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3-K-contact Wolf spaces

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Abstract. The aim of this paper is to give an easy explicit description of 3-K-contact structures on SO(3)-principal fibre bundles over Wolf quaternionic Kähler manifolds.

1. Introduction. In 1965 Wolf constructed examples of symmetric quaternionic Kähler manifolds $W(G), W(G)^*$ associated with every simple Lie group G (except SU(2)). This construction is based on the properties of the highest roots in a compact, simple Lie algebra. Every space $W(G)$ is a compact symmetric space and $W(G)^*$ is its non-compact dual space. It has been known since 1975 [K] that any quaternionic Kähler manifold (M, g_0) of positive scalar curvature admits a natural SO(3)-principal fibre bundle $p : P \to M$ such that (P, g) is a 3-Sasakian manifold and p is a Riemannian submersion. However, for a long time the analogous construction for quaternionic Kähler manifolds of negative scalar curvature was not given. Recently S. Tanno [T] proved that also in the case of negative scalar curvature the natural SO(3)-principal bundle admits a structure similar to a 3-Sasakian structure, called by him the nS-structure (compare also [J-1]).

In this paper we give an elementary description of the positive and negative 3-K-contact structures related to Wolf quaternionic Kähler spaces. We show that 3-K-contact structures are related to the real form $\mathfrak{so}(3)_\alpha$ of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})_\alpha \subset \mathfrak{g}^{\mathbb{C}}$ generated by the highest root α of $\mathfrak{g}$. We also give an alternative proof of the result of Bielawski [Bi] who, using Kronheimer's ideas, first explicitly described the metric of 3-Sasakian Wolf spaces (see [B-G]). We also remove a (slight) incorrectness of Bielawski's result (Bielawski gave the metric which is only homothetic to a 3-Sasakian metric) and give the description of negative 3-K-contact Wolf structures not considered by Bielawski. Our method is more elementary and in the spirit of Wolf's paper.

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2. Preliminaries. For the general facts concerning 3-K-contact and 3-Sasakian structures and quaternionic Kähler geometry we refer to [S], [B-G], [K], [Ku], [Sw], [T], [J-1], [J-2], [B]. We shall recall several facts proved by Wolf in [W]. Let $\mathfrak{g}$ be a compact, simple real Lie algebra and $\mathfrak{g}^{\mathbb{C}}$ its complexification. By $\langle\cdot,\cdot\rangle_{\mathrm{K}}$ we denote the Killing form on $\mathfrak{g}^{\mathbb{C}}$ and let σ be a real structure giving a compact real form $\mathfrak{g}$ of $\mathfrak{g}^{\mathbb{C}}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$. Fix a system of roots Δ with positive roots Δ_+. We write $\mathfrak{g}_\beta$ for the root space of $\beta \in \Delta$, i.e. $\mathfrak{g}_\beta = \{E \in \mathfrak{g}^{\mathbb{C}} : [H, E] = \beta(H)E \text{ for all } H \in \mathfrak{h}\}$. Let $\alpha \in \Delta_+$ be a highest root; it is characterized by the condition $[E_\alpha, E_\beta] = 0$ for all $\beta \in \Delta_+$. The following characterization of a highest root was given by Wolf [W]:

Proposition 1. *Let α be a root of a complex simple Lie algebra $\mathfrak{g}^{\mathbb{C}}$ relative to a Cartan subalgebra $\mathfrak{h}$. Then α is the maximal root for some choice of Δ_+ if and only if the eigenvalues of* $\mathrm{ad}(H_\alpha)$ *are $-\frac{1}{2}|\alpha|^2, 0, \frac{1}{2}|\alpha|^2$ off $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$. In that case the centralizer of H_α in $\mathfrak{g}^{\mathbb{C}}$ is a direct sum $\mathfrak{z}_1 \oplus \{H_\alpha\}$ of ideals, where $\mathfrak{z}_1$ centralizes $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$.*

The *centralizer* $\mathfrak{z}$ of H_α in $\mathfrak{g}^{\mathbb{C}}$ is

$$\mathfrak{z} = \mathfrak{h} \oplus \sum_{\beta \in \Phi} \mathfrak{g}_\beta$$

where $\Phi = \{\beta \in \Delta : \langle\alpha, \beta\rangle = 0\}$.

A *quaternionic Kähler structure* on a $4n$-dimensional manifold M, $n > 1$, consists of a metric g and a real rank-three subbundle $\mathcal{G}$ of $\mathrm{End}(TM)$ preserved by the Levi-Civita connection and locally generated by almost Hermitian structures I, J, K behaving under composition like the multiplicative pure imaginary quaternions. An equivalent definition of a quaternionic Kähler manifold (M, g) is that the holonomy group of g reduces to the group $\mathrm{Sp}(n)\,\mathrm{Sp}(1)$.

Of course, a hyper-Kähler manifold may be regarded as a special type of a quaternionic Kähler manifold with zero scalar curvature. We shall exclude this case, assuming that a quaternionic Kähler manifold is of non-zero scalar curvature.

If $n = 1$ then a 4-dimensional manifold (M, g) will be called *quaternionic Kähler* if (M, g) is Einstein and self-dual with non-zero scalar curvature.

Let (M, g) be a Riemannian manifold and let ξ be a unit Killing vector field on M. Define a tensor field ϕ by $\phi(X) = -\nabla_X \xi$ and a 1-form $\eta(X) := g(\xi, X)$. Then we call (M, g, ξ, ϕ, η) a *K-contact structure* if the following relation is satisfied:

$$\phi^2 = -\mathrm{id} + \eta \otimes \xi. \tag{K}$$

Assume that ξ is a Killing vector field of unit length on M. We shall find conditions under which the Killing vector field ξ defines a K-contact metric structure. Denote by $H = \ker\eta = \{X : g(\xi, X) = 0\}$ the distribution of horizontal vectors on M. The following lemma is well known.

