

Extension theorems of Sakai type for separately holomorphic and meromorphic functions

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Abstract. We first exhibit counterexamples to some open questions related to a theorem of Sakai. Then we establish an extension theorem of Sakai type for separately holomorphic/meromorphic functions.

1. Introduction. We first fix some notations and terminology. Throughout the paper, E denotes the unit disc of \mathbb{C} , and for any set $S \subset \mathbb{C}^n$, $\text{int } S$ (or equivalently $\text{int}_{\mathbb{C}^n} S$) denotes the interior of S . For any domain $D \subset \mathbb{C}^n$, we say that the subset $S \subset D$ *does not separate domains in D* if for every domain $U \subset D$, the set $U \setminus S$ is connected. Moreover, $\mathcal{O}(D)$ (resp. $\mathcal{M}(D)$) will denote the space of holomorphic (resp. meromorphic) functions on D . Finally, if S is a subset of $D \times G$, where $D \subset \mathbb{C}^p, G \subset \mathbb{C}^q$ are some open sets, then for $a \in D$ (resp. $b \in G$), the *fiber* $S(a, \cdot)$ (resp. $S(\cdot, b)$) is the set $\{w \in G : (a, w) \in S\}$ (resp. $\{z \in D : (z, b) \in S\}$).

In 1957 E. Sakai [9] claimed to have proved the following result.

THEOREM. *Let $S \subset E \times E$ be a relatively closed set such that $\text{int } S = \emptyset$ and S does not separate domains in $E \times E$. Let A (resp. B) be the set of all $a \in E$ (resp. $b \in E$) such that $\text{int}_{\mathbb{C}} S(a, \cdot) = \emptyset$ (resp. $\text{int}_{\mathbb{C}} S(\cdot, b) = \emptyset$). Put $X := \mathbb{X}(A, B; E, E) = (A \times E) \cup (E \times B)$. Then for every function $f : X \setminus S \rightarrow \mathbb{C}$ which is separately meromorphic on X , there exists an $\hat{f} \in \mathcal{M}(E \times E)$ such that $\hat{f} = f$ on $X \setminus S$.*

Unfortunately, it turns out, as reported in [4], that the proof of E. Sakai contains an essential gap. In the latter paper M. Jarnicki and the first author gave a correct proof of this theorem.

E. Sakai also claimed in [9] that the following question (the n -dimensional version of the Theorem) can be answered positively but he did not give any proof.

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QUESTION 1. For any $n \geq 3$, let $S \subset E^n$ be a relatively closed set such that $\text{int } S = \emptyset$ and S does not separate domains. Let $f : E^n \setminus S \rightarrow \mathbb{C}$ be such that for any $j \in \{1, \dots, n\}$ and for any $(a', a'') \in E^{j-1} \times E^{n-j}$ with $\text{int}_{\mathbb{C}} S(a', \cdot, a'') = \emptyset$, the function $f(a', \cdot, a'')$ extends meromorphically to E . Does f always extend meromorphically to E^n ?

In connection with the Theorem and Question 1, M. Jarnicki and the first author [4] posed two more questions:

QUESTION 2. Let \mathcal{A} be a subset of E^n ($n \geq 2$) which is plurithin at $0 \in E^n$ (see Section 2 below for the definition). For an arbitrary open neighborhood U of 0 , does there exist a non-empty relatively open subset C of a real hypersurface in U such that $C \subset U \setminus \mathcal{A}$?

QUESTION 3. Let $D \subset \mathbb{C}^p$, $G \subset \mathbb{C}^q$ ($p, q \geq 2$) be pseudoconvex domains and let $S \subset D \times G$ be a relatively closed set such that $\text{int } S = \emptyset$ and S does not separate domains in $D \times G$. Let A (resp. B) be the set of all $a \in D$ (resp. $b \in G$) such that $\text{int}_{\mathbb{C}^q} S(a, \cdot) = \emptyset$ (resp. $\text{int}_{\mathbb{C}^p} S(\cdot, b) = \emptyset$). Put $X := \mathbb{X}(A, B; D, G) = (A \times G) \cup (D \times B)$ and let $f : X \setminus S \rightarrow \mathbb{C}$ be separately meromorphic on X . Does there always exist a function $\hat{f} \in \mathcal{M}(D \times G)$ such that $\hat{f} = f$ on $X \setminus S$?

