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Smooth points of a semialgebraic set

by Jacek Stasica (Kraków)

Abstract. It is proved that the set of smooth points of a semialgebraic set is semialgebraic.

1. Introduction. The semialgebraicity of the smooth points of a semialgebraic set plays an important role in semialgebraic geometry. In [Ł1] S. Łojasiewicz proved that for locally semialgebraic sets the notions of a Nash smooth point and of an analytic point coincide. Moreover if $\Gamma \subset \mathbb{R}^n$ is an analytic submanifold, then for the germ Γ_a of Γ at $a \in \Gamma$ we have the equivalence:

$$\Gamma_a \text{ is Nash } \Leftrightarrow \ \Gamma_a \text{ is semialgebraic.}$$

Hence the semialgebraicity of the smooth points can be obtained following Łojasiewicz's method for the analogous theorem for semianalytic sets. The aim of this paper is to give a straightforward proof of the semialgebraicity of the smooth points of a semialgebraic set based on the properties of asymptotic analytic solutions proved in [S].

Recall that a subset of $\mathbb{R}^n$ is *semialgebraic* if it is described by polynomials on $\mathbb{R}^n$. Thus, the class of semialgebraic subsets of $\mathbb{R}^n$ is the algebra of subsets of $\mathbb{R}^n$ which is generated by the family of sets $\{P > 0\}$, where P is a polynomial. Equivalently, $E \subset \mathbb{R}^n$ is semialgebraic if there are polynomials P_i and P_{ij}, $i = 1, \ldots, p$, $j = 1, \ldots, q$, such that

$$E = \bigcup_{i=1}^{p} \{x \mid P_i(x) = 0,\ P_{ij}(x) > 0,\ j = 1, \ldots, q\}.$$

Let G be an open subset of $\mathbb{R}^n$. We say that an analytic function $f : G \to \mathbb{R}$ is a *Nash function at* $a \in G$ if $W(x, f(x)) = 0$ in a neighborhood of a for a polynomial $W \not\equiv 0$ in $\mathbb{R}^{n+1}_{x,t}$. A *Nash function on* G is an analytic function on G which is Nash at each point of G.

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2. Auxiliary results

LEMMA 1. *Let E be a semialgebraic subset of $\mathbb{R}^n$. If* $\operatorname{int} E = \emptyset$, *then some nonzero polynomial on $\mathbb{R}^n$ vanishes on E.*

Proof. We have $E = \bigcup_{i=1}^q B_i$ with $B_i = \{P_i = 0\} \cap \bigcap_j \{P_{ij} > 0\}$ for some polynomials P_i and P_{ij}. We can assume that each B_i is nonempty. Then $P_i \neq 0$, since otherwise B_i would be open. Hence $P_1 \cdot \ldots \cdot P_q$ is the required polynomial. ∎

LEMMA 2. *Every semialgebraic set $E \subset \mathbb{R}^n$ is contained in an algebraic set $V \subset \mathbb{R}^n$ of the same dimension.*

Proof. Let $\pi_\alpha : \mathbb{R}^n \to L_\alpha$ be the natural projections onto $L_\alpha = \mathbb{R}^{k+1}_{x_{\alpha_1},\ldots,x_{\alpha_{k+1}}}$, where $\alpha = (\alpha_1, \ldots, \alpha_{k+1})$, $1 \le \alpha_1 < \ldots < \alpha_{k+1} \le n$, and $k = \dim E$. Then each $\pi_\alpha(E)$ is semialgebraic of dimension at most k. Hence $\pi_\alpha(E) \subset \{P_\alpha = 0\}$ for a nonzero polynomial P_α. Therefore $E \subset V = \bigcap_\alpha \{P_\alpha \circ \pi_\alpha = 0\}$. Moreover $\dim V = k$, since otherwise V would contain a semialgebraic leaf Γ [1] of dimension $k+1$ and so there would exist an α such that $\operatorname{int} \pi_\alpha(\Gamma) \neq \emptyset$, which would imply $P_\alpha \equiv 0$. ∎

We say that a point $a \in E$ is a *smooth point* of dimension k of E if it has a neighborhood in E which is an analytic submanifold of dimension k. By definition the dimension of E is equal to the maximum of the dimensions of its smooth points.

Note that every polynomial on $\mathbb{R}^n$ of degree $r > 0$ is monic of degree r with respect to each of its variables in some coordinate system.