LEMMA 1. *Under the above assumptions the Killing vector field ξ gives a K-contact structure on M if and only if the tensor $J = \phi_{|H}$ is an almost complex structure on the bundle H, i.e. $J^2 = -\operatorname{id}_{|H}$.*

A K-contact structure (M, g, ξ) is called *Sasakian* if

$$R(X,\xi)Y = g(\xi, Y)X - g(X,Y)\xi \tag{S}$$

where R is the curvature tensor of (M, g).

A Riemannian manifold (M, g) with an almost complex structure $J \in \operatorname{End}(TM)$ is said to be an *almost Hermitian manifold* if $g(JX, JY) = g(X, Y)$ for all $X, Y \in TM$. The 2-form $\Omega(X, Y) = g(JX, Y)$ is called the *Kähler form* of an almost Hermitian manifold (M, g, J). An almost Hermitian manifold is called *almost Kähler* if its Kähler form is closed: $d\Omega = 0$.

If (M, g, ξ) is a regular K-contact structure (i.e. there exists a quotient manifold $M_* = M/\xi$) then (M, g_*, J_*) is an almost Kähler manifold, where g_* means an induced metric and J_* an induced almost complex structure. In that case (M, g, ξ) is Sasakian if and only if (M, g_*, J_*) is Kähler, i.e. if $\nabla^*\Omega_* = 0$ where ∇^* is the Levi-Civita connection of (M, g_*) and Ω_* is the Kähler form of (M, g_*, J_*).

Now let us recall the definition of (positive and negative) 3-K-contact structures (see [J-1]).

DEFINITION. Let (P, g) be a Riemannian manifold that admits three distinct K-contact structures (ϕ_i, ξ_i, η_i) such that

$$\text{(a) } g(\xi_i, \xi_j) = \delta_{ij}, \qquad \text{(b) } [\xi_i, \xi_j] = 2\varepsilon_{ijk}\xi_k, \qquad \text{(c) } \phi_i\xi_j = -\varepsilon_{ijk}\xi_k, \tag{2.1}$$

where $\phi_i = \nabla\xi_i$ and $\eta_i(X) = g(\xi_i, X)$. Denote by H the horizontal distribution $H = \ker\eta_1 \cap \ker\eta_2 \cap \ker\eta_3 = \bigcap \ker\eta_i$ and define the almost complex structures J_i on H by the formulas $J_i = -\phi_{i|H}$. We call (P, ξ_1, ξ_2, ξ_3) a *3-K-contact structure* (or *positive 3-K-contact structure*) if (for $i \neq j$)

$$J_i \circ J_j = \varepsilon_{ijk} J_k, \tag{2.2a}$$

and a *negative 3-K-contact structure* if (for $i \neq j$)

$$J_i \circ J_j = -\varepsilon_{ijk} J_k. \tag{2.2b}$$

A Riemannian manifold (P, g) with a positive (resp. negative) 3-K-contact structure is called a *positive* (resp. *negative*) *3-K-contact manifold.* Note that arbitrary unit Killing vector fields ξ_i satisfying (2.1)(c) and one of conditions (2.2) define K-contact structures on (P, g) (this follows from Lemma 1) so it is not necessary to include this condition in the definition above.

If each structure (M, g, ξ_i) is Sasakian and conditions (2.1) are satisfied then (2.2a) are automatically satisfied and we call such a positive 3-K-contact structure a 3-*Sasakian structure*.

In our paper [J-1] we have shown that if $\dim P \neq 11$ then every positive 3-K-contact structure on P is 3-Sasakian and every negative structure is a Tanno nS-structure.

3. **3-K-contact Wolf spaces.** Let G be a compact centreless Lie group. Choose a maximal torus T of G and denote by $\mathfrak{g}$ and $\mathfrak{t}$ the Lie algebras of G and T respectively. Let $\mathfrak{g}^{\mathbb{C}}$ and $\mathfrak{h}$ be the respective complexifications of $\mathfrak{g}$ and $\mathfrak{t}$. It is clear that $\mathfrak{h}$ is a Cartan subalgebra of a simple complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$. Let Δ be the set of roots of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{h}$ and fix a set of positive roots Δ_+. Let $\alpha \in \Delta_+$ be a highest root. Then $\alpha(X) = \langle H_\alpha, X\rangle$ for some $H_\alpha \in i\mathfrak{t} \subset \mathfrak{h}$. After rescaling the Killing form we can assume that $\langle\cdot,\cdot\rangle = (4/|\alpha|^2_{\mathrm{K}})\langle\cdot,\cdot\rangle_{\mathrm{K}}$ is an ad-invariant metric on $\mathfrak{g}^{\mathbb{C}}$ such that $|\alpha|^2 = \langle H_\alpha, H_\alpha\rangle = 4$. Note that $|\alpha|^2_{\mathrm{K}} = \langle H_\alpha, H_\alpha\rangle > 0$ and $\langle\cdot,\cdot\rangle = c\langle\cdot,\cdot\rangle_{\mathrm{K}}$ where $c > 0$. We can choose vectors $E_\alpha \in \mathfrak{g}_\alpha, E_{-\alpha} \in \mathfrak{g}_{-\alpha}$ such that $E_\alpha = \sigma(E_{-\alpha})$ and $\langle E_\alpha, E_{-\alpha}\rangle = -1$. Note that $\langle E_\alpha, E_\alpha\rangle = \langle E_{-\alpha}, E_{-\alpha}\rangle = 0$. It is easy to check that $[E_\alpha, E_{-\alpha}] = -H_\alpha$. Let us write $X_\alpha = (1/(\sqrt{2}i))(E_\alpha - E_{-\alpha})$, $Y_\alpha = (1/\sqrt{2})(E_\alpha + E_{-\alpha})$, $Z_\alpha = \frac{1}{2}iH_\alpha$. Then $X_\alpha, Y_\alpha, Z_\alpha \in \mathfrak{g}$ and the following equalities hold:

$$(3.1) \qquad [X_\alpha, Y_\alpha] = 2Z_\alpha, \qquad [X_\alpha, Z_\alpha] = -2Y_\alpha, \qquad [Y_\alpha, Z_\alpha] = 2X_\alpha.$$