This note has two purposes. The first one is to give counterexamples to the three open questions above. The second one is to describe the maximal domain to which the function f in Questions 1 and 3 can be meromorphically extended.

The paper is organized as follows. We begin Section 2 by collecting some background of pluripotential theory and introducing some notations. This is necessary to state the results afterwards.

Section 3 provides three counterexamples to the three open questions above.

The subsequent sections are devoted to the proof of a result in the positive direction. More precisely, we describe qualitatively the maximal domain of meromorphic extension of the function f in Questions 1 and 3. Section 4 develops auxiliary tools that will be used in Section 5 to prove the positive result.

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2. Background and statement of the results. We keep the main notation from [4].

Let $N \in \mathbb{N}$, $N \geq 2$, and let $\emptyset \neq A_j \subset D_j \subset \mathbb{C}^{n_j}$, where D_j is a domain, $j = 1, \dots, N$. We define an N -fold cross

$$X = \mathbb{X}(A_1, \dots, A_N; D_1, \dots, D_N) \\ := \bigcup_{j=1}^N A_1 \times \dots \times A_{j-1} \times D_j \times A_{j+1} \times \dots \times A_N \subset \mathbb{C}^{n_1 + \dots + n_N} = \mathbb{C}^n.$$

For an open set $\Omega \subset \mathbb{C}^n$ and $A \subset \Omega$, put

$$h_{A,\Omega} := \sup\{u : u \in \text{PSH}(\Omega), u \leq 1 \text{ on } \Omega, u \leq 0 \text{ on } A\},$$

where $\text{PSH}(\Omega)$ denotes the set of all plurisubharmonic functions on Ω . Put

$$\omega_{A,\Omega} := \lim_{k \rightarrow \infty} h_{A \cap \Omega_k, \Omega_k}^*$$

where $\{\Omega_k\}_{k=1}^\infty$ is a sequence of relatively compact open sets $\Omega_k \subset \Omega_{k+1} \subset \Omega$ with $\bigcup_{k=1}^\infty \Omega_k = \Omega$ (h^* denotes the upper semicontinuous regularization of h). We say that a subset $\emptyset \neq A \subset \mathbb{C}^n$ is *locally pluriregular* if $h_{A \cap \Omega, \Omega}^*(a) = 0$ for any $a \in A$ and for any open neighborhood Ω of a . We say that A is *plurithin* at a point $a \in \mathbb{C}^n$ if either $a \notin \bar{A}$, or $a \in \bar{A}$ and $\limsup_{A \setminus \{a\} \ni z \rightarrow a} u(z) < u(a)$ for a suitable function u plurisubharmonic in a neighborhood of a . For a good background of pluripotential theory, see the books [5] or [1].

For an N -fold cross $X := \mathbb{X}(A_1, \dots, A_N; D_1, \dots, D_N)$ let

$$\widehat{X} := \left\{ (z_1, \dots, z_N) \in D_1 \times \dots \times D_N : \sum_{j=1}^N \omega_{A_j, D_j}(z_j) < 1 \right\}.$$

Suppose that $S_j \subset (A_1 \times \dots \times A_{j-1}) \times (A_{j+1} \times \dots \times A_N)$, $j = 1, \dots, N$. Define the *generalized N -fold cross*

$$T = \mathbb{T}(A_1, \dots, A_N; D_1, \dots, D_N; S_1, \dots, S_N) \\ := \bigcup_{j=1}^N \{(z', z_j, z'') \in (A_1 \times \dots \times A_{j-1}) \times D_j \times (A_{j+1} \times \dots \times A_N) : \\ (z', z'') \notin S_j\}.$$

