, \hat{u} \in X$ and $\tau > 0$ be such that*

(i) $\mathcal{N}(\check{u}) = \mathcal{F}(\check{u}) = 0$;

(ii) $\mathcal{N}(\hat{u}) > \tau$;

(iii) $\sup_{\mathcal{N}(u)<\tau} \mathcal{F}(u) < \dfrac{\tau\mathcal{F}(\hat{u})}{\mathcal{N}(\hat{u})}$.

Then there exists $\rho \in \mathbb{R}$ such that (ii) *of Theorem* 2.5 *holds.*

3. Variational methods for ordinary differential inclusions. In the present section we will establish an appropriate variational framework for o.d.i.'s of the type (1.1). We follow Ribarska, Tsachev & Krastanov [10]. Our assumptions on the multifunction H are the following:

H. $H : [0, 1] \times \mathbb{R} \to 2^{\mathbb{R}}$ satisfies

(i) $H(x, \cdot) : \mathbb{R} \to 2^{\mathbb{R}}$ is u.s.c. with compact convex values for a.a. $x \in [0, 1]$;

(ii) $\min H, \max H : [0, 1] \times \mathbb{R} \to \mathbb{R}$ are $\mathcal{L} \otimes \mathcal{B}$-measurable;

(iii) $|\xi| \leq a(1 + |s|^{p-1})$ for a.a. $x \in [0, 1]$, all $s \in \mathbb{R}$ and $\xi \in H(x, s)$ ($a > 0$, $p > 1$).

We introduce the Hilbert space

$$X = \{u \in W^{1,2}([0, 1]) : u(0) = u(1)\},$$

endowed with the usual norm

$$\|u\| = \sqrt{\int_0^1 (|u'|^2 + |u|^2)\, dx} \quad \text{for all } u \in X.$$

A function $u \in X$ is a (*weak*) *solution* of problem (1.1) if there exists $u^* \in L^q([0, 1])$ (for some $q > 1$) such that

$$\int_0^1 (u'v' + uv - \lambda u^* v)\, dx = 0 \quad \text{for all } v \in X \tag{3.1}$$

and

$$u^*(x) \in H(x, u(x)) \quad \text{for a.a. } x \in [0, 1]. \tag{3.2}$$

We recall that X is compactly embedded into the space $C^0([0, 1])$ endowed with the maximum norm $\|\cdot\|_\infty$, and

$$\|u\|_\infty \leq \sqrt{2}\, \|u\| \quad \text{for all } u \in X. \tag{3.3}$$

Obviously, X is compactly embedded into $L^\nu([0, 1])$ endowed with the usual norm $\|\cdot\|_\nu$, for all $\nu \geq 1$.

We introduce for a.a. $x \in [0, 1]$ and all $s \in \mathbb{R}$ the Aumann-type set-valued integral

$$\int_0^s H(x, t)\, dt = \left\{ \int_0^s h(x, t)\, dt : h \text{ a measurable selection of } H \right\}$$

and set

$$\mathcal{H}(u) = \int_0^1 \min \int_0^u H(x, s)\, ds\, dx \quad \text{for all } u \in L^p([0, 1]).$$

Lemma 3.1. *The functional $\mathcal{H} : L^p([0, 1]) \to \mathbb{R}$ is well defined and Lipschitz on any bounded subset of $L^p([0, 1])$. Moreover, for all $u \in L^p([0, 1])$ and all $u^* \in \partial\mathcal{H}(u)$,*

$$u^*(x) \in H(x, u(x)) \quad \textit{for a.a. } x \in [0, 1].$$

Proof. We give an alternative representation for $\mathcal{H}$. For all $(x, s) \in [0, 1] \times \mathbb{R}$, set

$$h(x, s) = \begin{cases} \max H(x, s) & \text{if } s \leq 0, \\ \min H(x, s) & \text{if } s > 0. \end{cases}$$

Then $h : [0,1]\times\mathbb{R} \to \mathbb{R}$ is a measurable selection of H (see $\mathbf{H}$(ii)) and clearly

$$\min\int_0^s H(x,t)\,dt = \int_0^s h(x,t)\,dt \qquad \text{for a.a. } x \in [0,1] \text{ and all } s \in \mathbb{R}.$$