It follows that $\mathfrak{a} = \mathrm{span}_{\mathbb{R}}\{X_\alpha, Y_\alpha, Z_\alpha\}$ is a real subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{so}(3)$. If α is a highest root then $\mathrm{ad}(E_\alpha)^2 = 0$ and $\mathrm{ad}(E_{-\alpha})^2 = 0$ on the space $\sum_{\beta\in\Delta_+,\,\beta\neq\alpha}(\mathfrak{g}_\beta \oplus \mathfrak{g}_{-\beta})$ (see e.g. [K-S]).

Proposition 2. *Let* $\mathfrak{m} = \sum_{\beta\in\Delta_+,\,\beta\neq\alpha,\,\langle\beta,\alpha\rangle\neq0} \mathfrak{g} \cap (\mathfrak{g}_\beta \oplus \mathfrak{g}_{-\beta})$. *Then* $\mathrm{ad}(X_\alpha)\mathfrak{m} \subset \mathfrak{m}$, $\mathrm{ad}(Y_\alpha)\mathfrak{m} \subset \mathfrak{m}$, $\mathrm{ad}(Z_\alpha)\mathfrak{m} \subset \mathfrak{m}$ *and* $J_1 = \mathrm{ad}(X_\alpha)_{|\mathfrak{m}}, J_2 = \mathrm{ad}(Y_\alpha)_{|\mathfrak{m}}$, $J_3 = \mathrm{ad}(Z_\alpha)_{|\mathfrak{m}}$ *define on* $\mathfrak{m}$ *three complex structures which give a quaternion structure on* $\mathfrak{m}$, *i.e.* $J_i \circ J_j = \varepsilon_{ijk}J_k$ *if* $i \neq j$.

Proof. Since $X_\alpha, Y_\alpha, Z_\alpha \in \mathfrak{g}$ it is enough to prove that the analogous statement holds on $\mathfrak{m}^{\mathbb{C}} = \sum_{\beta\in\Delta_+,\beta\neq\alpha,\langle\beta,\alpha\rangle\neq0}(\mathfrak{g}_\beta \oplus \mathfrak{g}_{-\beta})$. Let $\gamma \in \Delta$, $\gamma \neq \alpha$ and $\langle\gamma, \alpha\rangle \neq 0$. Then $Z = [E_\alpha, E_\gamma] \in \gamma_{\alpha+\gamma}$ and if $Z \neq 0$ then $-\gamma \in \Delta_+$. Since from Proposition 1 we get

$$\langle\alpha+\gamma, \alpha\rangle = \langle\alpha, \alpha\rangle + \langle\gamma, \alpha\rangle = |\alpha|^2 - \tfrac{1}{2}|\alpha|^2 = \tfrac{1}{2}|\alpha|^2 \neq 0$$

it follows that $\mathrm{ad}(E_\alpha)\mathfrak{m}^{\mathbb{C}} \subset \mathfrak{m}^{\mathbb{C}}$. Similarly one can prove that $\mathrm{ad}(E_{-\alpha})\mathfrak{m}^{\mathbb{C}} \subset \mathfrak{m}^{\mathbb{C}}$. Thus $\mathrm{ad}(X_\alpha)\mathfrak{m}^{\mathbb{C}} \subset \mathfrak{m}^{\mathbb{C}}$ and $\mathrm{ad}(Y_\alpha)\mathfrak{m}^{\mathbb{C}} \subset \mathfrak{m}^{\mathbb{C}}$. Note that $\mathrm{ad}(H_\alpha)E_\gamma = \langle\alpha, \gamma\rangle E_\gamma = \pm\frac{1}{2}|\alpha|^2 E_\gamma$. Thus $\mathrm{ad}(Z_\alpha)E_\gamma = \pm iE_\gamma$ and $J_3^2 = -\mathrm{id}_{\mathfrak{m}}$. We also have

$$\mathrm{ad}(E_\alpha+E_{-\alpha})^2 = \mathrm{ad}(E_\alpha)^2 + \mathrm{ad}(E_{-\alpha})\circ\mathrm{ad}(E_\alpha) + \mathrm{ad}(E_\alpha)\circ\mathrm{ad}(E_{-\alpha}) + \mathrm{ad}(E_{-\alpha})^2.$$

Both $\mathrm{ad}(E_\alpha)^2$ and $\mathrm{ad}(E_{-\alpha})^2$ vanish on $\mathfrak{m}^{\mathbb{C}}$. Hence $\mathrm{ad}(E_\alpha + E_{-\alpha})^2(E_\gamma) = [E_\alpha,[E_{-\alpha},E_\gamma]] + [E_{-\alpha},[E_\alpha,E_\gamma]]$.