Let $M \subset T$ be a relatively closed set. We say that a function $f : T \setminus M \rightarrow \mathbb{C}$ is *separately holomorphic* and write $f \in \mathcal{O}_s(T)$ (resp. *separately meromorphic* and write $f \in \mathcal{M}_s(T)$) if for any $j \in \{1, \dots, N\}$ and $(a', a'') \in (A_1 \times \dots \times A_{j-1}) \times (A_{j+1} \times \dots \times A_N) \setminus S_j$ the function $f(a', \cdot, a'')$ can be holomorphically (resp. meromorphically) extended to D_j ⁽¹⁾.

We are now ready to state the results. The following propositions give negative answers to Questions 2, 3 and 1 respectively.

⁽¹⁾ Observe that we have changed a little the definition appearing in [2]–[4].

PROPOSITION A. *For any $n \geq 2$, there is an open dense subset A of E^n which is plurithin at 0 and there exists no non-empty relatively open subset C of a real hypersurface such that $C \subset E^n \setminus A$.*

PROPOSITION B. *Let $D \subset \mathbb{C}^p$, $G \subset \mathbb{C}^q$ ($p, q \geq 2$) be bounded pseudoconvex domains. Then there is a relatively closed set $S \subset D \times G$ with the following properties:*

- (i) $\text{int } S = \emptyset$ and S does not separate domains;
- (ii) let A (resp. B) be the set of all $a \in D$ (resp. $b \in G$) such that $\text{int}_{\mathbb{C}^p} S(a, \cdot) = \emptyset$ (resp. $\text{int}_{\mathbb{C}^q} S(\cdot, b) = \emptyset$) and put $X := \mathbb{X}(A, B; D, G)$; then there exists a function $f : X \setminus S \rightarrow \mathbb{C}$, separately holomorphic on X , such that there is no function $\hat{f} \in \mathcal{M}(D \times G)$ with $\hat{f} = f$ on $X \setminus S$.

PROPOSITION C. *For all $n \geq 3$, there is a relatively closed set $S \subset E^n$ with the following properties:*

- (i) $\text{int } S = \emptyset$ and S does not separate domains;
- (ii) for $1 \leq j \leq n$, let S_j denote the set of all $(a', a'') \in E^{j-1} \times E^{n-j}$ such that $\text{int}_{\mathbb{C}} S(a', \cdot, a'') \neq \emptyset$ and define the n -fold generalized cross $T := \mathbb{T}(E, \dots, E; E, \dots, E; S_1, \dots, S_n)$; then there is a function $f : T \setminus S \rightarrow \mathbb{C}$ which is separately holomorphic on T and there is no function $\hat{f} \in \mathcal{M}(E^n)$ such that $\hat{f} = f$ on $T \setminus S$.

PROBLEM. *Are the answers to Questions 1 and 3 affirmative if the condition on S is sharpened as follows: S does not separate lower-dimensional domains?*

Finally, we state a result in the positive direction.

THEOREM D. *For all $j \in \{1, \dots, N\}$ ($N \geq 2$), let D_j be a pseudoconvex domain in \mathbb{C}^{n_j} and let S be a relatively closed set of $D := D_1 \times \dots \times D_N$ with $\text{int } S = \emptyset$. For $j \in \{1, \dots, N\}$, let S_j denote the set of all $(a', a'') \in (D_1 \times \dots \times D_{j-1}) \times (D_{j+1} \times \dots \times D_N)$ such that $\text{int}_{\mathbb{C}^{n_j}} S(a', \cdot, a'') \neq \emptyset$ and define the N -fold generalized cross $T := \mathbb{T}(D_1, \dots, D_N; D_1, \dots, D_N; S_1, \dots, S_N)$. Let $f : T \setminus S \rightarrow \mathbb{C}$ be a function such that $f \in \mathcal{O}_s(T)$ (resp. $f \in \mathcal{M}_s(T)$).*