So, we may write

$$\mathcal{H}(u) = \int_0^1\int_0^u h(x,s)\,ds\,dx \qquad \text{for all } u \in L^p([0,1]). \tag{3.4}$$

Now we prove that $\mathcal{H}$ is well defined. For all $u \in L^p([0,1])$ we have

$$\begin{aligned}\int_0^1\left|\int_0^u h(x,s)\,ds\right|dx &\le \int_0^1 a\left(|u| + \frac{|u|^p}{p}\right)dx \qquad (\text{see } \mathbf{H}(\text{iii}))\\ &\le c_1\|u\|_p^p \qquad (c_1 > 0).\end{aligned}$$

We prove that $\mathcal{H}$ is Lipschitz on bounded sets: set $M > 0$ and choose $u, v \in L^p([0,1])$ such that $\|u\|_p, \|v\|_p \le M$. Then

$$\begin{aligned}|\mathcal{H}(u) - \mathcal{H}(v)| &\le \int_0^1\left|\int_v^u h(x,s)\,ds\right|dx \qquad (\text{see } (3.4))\\ &\le \int_0^1 a(1 + |u|^{p-1} + |v|^{p-1})|u-v|\,dx \qquad (\text{see } \mathbf{H}(\text{iii}))\\ &\le a(1 + 2M^{p-1})\|u-v\|_p.\end{aligned}$$

In particular, $\mathcal{H}$ is a l.L. functional. Choose $u \in L^p([0,1])$ and $u^* \in \partial\mathcal{H}(u)$. Then u^* is a bounded linear functional on $L^p([0,1])$, hence it can be identified with an element of $L^q([0,1])$ $(1/p + 1/q = 1)$. By the result of Chang [1, Theorem 2.2], we get, for a.a. $x \in [0,1]$,

(3.5)
$$u^*(x) \in \left[\liminf_{s\to 0,\,\tau\to 0^+}\frac{1}{\tau}\int_{u(x)+s-\tau}^{u(x)+s} h(x,t)\,dt,\ \limsup_{s\to 0,\,\tau\to 0^+}\frac{1}{\tau}\int_{u(x)+s}^{u(x)+s+\tau} h(x,t)\,dt\right].$$

An application of the mean value theorem yields, for all $s \in \mathbb{R}$ and $\tau > 0$,

$$\begin{aligned}\frac{1}{\tau}\int_{u(x)+s-\tau}^{u(x)+s} h(x,t)\,dt &\ge \inf_{u(x)+s-\tau\le t\le u(x)+s} h(x,t)\\ &\ge \inf_{u(x)+s-\tau\le t\le u(x)+s} \min H(x,t).\end{aligned}$$

Since $\min H(x,\cdot)$ is u.s.c. (by $\mathbf{H}$(i) and classical results of set-valued analysis), we have

$$\liminf_{s\to 0,\,\tau\to 0^+} \frac{1}{\tau} \int\limits_{u(x)+s-\tau}^{u(x)+s} h(x,t)\,dt \ge \liminf_{s\to 0,\,\tau\to 0^+} \Big[\inf_{u(x)+s-\tau\le t\le u(x)+s} \min H(x,t) \Big]$$

$$\ge \min H(x,u(x)) \quad \text{for a.a. } x\in[0,1].$$

An analogous argument leads to

$$\limsup_{s\to 0,\,\tau\to 0^+} \frac{1}{\tau} \int\limits_{u(x)+s}^{u(x)+s+\tau} h(x,t)\,dt \le \max H(x,u(x)) \quad \text{for a.a. } x\in[0,1].$$

Then (3.5) implies

$$u^*(x) \in [\min H(x,u(x)), \max H(x,u(x))] \quad \text{for a.a. } x\in[0,1].$$

The convexity of the set $H(x,u(x))$ (see $\mathbf{H}$(i)) finally implies (i). ∎

We define an energy functional for problem (1.1) by setting

$$\varphi(u) = \|u\|^2/2 - \mathcal{H}(u) \quad \text{for all } u\in X.$$

Lemma 3.2. *The functional $\varphi : X\to\mathbb{R}$ is l.L. Moreover, for all critical points $u\in X$ of φ, u is a solution of problem* (1.1).

Proof. The functionals $u\mapsto \|u\|^2/2$ and $\mathcal{H}$ are l.L. (see Lemmata 2.1 and 3.1, respectively), hence so is φ.