Now assume that $\gamma \in \Delta_+$. Then

$$\begin{aligned}[E_\alpha,[E_{-\alpha},E_\gamma]] + [E_{-\alpha},[E_\alpha,E_\gamma]] &= -[E_\gamma,[E_\alpha,E_{-\alpha}]] + 2[E_{-\alpha},[E_\alpha,E_\gamma]] \\ &= -[E_\gamma,[E_\alpha,E_{-\alpha}]] = [E_\gamma,H_\alpha] \\ &= -\langle\alpha,\gamma\rangle E_\gamma = -\tfrac{1}{2}|\alpha|^2 E_\gamma\end{aligned}$$

where we used the fact that if $\alpha \in \Delta_+$ is a highest root and $\gamma \in \Delta_+$, $\langle\alpha,\gamma\rangle \neq 0$ then $\langle\alpha,\gamma\rangle > 0$ since otherwise $\alpha+\gamma$ would be a positive root, a contradiction. If $-\gamma \in \Delta_+$ then

$$\begin{aligned}[E_\alpha,[E_{-\alpha},E_\gamma]] + [E_{-\alpha},[E_\alpha,E_\gamma]] &= -[E_\gamma,[E_{-\alpha},E_\alpha]] + 2[E_\alpha,[E_\alpha,E_\gamma]] \\ &= [E_\gamma,[E_\alpha,E_{-\alpha}]] = -[E_\gamma,H_\alpha] \\ &= \langle\alpha,\gamma\rangle E_\gamma = -\tfrac{1}{2}|\alpha|^2 E_\gamma.\end{aligned}$$

Recall that $|\alpha|^2 = 4$. Consequently, $\mathrm{ad}(E_\alpha + E_{-\alpha})^2_{|\mathfrak{m}} = -2\,\mathrm{id}_{|\mathfrak{m}}$ and $\mathrm{ad}(Y_\alpha)^2_{|\mathfrak{m}} = -\mathrm{id}_{|\mathfrak{m}}$. Thus $J_2^2 = -\mathrm{id}_{|\mathfrak{m}}$. Analogously one can prove that $J_1^2 = -\mathrm{id}_{|\mathfrak{m}}$.

Now we show that $J_1 \circ J_2 = -J_3$. We have

$$\begin{aligned}[E_\alpha - E_{-\alpha},[E_\alpha + E_{-\alpha},E_\gamma]] &= [E_\alpha,[E_{-\alpha},E_\gamma]] - [E_{-\alpha},[E_\alpha,E_\gamma]] \\ &= -[E_\gamma,[E_\alpha,E_{-\alpha}]] = -[H_\alpha,E_\gamma]\end{aligned}$$

and consequently $J_1 \circ J_2 = J_3$. It follows easily that $J_i \circ J_j = \varepsilon_{ijk} J_k$ if $i \neq j$. ■

Now consider the group G with the bi-invariant metric g induced by $-\langle\cdot,\cdot\rangle$. Note that g is positive definite. Write $\mathfrak{l} = \{H \in \mathfrak{t} : \alpha(H) = 0\} \oplus \sum_{\beta\in\Delta_+,\,\langle\alpha,\beta\rangle=0} \mathfrak{g} \cap (\mathfrak{g}_\beta \oplus \mathfrak{g}_{-\beta})$. Then $\mathfrak{l}$ is a Lie subalgebra of $\mathfrak{g}$ and $[\mathfrak{a},\mathfrak{l}] = 0$. Note that $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{a} \oplus \mathfrak{m}$ and $[\mathfrak{l} \oplus \mathfrak{a},\mathfrak{m}] \subset \mathfrak{m}$, $[\mathfrak{m},\mathfrak{m}] \subset \mathfrak{l} \oplus \mathfrak{a}$. Let L, A be the connected subgroups of G corresponding to the Lie subalgebras $\mathfrak{l}, \mathfrak{a}$ respectively. Let $t > 0$ and let g_t be a left-invariant metric on G defined by (we identify g_t with a metric on $\mathfrak{g}$) $g_t = g_0 + g_1 + tg_2$ where $g_0 = g_{|\mathfrak{l}}$, $g_1 = g_{|\mathfrak{a}}$, $g_2 = g_{|\mathfrak{m}}$. Let $p : G \to G/L$ be the natural projection. Then the metric $g_1 + tg_2$ on $\mathfrak{a} \oplus \mathfrak{m}$ induces an invariant metric h_t on the coset space G/L such that $p : (G,g_t) \to (G/L,h_t)$ is a Riemannian submersion. The left-invariant vector fields $X_\alpha, Y_\alpha, Z_\alpha \in \mathfrak{g}$ are Killing vector fields with respect to the metric g_t.

In fact it is easy to check that if $A \in \mathfrak{a}$ then $g_t(\mathrm{ad}(A)X,Y) + g_t(X,\mathrm{ad}(A)Y) = 0$ for all $X, Y \in \mathfrak{g}$. Since they are horizontal with respect to the Riemannian submersion p and $[\mathfrak{l},\mathfrak{a}] = 0$ it follows that there exist Killing vector fields ξ_1, ξ_2, ξ_3 on G/L which are p-related to $X_\alpha, Y_\alpha, Z_\alpha$ respectively.

We show that for an appropriate choice of t the fields ξ_1, ξ_2, ξ_3 define on $M = G/L$ a positive 3-K-contact structure (in fact Sasakian). Define the three 1-forms on G by

$$\theta_1(X) = g_t(X_\alpha, X), \quad \theta_2(X) = g_t(Y_\alpha, X), \quad \theta_3(X) = g_t(Z_\alpha, X).$$