- (i) *Then there are an open dense set Ω of D and exactly one function $\hat{f} \in \mathcal{O}(\Omega)$ such that $\hat{f} = f$ on $(T \cap \Omega) \setminus S$.*
- (ii) *In the case where $N = 2$, (i) can be strengthened as follows. Let Ω_j be a relatively compact pseudoconvex subdomain of D_j ($j = 1, 2$). Then there are an open dense set A_j in Ω_j and exactly one function $\hat{f} \in \mathcal{O}(\hat{X})$ (resp. $\hat{f} \in \mathcal{M}(\hat{X})$), where $X := \mathbb{X}(A_1, A_2; \Omega_1, \Omega_2)$, such that $\hat{f} = f$ on $(T \cap \hat{X}) \setminus S$.*

A remark is in order. In contrast with the other usual extension theorems (see [1]–[4] and the references therein), the domain of meromorphic/holomorphic extension of the function f in Theorem D depends on f .

3. Three counterexamples. In what follows we will fix a function $v \in \mathcal{SH}(2E)$ such that $v(0) = 0$ and the complete polar set $\{z \in 2E : v = -\infty\}$ is dense in $2E$. For example one can choose v of the form

$$(3.1) \quad v(z) := \sum_{k=1}^{\infty} \frac{\log(|z - q_k|/4)}{d_k} - \sum_{k=1}^{\infty} \frac{\log(|q_k|/4)}{d_k},$$

where $(\mathbb{Q} + i\mathbb{Q}) \cap 2E^* = \{q_1, q_2, \dots, q_k, \dots\}$, $E^* := E \setminus \{0\}$, and $\{d_k\}_{k=1}^{\infty}$ is any sequence of positive real numbers such that $\sum_{k=1}^{\infty} \frac{\log(|q_k|/4)}{d_k}$ is finite.

For any positive integer $n \geq 2$, define a new function $u \in \text{PSH}((2E)^n)$ and a subset \mathcal{A} of E^n as follows:

$$(3.2) \quad u(z) := \sum_{k=1}^n v(z_k), \quad z = (z_1, \dots, z_n) \in (2E)^n,$$

$$\mathcal{A} = \mathcal{A}_n := \{z \in E^n : u(z) < -1\}.$$

Observe that \mathcal{A} is an open dense subset of E^n because \mathcal{A} contains the set $\{z \in E : v = -\infty\} \times \dots \times \{z \in E : v = -\infty\}$, which is dense in E^n by our construction (3.1) above.

PROPOSITION 3.1. *Let \mathcal{S} be any closed set contained in the closed set $E^n \setminus \mathcal{A}$. Then \mathcal{S} does not separate domains.*

Taking this proposition for granted, we are now able to complete the proof of Proposition A.

Proof of Proposition A. It is clear from (3.1) and (3.2) that the open dense set \mathcal{A} is plurithin at $0 \in E^n$. By Proposition 3.1, the closed set $E^n \setminus \mathcal{A}$ does not separate domains. Therefore this set cannot contain any open subset of a real hypersurface. Thus \mathcal{A} has all the desired properties. ■

Proof of Proposition 3.1. One first observes that

$$\mathcal{S} \subset E^n \setminus \mathcal{A} = \{z \in E^n : u(z) \geq -1\}.$$

For any four vectors $a := (a_1, \dots, a_n)$, $b := (b_1, \dots, b_n)$, $c := (c_1, \dots, c_n)$, $d := (d_1, \dots, d_n)$ in \mathbb{R}^n with the property that $a_k < b_k$ and $c_k < d_k$, $k = 1, \dots, n$, one defines an open cube in \mathbb{C}^n by

$$\Delta = \Delta(a, b, c, d)$$

$$:= \{z \in \mathbb{C}^n : a_k < \text{Re } z_k < b_k, c_k < \text{Im } z_k < d_k, k = 1, \dots, n\}.$$

It is clear that the intersection of two such cubes is either empty or a cube.