LEMMA 3. *Let $E \subset \mathbb{R}^n_x$ be a semialgebraic set of dimension $\le k < n$. Then in some linear coordinate system, E is contained in a Weierstrass set*

$$(*)\qquad \{P_{k+1}(u, x_{k+1}) = \ldots = P_n(u, x_n) = 0\},$$

where $u = (x_1, \ldots, x_k)$ and P_j is a monic polynomial on $\mathbb{R}^{k+1}_{u,x_j}$ for $j = k+1, \ldots, n$. (*Such a coordinate system will be called a regular system for E.*)

Proof. By the previous lemma it suffices to give the proof for an algebraic set $V \supset E$ of dimension k. Let $\mathcal{P}_n \supset \ldots \supset \mathcal{P}_{k+1} \supset \mathcal{P}_k$ denote the rings of polynomials on $\mathbb{R}^n_{u,x_{k+1},\ldots,x_n}, \ldots, \mathbb{R}^{k+1}_{u,x_{k+1}}, \mathbb{R}^k_u$ (after suitable identifications). Denote by R_j the ring of restrictions of the polynomials from $\mathcal{P}_j$ to V. Let $I = \{P \in \mathcal{P}_n \mid P_{|V} = 0\}$. Changing coordinate systems in $\mathbb{R}^n, \ldots, \mathbb{R}^{k+1}$ successively we find in $I \cap \mathcal{P}_j$ monic polynomials with respect to x_j. This means in particular that $w_j = x_{j|V}$ is integral over R_{j-1}. Evidently $R_j = R_{j-1}[w_j]$; it follows that $w_n, \ldots, w_{k+1}$ are integral over R_k, which means that there exist monic polynomials $P_j \in \mathcal{P}_k[x_j]$, $j = n, \ldots, k+1$, with $P_{j|V} = 0$. ∎

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[1] A *semialgebraic leaf* is any analytic submanifold which is a semialgebraic set.

Let us recall two theorems (for proofs see [S]) useful for the proof of the main theorem.

THEOREM 1. *Let $Q(x,t)$ be a complex polynomial on $\mathbb{R}^{n+1}_{x,t}$, monic in t. Then there exists $m > 0$ such that if f, g are complex, continuous roots: $Q(x, f(x)) = Q(x, g(x)) = 0$ in a neighborhood of $a \in \mathbb{R}^n$, then the following implication holds:*

$$f(x) - g(x) = O(|x-a|^m) \ \Rightarrow \ f = g$$

in a neighborhood of a.

THEOREM 2. *Let Q be a real polynomial on $\mathbb{R}^{n+1}_{x,t}$, monic in t, and let $L > 0$. Then there exists $N \in \mathbb{N}$ such that the following implication holds: if $Q(x, \psi(x)) = O(|x-a|^N)$ as $x \to a$, with some real-analytic function ψ defined in a neighborhood of $a \in \mathbb{R}^n$, then there exists a Nash function φ defined in a neighborhood of $a \in \mathbb{R}^n$ such that $Q(x, \varphi(x)) \equiv 0$ and $\varphi - \psi = O(|x-a|^L)$ as $x \to a$.*

3. Main result. We say that a submanifold $\Gamma \subset \mathbb{R}^n$ is *topographic* if it is the graph of an analytic mapping of an open subset of $\mathbb{R}^k$ into $\mathbb{R}^{n-k}$.

THEOREM 3. *Let $E \subset \mathbb{R}^{k+l}_{u,v}$, $l = n-k$, be a semialgebraic set contained in the Weierstrass set $(*)$. Then the set*

$$\Lambda = \{x \in E \mid U \cap E \text{ is a } k\text{-topographic submanifold for some nbd } U \text{ of } x\}$$

is semialgebraic.

Proof. For $a \in \mathbb{R}^k_u$, $b \in \mathbb{R}^l_v$ and $\delta, \varepsilon > 0$ set $U_{ab\delta\varepsilon} = B(a,\delta) \times B(b,\varepsilon)$, where $B(a,\delta) = \{u \mid |u-a| < \delta\}$, $B(b,\varepsilon) = \{v \mid |v-b| < \varepsilon\}$. Let $(a,b) \in E$. The set $E_{ab\delta\varepsilon} = E \cap U_{ab\delta\varepsilon}$ is the graph of some continuous function $B(a,\delta) \to \mathbb{R}^l_v$ if and only if $\overline{E} \cap U_{ab\delta\varepsilon} \subset E$ and for $u \in B(a,\delta)$ we have $(u,v) \in \overline{E} \cap U_{ab\delta\varepsilon}$ for exactly one v. Thus the set

$$F = \{(a,b,\delta,\varepsilon) \mid E_{ab\delta\varepsilon} \text{ is the graph of some continuous function on } B(a,\delta)\}$$

is semialgebraic.