Note that the forms θ_i are left-invariant, $\theta_i(Y) = 0$ if $Y \in \mathfrak{l}$ and $\mathrm{ad}_l^*(\theta_i) = \theta_i$ for any $l \in L$. Thus (see for example [O-T, p. 139]) $\theta_i = p^*\eta_i$ where η_i are the one-forms on M defined by $\eta_i(X) = h_t(\xi_i, X)$. Let $X, Y \in \mathfrak{l} \oplus \mathfrak{a}$. Then

$$d\theta_i(X, Y) = -\theta_i([X, Y]). \tag{3.2}$$

The group A is a totally geodesic subgroup of (G, g_t). Consequently, the orbits of the action of A on M are totally geodesic submanifolds of M (the fundamental Killing vector fields of A have constant length). From (3.2) we get (setting $T_1 = X_\alpha$, $T_2 = Y_\alpha$, $T_3 = Z_\alpha$)

$$d\theta_i(X, Y) = -g_t(T_i, [X, Y]). \tag{3.3}$$

Note that $d\theta_i(X, Y) = 0$ if $X \in \mathfrak{a}$ and $Y \in \mathfrak{m}$. We also have

$$d\theta_i(X, Y) = \langle T_i, [X, Y]\rangle = -\langle \mathrm{ad}(X)T_i, Y\rangle = g_t(\mathrm{ad}(X)T_i, Y) \quad \text{if } X, Y \in \mathfrak{a}, \tag{3.4a}$$

$$d\theta_i(X, Y) = \langle T_i, [X, Y]\rangle = -\langle \mathrm{ad}(X)T_i, Y\rangle = \frac{1}{t}\, g_t(\mathrm{ad}(X)T_i, Y) \quad \text{if } X, Y \in \mathfrak{m}. \tag{3.4b}$$

Thus if $X, Y \in \mathfrak{m} \oplus \mathfrak{a}$ and $g \in G$ and $x = p(g) \in M$ then $p(X_g) \in T_xM$, $p(Y_g) \in T_xM$ and

$$d\eta_i(p(X), p(Y))_{x)} = g_t(\mathrm{ad}(X)T_i, Y) \qquad \text{if } X, Y \in \mathfrak{a}, \tag{3.5a}$$

$$d\eta_i(p(X), p(Y))_x = \frac{1}{t}\, g_t(\mathrm{ad}(X)T_i, Y) \quad \text{if } X, Y \in \mathfrak{m}. \tag{3.5b}$$

Consequently, since $p^*d\eta_i = d\theta_i$ and $d\eta_i(X, Y) = 2h_t(\nabla^t_X \xi_i, Y)$ we obtain (note that $p : (G, g_t) \to (M, h_t)$ is a Riemannian submersion)

$$\nabla^t_{p(X)}\xi_i = -\frac{1}{2}\, p(\mathrm{ad}(T_i)(X)) \qquad \text{if } X \in \mathfrak{a}, \tag{3.6a}$$

$$\nabla^t_{p(X)}\xi_i = -\frac{1}{2t}\, p(\mathrm{ad}(T_i)(X)) \quad \text{if } X \in \mathfrak{m}, \tag{3.6b}$$

where by ∇^t we denote the Levi-Civita connection of (M, h_t). If we identify the space T_xM with $\mathfrak{a} \oplus \mathfrak{m}$ by means of p then

$$\nabla^t \xi_{i|\mathfrak{a}} = -\frac{1}{2}\ \mathrm{ad}(T_i)_{|\mathfrak{a}}, \tag{3.7a}$$

$$\nabla^t \xi_{i|\mathfrak{m}} = -\frac{1}{2t}\ \mathrm{ad}(T_i)_{|\mathfrak{m}}. \tag{3.7b}$$

Note that if $p(g) = p(g_1)$ then $g_1 = gl$ where $l \in L$. Thus if we identify $(\mathfrak{a} \oplus \mathfrak{m})_g = d_eL_g(\mathfrak{a} \oplus \mathfrak{m}) \subset T_gG$ and $(\mathfrak{a} \oplus \mathfrak{m})_{g_1} = d_eL_{g_1}(\mathfrak{a} \oplus \mathfrak{m}) \subset T_{g_1}G$ with $T_{gL}G/L$ by means of p and $X \in \mathfrak{a} \oplus \mathfrak{m}$ then a vector $p(X_g) \in T_{gL}G/L$ is represented by a vector $(\mathrm{Ad}(l)X)_{g_1} \in \mathfrak{m}_{g_1}$. However $[L, A] = \{e\}$ and consequently (3.7) does not depend on the choice of the isomorphism $(\mathfrak{a} \oplus \mathfrak{m})_g = T_{gL}G/L$.

Now consider the Lie algebra $\mathfrak{g}_0 = \mathfrak{l} \oplus \mathfrak{a} \oplus i\mathfrak{m} \subset \mathfrak{g}^{\mathbb{C}}$. To this Lie algebra corresponds a connected Lie subgroup G_0 of the Lie group $G^{\mathbb{C}}$. We call G_0 the *dual group* of G. On the group $G^{\mathbb{C}}$ we have a bi-invariant metric g induced by the Killing form $\langle \cdot, \cdot \rangle_{\mathrm{K}}$ on $g^{\mathbb{C}}$, i.e.

$$g(X, Y)_e = -\frac{1}{|\alpha|^2} \langle X, Y \rangle_{\mathrm{K}} = -\langle X, Y \rangle.$$

Let $t > 0$ and let g_t be a left-invariant metric on G_0 defined by (we identify g_t with a metric on $\mathfrak{g}_0$) $g_t = g_0 + g_1 + t g_2$ where $g_0 = g_{|\mathfrak{l}}$, $g_1 = g_{|\mathfrak{a}}$, $g_2 = -g_{|i\mathfrak{m}}$. Note that g_t is a positive-definite metric on G_0. Let $p_0 : G_0 \to G_0/L$ be a natural projection. Then the metric $g_1 + t g_2$ on $\mathfrak{a} \oplus \mathfrak{m}$ induces an invariant metric h_t on the coset space G_0/L such that $p_0 : (G_0, g_t) \to (G_0/L, h_t)$ is a Riemannian submersion. The left-invariant vector fields $X_\alpha, Y_\alpha, Z_\alpha \in \mathfrak{a} \subset \mathfrak{g}_0$ are Killing vector fields with respect to the metric g_t on G_0. It follows that there exist Killing vector fields ξ_1, ξ_2, ξ_3 on $M_0 = G_0/L$ which are p_0-related to T_1, T_2, T_3 respectively.