One first shows that for any cube $\Delta \subset E^n$ the open set $\Delta \setminus \mathcal{S}$ is connected. Indeed, pick two points $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ in $\Delta \setminus \mathcal{S}$. Since $\{z \in E : v(z) = -\infty\}$ is dense in E , we can choose $z' = (z'_1, \dots, z'_n)$ and $w' = (w'_1, \dots, w'_n)$ in $\Delta \setminus \mathcal{S}$ such that

(i) the segments $\gamma_1(t) := (1 - t)z + tz'$ and $\gamma_3(t) := (1 - t)w + tw'$, $0 \leq t \leq 1$, are contained in $\Delta \setminus \mathcal{S}$;

(ii) z'_1, \dots, z'_n and w'_1, \dots, w'_n are in $\{z \in E : v(z) = -\infty\}$.

Consider now $\gamma_2 : [0, 1] \rightarrow \Delta$ given by

$$\gamma_2(t) := (w'_1, \dots, w'_j, (j + 1 - nt)z'_{j+1} + (nt - j)w'_{j+1}, z'_{j+2}, \dots, z'_n)$$

for $t \in [j/n, (j + 1)/n]$ and $j = 0, \dots, n - 1$. By (3.2) and property (ii) above, $\gamma_2(t) \in \{z \in E^n : u(z) = -\infty\}$ for $t \in [0, 1]$. This implies that $\gamma_2 : [0, 1] \rightarrow \Delta \setminus \mathcal{S}$.

Observe that $\gamma_2(0) = z'$ and $\gamma_2(1) = w'$. By (i), the new path $\gamma : [0, 1] \rightarrow \Delta \setminus \mathcal{S}$ given by

$$\gamma(t) := \begin{cases} \gamma_1(3t), & t \in [0, 1/3], \\ \gamma_2(3t - 1), & t \in [1/3, 2/3], \\ \gamma_3(3t - 2), & t \in [2/3, 1], \end{cases}$$

satisfies $\gamma(0) = z$ and $\gamma(1) = w$, and $\Delta \setminus \mathcal{S}$ is therefore connected.

Now let U be any subdomain of E^n . We wish to show that $U \setminus \mathcal{S}$ is connected. To do this, pick points $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ in $U \setminus \mathcal{S}$. Since U is arcwise connected, there is a continuous function $\gamma : [0, 1] \rightarrow U$ such that $\gamma(0) = z$ and $\gamma(1) = w$.

By the Heine–Borel Theorem, the compact set $\mathcal{L} := \gamma([0, 1])$ can be covered by a finite number of cubes Δ_l ($1 \leq l \leq N$) with $\Delta_l \subset U$ and $\Delta_l \cap \mathcal{L} \neq \emptyset$. Since the path \mathcal{L} is connected, the union $\bigcup_{l=1}^N \Delta_l$ is also connected.

Suppose without loss of generality that $z \in \Delta_1$ and $w \in \Delta_N$. From the discussion above, if $\Delta_1 \cap \Delta_2 \neq \emptyset$ then $(\Delta_1 \setminus \mathcal{S}) \cap (\Delta_2 \setminus \mathcal{S}) = (\Delta_1 \cap \Delta_2) \setminus \mathcal{S}$ is connected, and hence $(\Delta_1 \cup \Delta_2) \setminus \mathcal{S}$ is also connected. Repeating this argument at most N times and using the connectivity of $\bigcup_{l=1}^N \Delta_l$, we finally conclude that $\bigcup_{l=1}^N \Delta_l \setminus \mathcal{S} (\subset U \setminus \mathcal{S})$ is also connected. This completes the proof. ■

COROLLARY 3.2. (i) *If $\mathcal{S}_1, \dots, \mathcal{S}_N$ are relatively closed subsets of E^n which do not separate domains, then the union $\bigcup_{l=1}^N \mathcal{S}_l$ does not separate domains either.*