Take m from Theorem 1, the same for all polynomials P_j, and $N > m$ from Theorem 2 also the same for all polynomials P_j, and $L = m$. For $c = \{c_\alpha\}_{|\alpha| \le N}$, where $c_\alpha \in \mathbb{R}^l$ and $\alpha \in \mathbb{N}^k$, we define the polynomial mapping $P_c : u \mapsto \sum_{|\alpha|} c_\alpha u^\alpha$. In what follows we write $(c_0, c') = c$. For $a \in \mathbb{R}^k_u$, $c, C, N, \delta > 0$ we define the following subset of $\mathbb{R}^{k+l}_{u,v}$:

$$W_{acN\delta} = \{(u,v) \mid u \in B(a,\delta) \text{ and } |v - P_c(u-a)| \le C|u-a|^N\}.$$

It is enough to show that

$$(**)\qquad \Lambda = \{(a,b) \in \mathbb{R}^{k+l}_{u,v} \mid (a,b,\delta,\varepsilon) \in F \text{ and } E_{ab\delta\varepsilon} \subset W_{a(b,c')CN\delta} \text{ with some } \delta,\varepsilon,c',C\}$$

because the last set is semialgebraic. Let $(a,b) \in \Gamma$. Obviously for some sufficiently small $\delta,\varepsilon > 0$ we have $(a,b,\delta,\varepsilon) \in F$ and the set $E_{ab\delta\varepsilon}$ is the graph of some Nash function φ; taking c' such that $x \mapsto P_{(b,c')}(x-a)$ is the Nth Taylor polynomial for φ we have $E_{ab\delta\varepsilon} = \varphi \subset W_{a(b,c')CN\delta}$ with some C when δ is sufficiently small.

Now, assume that the condition $(**)$ holds. Then $E_{ab\delta\varepsilon}$ is the graph of some continuous function $\psi : B(a,\delta) \to \mathbb{R}^l_v$. Hence $Q(u,\psi(u)) = 0$ on $B(a,\delta)$, where $Q = (P_{k+1},\ldots,P_n)$. The graph of ψ is contained in the set $W_{acCN\delta}$. Thus $\psi(u) - P_c(u-a) = O(|u-a|^N)$. Therefore $Q(u,P_c(u-a)) = O(|u-a|^N)$. By Theorem 2 there exists an analytic function φ defined in a neighborhood of a such that $Q(u,\varphi(u)) \equiv 0$ and $\varphi(u) - P_c(u-a) = O(|u-a|^m)$. Hence $\varphi - \psi = O(|u-a|^m)$ and according to Theorem 1 we have $\varphi = \psi$ in a neighborhood of a. ∎

THEOREM 4. *Let $E \subset \mathbb{R}^n$ be a semialgebraic set. Then the set $E^{(k)}$ of its smooth points of dimension k is semialgebraic as well.*

Proof. It is enough to prove the theorem for $k = \dim E$, because we can proceed by induction.

For each isomorphism $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ we define

$$\lambda(\varphi) = \varphi^{-1}(e_{k+1}) \wedge \ldots \wedge \varphi^{-1}(e_n) \in \Lambda^{n-k}\mathbb{R}^n,$$

where $e_1,\ldots,e_n$ is a canonical basis of $\mathbb{R}^n$. We say that a sequence $\varphi_1,\ldots,\varphi_r$ of linear isomorphisms of $\mathbb{R}^n$ is *complete* if $\lambda(\varphi_1),\ldots,\lambda(\varphi_r)$ generate $\Lambda^{n-k}\mathbb{R}^n$. One can easily prove that:

(1) there exists a complete sequence,

(2) any sequence that is sufficiently close (in the natural topology) to a complete one is complete,

(3) if $\varphi_1,\ldots,\varphi_r$ is a complete sequence and the set E is smooth of dimension k at a then $\varphi_\nu(E)$ is k-topographic at $\varphi_\nu(a)$ for some ν.

We find a complete sequence $\varphi_1,\ldots,\varphi_r$ such that each φ_i is a regular system for E. According to Theorem 3 the set R_ν of points at which $\varphi_\nu(E)$ is k-topographic is a semialgebraic set. Hence

$$E^{(k)} = \bigcup \varphi_\nu^{-1}(R_\nu)$$

is semialgebraic as a finite sum of semialgebraic sets. ∎

References

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Institute of Mathematics
Jagiellonian University
Reymonta 4
30-059 Kraków, Poland
E-mail: jacek.stasica@im.uj.edu.pl

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