Define three 1-forms on G by $\theta_i(X) = g_t(T_i, X)$. Note that the forms θ_i are left-invariant, $\theta_i(Y) = 0$ if $Y \in \mathfrak{l}$ and $\mathrm{ad}_l^* \theta_i = \theta_i$ for any $l \in L$. Thus $\theta_i = p^* \eta_i$ where η_i are one-forms on M defined by $\eta_i(X) = h_t(\xi_i, X)$. Let $X, Y \in \mathfrak{l} \oplus \mathfrak{a}$. Then as above

$$d\theta_i(X, Y) = -\theta_i([X, Y]) = -g_t(T_i, [X, Y]). \tag{3.8}$$

Note that $d\theta_i(X, Y) = 0$ if $X \in \mathfrak{a}$ and $Y \in i\mathfrak{m}$. We also have

$$d\theta_i(X, Y) = \langle T_i, [X, Y] \rangle = -\langle \mathrm{ad}(X) T_i, Y \rangle = g_t(\mathrm{ad}(X) T_i, Y) \quad \text{if } X, Y \in \mathfrak{a}, \tag{3.9a}$$

$$d\theta_i(X, Y) = \langle T_i, [X, Y] \rangle = -\langle \mathrm{ad}(X) T_i, Y \rangle = -\frac{1}{t} g_t(\mathrm{ad}(X) T_i, Y) \quad \text{if } X, Y \in i\mathfrak{m}. \tag{3.9b}$$

Thus if $X, Y \in i\mathfrak{m} \oplus \mathfrak{a}$ and $g \in G_0$ and $x = p_0(g) \in M_0$ then $p_0(X_g) \in T_x M_0$, $p_0(Y_g) \in T_x M_0$ and

$$d\eta_i(p_0(X), p_0(Y))_{x)} = g_t(\mathrm{ad}(X) T_i, Y) \qquad \text{if } X, Y \in \mathfrak{a}, \tag{3.10a}$$

$$d\eta_i(p_0(X), p_0(Y))_x = -\frac{1}{t} g_t(\mathrm{ad}(X) T_i, Y) \quad \text{if } X, Y \in i\mathfrak{m}. \tag{3.10b}$$

Consequently, since $d\eta_i = p_0^* d\theta_i$ and $d\eta_i(X, Y) = 2h_t(\nabla^t_X \xi_i, Y)$ we obtain (note that $p_0 : (G_0, g_t) \to (M_0, h_t)$ is a Riemannian submersion and we identify TM_0 with $\mathfrak{a} \oplus i\mathfrak{m}$ by means of p_0)

$$\nabla^t \xi_{i|\mathfrak{a}} = -\frac{1}{2} \, \mathrm{ad}(T_i)_{|\mathfrak{a}}, \tag{3.11a}$$

(3.11b) $$\nabla^t \xi_{i|i\mathfrak{m}} = \frac{1}{2t}\,\mathrm{ad}(T_i)_{|i\mathfrak{m}}.$$

Hence we can prove

THEOREM 1. *Let G be a compact, simple and centreless Lie group and let G_0 be its dual group. Then $(G/L, h_t, \xi_1, \xi_2, \xi_3)$ is a positive 3-K-contact structure and $(G_0/L, h_t, \xi_1, \xi_2, \xi_3)$ is a negative 3-K-contact structure if and only if $t = 1/2$.*

Proof. Note that in both cases considered above we have $\nabla^t_{\xi_i}\xi_j = \varepsilon_{ijk}\xi_k$. Thus conditions (2.1) of the definition of 3-K-contact structure are satisfied. If we identify T_xM with $\mathfrak{a} \oplus \mathfrak{m}$ (respectively T_xM_0 with $\mathfrak{a} \oplus i\mathfrak{m}$) by means of p (resp. p_0) then the space H described in the definition coincides with $\mathfrak{m}$ (resp. $i\mathfrak{m}$). With this identification $J_i = \nabla^t \xi_{i|H}$ equals

$$\nabla^t \xi_{i|\mathfrak{m}} = -\frac{1}{2t}\,\mathrm{ad}(T_i)_{|\mathfrak{m}}$$

in the first case and

$$\nabla^t \xi_{i|i\mathfrak{m}} = \frac{1}{2t}\,\mathrm{ad}(T_i)_{|i\mathfrak{m}}$$

in the second case. From Proposition 2 it follows that if $t = 1/2$ then $\nabla^t \xi_i$ defines on the space $H_i = \{X \in TM(TM_0) : h_t(\xi_i, X) = 0\}$ an almost complex structure (i.e. $(\nabla^t \xi_{i|H_i})^2 = -\mathrm{id}_{H_i}$). Consequently, each field ξ_i defines a K-contact structure on $(M, h_{1/2})$ (resp. on $(M_0, h_{1/2})$).

Now from (3.7) and (3.11) it follows that for $t = 1/2$ we have

$$-\nabla^t \xi_{i|\mathfrak{m}} = \frac{1}{2t}\,\mathrm{ad}(T_i)_{|\mathfrak{m}} = J_i$$

and respectively

$$-\nabla^t \xi_{i|i\mathfrak{m}} = -\frac{1}{2t}\,\mathrm{ad}(T_i)_{|i\mathfrak{m}} = iJ_i i$$

where J_i is defined in Proposition 2 and $iJ_i i(X) = i(J_i(iX))$ for $X \in i\mathfrak{m}$. Consequently, it follows from Proposition 2 that $(M, h_{1/2}, \xi_1, \xi_2, \xi_3)$ is a positive 3-K-contact structure and that $(M_0, h_{1/2}, \xi_1, \xi_2, \xi_3)$ is a negative 3-K-contact structure. ■