(ii) *Let \mathcal{A}, \mathcal{S} be as in Proposition 3.1. Then for any closed sets F_1 in \mathbb{C}^p and F_2 in \mathbb{C}^q ($p, q \geq 0$), the closed set $F_1 \times \mathcal{S} \times F_2$ does not separate domains in $\mathbb{C}^p \times E^n \times \mathbb{C}^q$.*

Proof. To prove (i), let U be any subdomain of E^n . Since $U \setminus (\bigcup_{l=1}^N \mathcal{S}_l) = (\dots (U \setminus \mathcal{S}_1) \dots \setminus \mathcal{S}_N)$, part (i) follows from the hypothesis on the \mathcal{S}_l 's.

To prove part (ii), consider any subdomain U of $\mathbb{C}^p \times E^n \times \mathbb{C}^q$ and let $(z_1, w_1, t_1), (z_2, w_2, t_2)$ be two points in $U \setminus (F_1 \times \mathcal{S} \times F_2)$. Since \mathcal{A} is an open dense subset of E^n , $\text{int}(F_1 \times \mathcal{S} \times F_2) = \emptyset$, and therefore we can employ the

compactness argument that we already used in the proof of Proposition 3.1. Consequently, one is reduced to the case where U is a cube in \mathbb{C}^{p+n+q} .

Another reduction is in order. Since $U \setminus (F_1 \times \mathcal{S} \times F_2)$ is open and \mathcal{A} is dense in E^n , by replacing w_1 (resp. w_2) by w'_1 (resp. w'_2) close to w_1 (resp. w_2), we may suppose that $w_1, w_2 \in E^n \setminus \mathcal{S}$.

Write the cube U as the product $\Delta_1 \times \Delta_2 \times \Delta_3$, where Δ_1 (resp. Δ_2 and Δ_3) is a cube in \mathbb{C}^p (resp. \mathbb{C}^n and \mathbb{C}^q). By Proposition 3.1, there is a continuous path $\gamma_2 : [0, 1] \rightarrow \Delta_2 \setminus \mathcal{S}$ such that $\gamma_2(0) = w_1$ and $\gamma_2(1) = w_2$.

We now consider the path $\gamma : [0, 1] \rightarrow \Delta_1 \times \Delta_2 \times \Delta_3 \setminus \mathcal{S}$ where $\gamma(t) := (\gamma_1(t), \gamma_2(t), \gamma_3(t))$ and $\gamma_1(t) := (1 - t)z_1 + tz_2$, $\gamma_3(t) := (1 - t)t_1 + tt_2$, $t \in [0, 1]$. It is easy to see that $\gamma(0) = (z_1, w_1, t_1)$ and $\gamma(1) = (z_2, w_2, t_2)$, which finishes the proof. ■

The following two lemmas will be crucial for the proof of Propositions B and C.

LEMMA 3.3. *For an open set $\Omega \subset \mathbb{C}^n$ and $A \subset \Omega$, we have either $\omega_{A, \Omega} \equiv 0$ or $\sup_{\Omega} \omega_{A, \Omega} = 1$.*

Proof. We first prove the lemma in the case where Ω is bounded. Suppose in order to get a contradiction that $\sup_{\Omega} h^*_{A, \Omega} = M$ with $0 < M < 1$. By the definition of $h^*_{A, \Omega}$, it follows that

$$\begin{aligned} \{u : u \in \text{PSH}(\Omega), u \leq 1 \text{ on } \Omega, u \leq 0 \text{ on } A\} \\ = \{u : u \in \text{PSH}(\Omega), u \leq M \text{ on } \Omega, u \leq 0 \text{ on } A\}. \end{aligned}$$

Therefore, $h^*_{A, \Omega}(z) \leq M h^*_{A, \Omega} < h^*_{A, \Omega}$ for any $z \in \Omega$ with $h^*_{A, \Omega}(z) > 0$, and we obtain the desired contradiction.