Now let $u\in X$ be a critical point of φ. So

$$0\in\partial\varphi(u). \tag{3.6}$$

Set

$$\langle A(u), v\rangle = \int\limits_0^1 (u'v' + uv)\,dx \quad \text{for all } u,v\in X. \tag{3.7}$$

By (ii) of Lemma 2.1, and (i) and (ii) of Lemma 2.3, condition (3.6) implies

$$0\in A(u) - \partial\mathcal{H}(u),$$

that is, there exists $u^*\in\partial\mathcal{H}(u)$ such that

$$A(u) = u^* \quad \text{in } X^*. \tag{3.8}$$

We extend u^* to an element of $L^q([0,1])$. Here, we regard X as a closed subspace of $L^p([0,1])$.

First, we observe that u^*, as a linear functional on X, is continuous also with respect to the topology induced by the norm $\|\cdot\|_p$. Indeed, by Lemma 3.1, $\mathcal{H}$ admits a Lipschitz constant L around u with respect to $\|\cdot\|_p$. Then, by (ii) of Lemma 2.2, we get

$$\langle u^*, v\rangle \le L\|v\|_p \quad \text{for all } v\in X.$$

Moreover, $\mathcal{H}^\circ(u;\cdot)$ is subadditive and positively homogeneous on $L^p([0,1])$ and

$$\langle u^*, v\rangle \le \mathcal{H}^\circ(u;v) \quad \text{for all } v\in X. \tag{3.9}$$

By the Hahn–Banach theorem, u^* extends to a bounded linear functional defined on $L^p([0,1])$ satisfying (3.9) for all $v\in L^p([0,1])$. This implies two facts. First, we may assume $u^*\in L^q([0,1])$ and rephrase (3.8) as

$$\int_0^1 (u'v' + uv - u^*v)\,dx = 0 \quad \text{for all } v\in X.$$

Second, by Lemma 3.1 we have

$$u^*(x)\in H(x,u(x)) \quad \text{for a.a. } x\in[0,1].$$

Thus, (3.1) and (3.2) are fulfilled and u is a solution of (1.1). ∎

Remark. We owe the reader a clarification. Hypothesis **H**(iii) may seem unnecessary at first sight, as growth conditions based on powers are usually not required in dealing with o.d.e.'s (or o.d.i.'s). Actually, we could have proved that the functional φ is l.L. on $C^0([0,1])$ under looser assumptions. But the elements of the dual space of $C^0([0,1])$ cannot, in general, be represented as functions defined almost everywhere on $[0,1]$, so we could not get a relation like (i) of Lemma 3.1 and the subsequent Lemma 3.2. This is why we need to use the space $L^p([0,1])$ (whose dual space is $L^q([0,1])$) and hence condition **H**(iii).

4. Main result and examples. As we pointed out in the Introduction, we are going to deal with a special case of problem (1.1), depending on two positive parameters λ and μ:

$$\begin{cases} -u'' + u \in \lambda F(u) + \mu G(x,u) & \text{in } [0,1],\\ u(0) = u(1), \\ u'(0) = u'(1). \end{cases} \tag{4.1}$$

We will assume that:

$\mathbf{H}_F$. $F:\mathbb{R}\to 2^{\mathbb{R}}$ satisfies

(i) F is u.s.c. with compact convex values;
(ii) $|\xi|\le a(1+|s|^{p-1})$ for all $s\in\mathbb{R}$ and $\xi\in F(s)$ $(a>0,\ p>1)$;
(iii) $\displaystyle\max_{|t|\le 2\sqrt{\tau}} \min\int_0^t F(s)\,ds < \frac{2\tau}{k^2}\min\int_0^k F(s)\,ds \ \left(k>0,\ 0<\tau<\frac{k^2}{4}\right)$;
(iv) $\displaystyle\min\int_0^t F(s)\,ds \le b(1+|t|^q)$ for all $t\in\mathbb{R}$ $(b>0,\ 1<q<2)$;

$\mathbf{H}_G$. $G : [0,1] \times \mathbb{R} \to 2^{\mathbb{R}}$ satisfies

(i) $G(x, \cdot) : \mathbb{R} \to 2^{\mathbb{R}}$ is u.s.c. with compact convex values for a.a. $x \in [0,1]$;
(ii) $\min G, \max G : [0,1] \times \mathbb{R} \to \mathbb{R}$ are $\mathcal{L} \otimes \mathcal{B}$-measurable;
(iii) $|\xi| \le a(1 + |s|^{p-1})$ for a.a. $x \in [0,1]$, all $s \in \mathbb{R}$ and $\xi \in G(x,s)$.