Note that the spaces G/L are SO(3) or Sp(1) bundles over the symmetric quaternionic spaces $W(G)$, and G/L are exactly the spaces

$$\mathrm{Sp}(n)/\mathrm{Sp}(n-1) = S^{4n-1}, \quad \mathrm{SU}(m)/S(U(m-2)\times U(1)),$$
$$\mathrm{SO}(k)/\mathrm{SO}(k-4)\times \mathrm{Sp}(1), \quad G_2/\mathrm{Sp}(1), \quad F_4/\mathrm{Sp}(3),$$
$$E_6/\mathrm{SU}(6), \quad E_7/\mathrm{Spin}(12), \quad E_8/E_7,$$

where $n \geq 1$, $m \geq 3$, $k \geq 7$, and G/L is an Sp(1) bundle only in the first case of $\mathrm{Sp}(n)/\mathrm{Sp}(n-1) = S^{4n-1}$. Note that this space admits a $\mathbb{Z}_2$ quotient $\mathrm{Sp}(n)/\mathrm{Sp}(n-1)\times \mathbb{Z}_2 = \mathbb{RP}^{4n-1}$ which is also a 3-Sasakian space.

The holonomy representation of $W(G)$ with symmetric metric is the representation ad of the group LA on the space $\mathfrak{m}$ with quaternionic structure given by J_1, J_2, J_3 where $A = \mathrm{Sp}(1)$ and the action $A \ni a \mapsto \mathrm{Ad}(a)_{|\mathfrak{m}}$ coincides with the standard representation of the group $\mathrm{Sp}(1) = \{q \in \mathbb{H} : q\overline{q} = 1\}$ on the space $\mathbb{H}^n$ where $n = \frac{1}{4}\dim \mathfrak{m}$. Consequently, $L\,\mathrm{Sp}(1) \subset \mathrm{Sp}(n)\,\mathrm{Sp}(1)$.

Now our aim is to give a precise description of twistor spaces of Wolf spaces (see [S], [Sw], [J-1]). In the negative case we obtain homogeneous almost Kähler manifolds which are not Kähler. In the positive case we get Einstein Kähler spaces G/LT of positive scalar curvature where T is the one-dimensional torus group. We only give the proof for the negative case, the positive case being similar.

PROPOSITION 3. *The homogeneous spaces G_0/LT, where T is the one-parameter subgroup of* $\mathrm{Sp}(1) = A$ *generated by $Z_\alpha \in \mathfrak{a}$ with metric induced by the metric $m = g_{|\mathfrak{a}_1} - \frac{1}{2}g_{|i\mathfrak{m}}$ on the space $\mathfrak{m}_0 = \mathfrak{a}_1 \oplus i\mathfrak{m}$ where $g = -(4/|\alpha|^2_{\mathrm{K}})\langle\cdot,\cdot\rangle_{\mathrm{K}}$ and $\mathfrak{a}_1 = \mathrm{span}_{\mathbb{R}}\{X_\alpha, Y_\alpha\}$, are strictly almost Kähler homogeneous spaces.*

Proof. Let π_* be the natural projection $\pi_* : G_0/L \to G_0/LT$. Since G_0/LT is the quotient of G_0/L by the one-parameter group of isometries generated by the Killing vector field ξ_3 and $((G_0/L, h_{1/2}), \xi_3)$ is a K-contact structure it follows that G_0/LT with the induced metric g_* and an almost Hermitian structure J_* such that $g(J_*\pi_*X, \pi_*(Y)) = d\eta_1(X,Y)$ is an almost Kähler manifold with a Kähler form $\Omega_*(X,Y) = g_*(J_*X,Y)$. It is not Sasakian, since $R(X,\xi_i)\xi_j = 2\varepsilon_{ijk}\phi_k(X)$ for a horizontal vector X. ■

REMARK. Note that the metric on the 3-K-contact space is uniquely determined as $g_t = g_0 + g_1 + tg_2$ where $g = (4/|\alpha|^2_{\mathrm{K}})\langle\cdot,\cdot\rangle_{\mathrm{K}}$, $g_0 = g_{|\mathfrak{l}}$, $g_1 = g_{|\mathfrak{a}}$, $g_2 = \varepsilon g_{|\mathfrak{m}_1}$ with $\varepsilon = 1$ and $\mathfrak{m}_1 = \mathfrak{m}$ in the case of a positive 3-K-contact space G/L and $\varepsilon = -1$ and $\mathfrak{m}_1 = i\mathfrak{m}$ in the case of a negative 3-K-contact space G_0/L, whereas the metric on the almost Kähler space G_0/LT is given up to homothety, i.e. we can also choose the metric $m = g_{|\mathfrak{a}_1} - \frac{1}{2}g_{|i\mathfrak{m}}$ on the space $\mathfrak{m}_0 = \mathfrak{a}_1 \oplus i\mathfrak{m}$ where $g = -\langle\cdot,\cdot\rangle_{\mathrm{K}}$ and the twistor space with this metric is still almost Kähler.

Our last aim is to give a precise description of the reduction of the principal bundle $\mathrm{SO}(M)$ of orthonormal oriented frames of the Wolf spaces $W(G), W(G)^*$ to the LA-structure $P(LA, M)$, and to describe the Levi-Civita connection in P. We denote by g_* the symmetric metric on $W(G)$ or $W(G)^*$, i.e. g_* is induced by the metric $-\langle\cdot,\cdot\rangle$ on $\mathfrak{m}$ or $\langle\cdot,\cdot\rangle$ on $i\mathfrak{m}$. Write $K = LA$ and denote by $\mathfrak{k}$ the Lie algebra of K. By $\pi : G/K \to M$ or $\pi : G_0/K \to M$ we mean the natural projection.