The general case is analogous using the definition of $\omega_{A, \Omega}$. ■

LEMMA 3.4. *Let $\Omega_1 \subsetneq \Omega_2$ be two domains of \mathbb{C}^n such that Ω_2 is pseudoconvex. Assume that there is an upper bounded function $\phi \in \text{PSH}(\Omega_2)$ satisfying $\Omega_1 = \{z \in \Omega_2 : \phi(z) < 0\}$. Then there is a function $f \in \mathcal{O}(\Omega_1)$ such that there is no function $\widehat{f} \in \mathcal{M}(\Omega_2)$ with $\widehat{f} = f$ on Ω_1 .*

Proof. It is clear from the hypothesis that Ω_1 is also pseudoconvex. Let $\partial\Omega_1$ be the boundary of Ω_1 in Ω_2 and let S be a countable dense subset of $\partial\Omega_1$. It is a classical fact that there is a function $f \in \mathcal{O}(\Omega_1)$ such that

$$(3.3) \quad \lim_{z \in \Omega_1, z \rightarrow w} |f(z)| = \infty, \quad w \in S.$$

We will show that this is the desired function. Indeed, suppose in order to get a contradiction that there is a function $\widehat{f} \in \mathcal{M}(\Omega_2)$ satisfying $\widehat{f} = f$ on Ω_1 . Because of (3.3), $S \cap \Omega_2$ and then $\partial\Omega_1 \cap \Omega_2$ are contained in the pole set of \widehat{f} (i.e. the union of the set of all poles of \widehat{f} and the set of all indeterminacy points of \widehat{f}). Therefore, for any point $w \in \partial\Omega_1 \cap \Omega_2$, there is a small open neighborhood U of w and a complex analytic subset C

of codimension one such that $U \setminus C \subset \Omega_1$. Since $\phi \in \text{PSH}(U)$ is upper bounded, $\phi(w) = \limsup_{z \in U \setminus C, z \rightarrow w} \phi(z) = \phi(w) \leq 0$ for all $w \in \partial\Omega_1$. Since $\Omega_1 \subsetneq \Omega_2$, ϕ is non-constant and therefore $\phi(w) < 0$ for all $w \in \partial\Omega_1$, which is a contradiction. ■

We are now ready to prove Propositions B and C.

Proof of Proposition B. Suppose, without loss of generality, that $D = E^p$ and $G = E^q$. The general case is analogous. Let F_p (resp. F_q) be any closed ball contained in the open set \mathcal{A}_p (resp. \mathcal{A}_q). Define the relatively closed set S by

$$(3.4) \quad S := (E^p \setminus \mathcal{A}_p) \times F_q \cup F_p \times (E^q \setminus \mathcal{A}_q).$$

We now check the properties (i) and (ii) of Proposition B. First, $\text{int } S = \emptyset$ because \mathcal{A}_p (resp. \mathcal{A}_q) is open dense set in E^p (resp. E^q). Second, by Proposition 3.1 and Corollary 3.2(ii), the two relatively closed sets $(E^p \setminus \mathcal{A}_p) \times F_q$ and $F_p \times (E^q \setminus \mathcal{A}_q)$ do not separate domains. By Corollary 3.2(i), the union S also enjoys this property. Thus S satisfies (i).

Using (3.4), a direct computation entails that $A = \mathcal{A}_p$ and $B = \mathcal{A}_q$ and A, B are open, in particular they are locally pluriregular.