Our multiplicity result for the solutions of problem (4.1) is as follows:

THEOREM 4.1. *Let $\mathbf{H}_F$ be satisfied. Then there exist a nondegenerate interval $[\alpha, \beta] \subset]0, +\infty[$ and $r > 0$ with the following property: for any $\lambda \in [\alpha, \beta]$ and any multifunction G satisfying $\mathbf{H}_G$, there exists $\delta > 0$ such that, for all $\mu \in [0, \delta]$, problem* (4.1) *admits at least three solutions with norms in $W^{1,2}([0,1])$ less than r.*

Proof. We are going to apply Theorem 2.5. With this aim in mind, we define X as in Section 3 and we set $I = [0, +\infty[$ and

$$\mathcal{N}(u) = \frac{\|u\|^2}{2}, \quad \mathcal{F}(u) = \int_0^1 \min \int_0^u F(s)\, ds\, dx \qquad \text{for all } u \in X.$$

We know that $\mathcal{N} \in C^1(X)$ is a weakly l.s.c. functional whose derivative (the operator A defined in (3.7)) is of type $(S)_+$. By Lemma 3.1, $\mathcal{F} : X \to \mathbb{R}$ is Lipschitz on bounded subsets of X.

We now prove that the gradient $\partial\mathcal{F} : X \to 2^{X^*}$ is compact. Let us fix a bounded sequence (u_n) in X and $u_n^* \in \partial\mathcal{F}(u_n)$ for all $n \in \mathbb{N}$. Let $L > 0$ be a Lipschitz constant for $\mathcal{F}$, restricted to a bounded set where the sequence (u_n) lies; then, by (ii) of Lemma 2.2 and the definition of $\partial\varphi(u_n)$, we have

$$\|u_n^*\|_* \le L \qquad \text{for all } n \in \mathbb{N}.$$

Up to a subsequence, (u_n^*) weakly converges to some u^* in X^*. We shall prove that the convergence is strong, arguing by contradiction: let us assume that there is $\varepsilon > 0$ such that

$$\|u_n^* - u^*\|_* > \varepsilon \qquad \text{for all } n \in \mathbb{N}.$$

Hence, for all $n \in \mathbb{N}$ we can find $v_n \in X$ with $\|v_n\| < 1$ such that

$$\langle u_n^* - u^*, v_n \rangle > \varepsilon. \tag{4.2}$$

Passing if necessary to a subsequence, we can assume that $v_n \rightharpoonup v$ in X, while $v_n \to v$ in both $L^1([0,1])$ and $L^p([0,1])$. From $\mathbf{H}_F$(ii) we easily get

$$\begin{aligned}\langle u_n^* - u^*, v_n \rangle &= \langle u_n^*, v_n - v \rangle + \langle u_n^* - u^*, v \rangle + \langle u^*, v - v_n \rangle \\ &\le c_2(\|v_n - v\|_1 + \|v_n - v\|_p) + \langle u_n^* - u^*, v \rangle + \langle u^*, v - v_n \rangle\end{aligned}$$

($c_2 > 0$), and the latter tends to 0 as $n \to \infty$, contrary to (4.2).

Now we prove that, for all $\lambda \geq 0$, the functional $\mathcal{N} - \lambda\mathcal{F}$ is coercive (that is, (i) of Theorem 2.5 holds). For all $u \in X$ we have

$$\begin{aligned}\mathcal{N}(u) - \lambda\mathcal{F}(u) &\geq \|u\|^2/2 - \lambda\int_0^1 b(1+|u|^q)\,dx \quad (\text{see } \mathbf{H}_F(\text{iv}))\\ &\geq \|u\|^2/2 - \lambda c_3(1+\|u\|^q) \quad (c_3 > 0),\end{aligned}$$

and the latter tends to $+\infty$ as $\|u\| \to \infty$ (recall that $q < 2$).