Define $P = G$ in the positive case and $P = G_0$ in the negative case. Define the horizontal distribution $\mathcal{H} \subset TG$ (resp. TG_0) by $\mathcal{H}_g = d_eL_g(\mathfrak{m}_1)$ and the vertical distribution by $\mathcal{V}_g = d_e L_g(\mathfrak{k})$. Let θ^{C} be a Cartan form on G (resp. G_0) defined on $X \in T_g G$ (resp. $T_g G_0$) as follows: $\theta^{\mathrm{C}}(X) = d_e L_{g^{-1}}(X) \in \mathfrak{g}$ (resp $\mathfrak{g}_0$). Denote by $p_{\mathfrak{k}}, p_{\mathfrak{m}_1}$ the projections onto $\mathfrak{k}, \mathfrak{m}_1$ with respect to the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ and $\mathfrak{g}_0 = \mathfrak{k} \oplus i\mathfrak{m}$. Then the connection form ω with horizontal distribution $\mathcal{H}$ is defined by $\omega = p_{\mathfrak{k}} \circ \theta^{\mathrm{C}}$.

We shall treat G (resp. G_0) as a subbundle of the bundle $\mathrm{SO}(M)$ by identifying an element $g \in G$ with the mapping $u_g : \mathfrak{m}_1 \to T_{gK} M$ given by $u_g(X) = \pi(d_e L_g(X))$. Then the canonical form of $P \subset \mathrm{SO}(M)$ is $\theta = p_{\mathfrak{m}_1} \circ \theta^{\mathrm{C}}(X)$ since

$$\theta(X) = u_g^{-1}(\pi(X)) = p_{\mathfrak{m}_1} \circ \theta^{\mathrm{C}}(X).$$

Since $[\mathfrak{m}_1, \mathfrak{m}_1] \subset \mathfrak{k}$ it follows easily that $\Theta(X,Y) = d\theta(hX, hY) = 0$ where $hX \in \mathcal{H}$ denotes the $\mathcal{H}$-component of X with respect to the decomposition $TG = \mathcal{V} \oplus \mathcal{H}$.

Thus the connection Γ given by ω is a torsionless connection in the principal bundle of oriented orthonormal frames, i.e. Γ is the Levi-Civita connection of (M, g_*). Note that we treat K as a subgroup of $\mathrm{SO}(\mathfrak{m}_1)$ (where on $\mathfrak{m}_1$ we have the metric $-\langle \cdot, \cdot \rangle$ if $\mathfrak{m}_1 = \mathfrak{m}$, and $\langle \cdot, \cdot \rangle$ if $\mathfrak{m}_1 = i\mathfrak{m}$), via the representation $\mathrm{Ad} : K \to \mathrm{SO}(\mathfrak{m}_1)$.

Let $G_1 = G$ or $G_1 = G_0$ and $(a, b) \ni t \mapsto x_t \in M$ be a smooth path in M such that

$$x(a) = eK = x_0 \in G_1/K = M$$

and let $Y \in T_{x_0} M$. Then there exists $Y^* \in \mathfrak{m}_1 \subset \mathfrak{g} = T_e G_1$ such that $Y = \pi(Y^*)$. Let g_t be a horizontal lift of x_t to the K-principal bundle G_1 over M with connection Γ, i.e. $\pi(g_t) = x_t$ and $\omega(\dot{g}_t) = 0$. Then $Y_t = \pi(Y_t^*)$ is a parallel field along x_t where $Y_t^* = d_e L_{g_t}(Y^*) \in \mathcal{H}_{g_t}$. Note that $Y_a = Y$. If $x_a = x_b = x_0$ then $g_b = k \in K$ and under the identification $\mathcal{H}_e = \mathcal{H}_k$ we obtain $Y_b^* = \mathrm{ad}(k) Y^*$.

Consequently, the holonomy group coincides exactly with K and the holonomy representation is $K \ni k \mapsto \mathrm{ad}(k)_{|\mathfrak{m}_1} \in \mathrm{SO}(\mathfrak{m}_1)$ (for the details see [H, p. 207]). Recall that the endomorphisms $J_i : \mathfrak{m}_1 \to \mathfrak{m}_1$ are described in Proposition 2. It is easy to see that the bundle $\mathcal{G} \subset \mathrm{End}(TM)$ of endomorphisms defining the quaternionic structure on M is generated by the endomorphisms $\pi(J_i \circ u_g^{-1})$ where $u_g \in G_1$, $i \in \{1, 2, 3\}$ (see the construction of $\mathcal{G}$ in [J-1], [J-2]). We have

PROPOSITION 4. *The principal bundle* $\mathrm{SO}(M)$ *and the Levi-Civita connection of a quaternionic Kähler Wolf space* $W(G)$ (*resp.* $W(G)^*$) *admit a reduction to a* K*-structure* $G_1 \subset \mathrm{SO}(M)$ *with Levi-Civita connection form* $\omega = p_{\mathfrak{k}} \circ \theta^{\mathrm{C}}$. *The bundle* $\mathcal{G}$ *is generated by the endomorphisms* $\pi(J_i \circ u_g^{-1})$ *where* $i \in \{1, 2, 3\}$.

It follows that our construction of positive and negative 3-K-contact structures coincides with the one given in [J-2]. The only difference is that we consider a K-reduction $G, G_0 \subset \mathrm{SO}(M)$ instead of an $\mathrm{Sp}(n)\,\mathrm{Sp}(1)$-reduction $Q \subset \mathrm{SO}(M)$. It is clear that $K \subset \mathrm{Sp}(n)\,\mathrm{Sp}(1)$. Consequently, the positive structure is 3-Sasakian and the negative structure is the Tanno nS-structure. In the case $4n \neq 8$ this also follows directly from [J-1]. Let us remark here that K coincides with $\mathrm{Sp}(n)\,\mathrm{Sp}(1)$ only in the case of $M = \mathbb{HP}^n = \mathrm{Sp}(n+1)/\mathrm{Sp}(n)\,\mathrm{Sp}(1)$ and its dual Wolf space (see [A]).

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