We prove that (ii) of Theorem 2.5 (the minimax inequality) holds; we will not argue directly, but use Lemma 2.6. Set

$$\check{u}(x) = 0, \quad \hat{u}(x) = k \quad \text{for all } x \in [0,1] \quad (k \text{ as in } \mathbf{H}_F(\text{iii})).$$

Then, clearly, (i) of Lemma 2.6 holds. Moreover, we have

$$\mathcal{N}(\hat{u}) = \frac{k^2}{2},$$

so (ii) of Lemma 2.6 holds as well. Finally, we observe that

$$\mathcal{F}(\hat{u}) = \min\int_0^k F(s)\,ds$$

and that for all $u \in X$ with $\mathcal{N}(u) < \tau$ we have

$$\|u\|_\infty < 2\sqrt{\tau} \quad (\text{see (3.3)}),$$

so

$$\begin{aligned}\mathcal{F}(u) &\leq \max_{|t|\leq 2\sqrt{\tau}} \min\int_0^t F(s)\,ds\\ &< \frac{2\tau}{k^2}\min\int_0^k F(s)\,ds \quad (\text{see } \mathbf{H}_F(\text{iii}))\\ &= \frac{\tau\mathcal{F}(\hat{u})}{\mathcal{N}(\hat{u})},\end{aligned}$$

so (iii) of Lemma 2.6 is satisfied. Then we get (ii) of Theorem 2.5 for some $\rho \in \mathbb{R}$.

All assumptions of our abstract result are fulfilled. Let $[\alpha, \beta]$ $(0 < \alpha < \beta)$ and $r > 0$ be as in Theorem 2.5. Let us choose $\lambda \in [\alpha, \beta]$ and a multifunction $\mathcal{G}$ satisfying $\mathbf{H}_G$. Set

$$\mathcal{G}(u) = \int_0^1 \min\int_0^u G(s)\,ds \quad \text{for all } u \in X.$$

By Lemma 3.1 and an argument analogous to that used for $\mathcal{F}$, the functional $\mathcal{G} : X \to \mathbb{R}$ turns out to be l.L. and its gradient $\partial\mathcal{G}$ is compact.

Thus there is $\delta > 0$ such that, for all $\mu \in [0, \delta]$, the functional

$$\varphi_{\lambda,\mu} = \mathcal{N} - \lambda\mathcal{F} - \mu\mathcal{G}$$

admits at least three critical points $u_0, u_1, u_2 \in X$ with

$$\|u_i\| < r \qquad (i = 0, 1, 2).$$

For all $\lambda > 0$ and $\mu \geq 0$, the multifunction H defined by putting

$$H(x, s) = \lambda F(s) + \mu G(x, s) \quad \text{ for all } (x, s) \in [0, 1] \times \mathbb{R}$$

satisfies **H**. So, by Lemma 3.2, u_i $(i = 0, 1, 2)$ is a solution of problem (4.1). ∎

We present two simple examples of multifunctions F satisfying hypotheses $\mathbf{H}_F$:

EXAMPLE 4.2. Set, for all $s \in \mathbb{R}$,

$$F(s) = \begin{cases} \{-s^2\} & \text{if } s < 1, \\ [-1, 1] & \text{if } s = 1, \\ \{\sqrt{s}\} & \text{if } s > 1. \end{cases}$$

Hypotheses $\mathbf{H}_F$ are satisfied with $a = \tau = b = 1$, $p = 3$, $k = 2$ and $q = 3/2$. So, for each multifunction G satisfying $\mathbf{H}_G$, problem (4.1) admits at least three solutions (uniformly bounded) for λ and μ lying in appropriate intervals.

EXAMPLE 4.3. Set, for all $s \in \mathbb{R}$,

$$F(s) = \begin{cases} \{0\} & \text{if } s < 0, \\ [\ln(s + 1/2), 0] & \text{if } 0 \leq s < 1/2, \\ \{\ln(s + 1/2)\} & \text{if } s \geq 1/2. \end{cases}$$

Hypotheses $\mathbf{H}_F$ are satisfied with $a, b > 0$ large enough, $p = 2$, $\tau = 1/16$, $k = 1$ and $q = 3/2$. So, for each multifunction G satisfying $\mathbf{H}_G$, problem (4.1) admits at least three solutions (uniformly bounded) for λ and μ lying in appropriate intervals.

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Antonio Iannizzotto
Dipartimento di Matematica e Informatica
Università degli Studi di Catania
Viale A. Doria 6
95125 Catania, Italy
E-mail: iannizzotto@dmi.unict.it

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