By the classical Cross Theorem (see for instance [7] or [1]), the envelope of holomorphy of X is given by

$$\widehat{X} := \{(z, w) \in E^p \times E^q : \omega_{A, E^p}(z) + \omega_{B, E^q}(w) < 1\}.$$

We now show that $h_{\mathcal{A}_n, E^n}^*(0) > 0$ for $n \geq 2$. Indeed, let $M := \sup_{E^n} u$, where u is defined in (3.2). Observe that $M > 0$ since $u(0) = 0$. Consider the function $\tilde{u} \in \text{PSH}(E^n)$ given by

$$\tilde{u}(z) := \frac{u(z) - M}{M + \frac{1}{2}} + 1 \quad \text{for } z \in E^n.$$

It can be easily checked that $\tilde{u}(z) \leq 1$ on E^n and $\tilde{u}(z) \leq 0$ on \mathcal{A}_n . Thus $\tilde{u}(0) \leq h_{\mathcal{A}_n, E^n}^*(0)$. On the other hand, $\tilde{u}(0) = 1/(2M + 1) > 0$. Hence our assertion above follows.

We next show that $\widehat{X} \subsetneq E^p \times E^q$. Indeed, we have

$$\{w \in E^q : (0, w) \in \widehat{X}\} \subset \{w \in E^q : h_{\mathcal{A}_q, E^q}^*(w) < 1 - h_{\mathcal{A}_p, E^p}^*(0)\}.$$

Since $h_{\mathcal{A}_q, E^q}^*(0) > 0$ and $h_{\mathcal{A}_p, E^p}^*(0) > 0$, Lemma 3.3 applies, and consequently, the latter set is strictly contained in E^q . This proves our assertion.

We are now ready to complete the proof. By Lemma 3.4, there is a holomorphic function f in \widehat{X} which cannot be meromorphically extended to $E^p \times E^q$. Therefore, there is no meromorphic function $\hat{f} \in \mathcal{M}(E^p \times E^q)$ such that $\hat{f} = f$ on

$$X \setminus S = ((\mathcal{A}_p \setminus F_p) \times E^q) \cup (\mathcal{A}_p \times \mathcal{A}_q) \cup (E^p \times (\mathcal{A}_q \setminus F_q)),$$

which is a set of unicity for meromorphic functions. The proof is thereby finished. ■

Proof of Proposition C. In order to simplify the notation, we only consider the case $n = 3$, the general case $n > 3$ is analogous. Let \mathcal{B} be the following open dense subset of E :

$$\mathcal{B} := \{z \in E : v(z) < -1/2\},$$

where v is given by (3.1). Then by virtue of (3.2), it can be checked that $(E \setminus \mathcal{B}) \times (E \setminus \mathcal{B}) \subset E^2 \setminus \mathcal{A}_2$. Fix any closed ball F contained in the open set \mathcal{B} . Next apply Proposition 3.1 and Corollary 3.2 to the relatively closed set $\mathcal{S} := (E \setminus \mathcal{B}) \times (E \setminus \mathcal{B})$. Consequently, the set

$$(3.5) \quad S := (F \times (E \setminus \mathcal{B}) \times (E \setminus \mathcal{B})) \cup ((E \setminus \mathcal{B}) \times F \times (E \setminus \mathcal{B})) \\ \cup ((E \setminus \mathcal{B}) \times (E \setminus \mathcal{B}) \times F)$$

does not separate domains in E^3 . Moreover, since \mathcal{B} is an open dense subset of E , we see that $\text{int } S = \emptyset$ and S is relatively closed. Hence S satisfies property (i).

To verify (ii), one first computes the following sets using (3.5):

$$(3.6) \quad S_1 = S_2 = S_3 = (E \setminus \mathcal{B}) \times (E \setminus \mathcal{B}), \\ T = (\mathcal{B} \times E \times E) \cup (E \times \mathcal{B} \times E) \cup (E \times E \times \mathcal{B}).$$

Next, by the product property for the relative extremal function [6], we have $h_{\mathcal{B} \times \mathcal{B}, E^2}^*(0) = h_{\mathcal{B}, E}^*(0)$. Since $\mathcal{B} \times \mathcal{B} \subset \mathcal{A}_2$ and we have shown in Proposition B that $h_{\mathcal{A}_2, E^2}^*(0) > 0$, it follows that $h_{\mathcal{B}, E}^*(0) > 0$.

Consider now the domain of holomorphy

$$(3.7) \quad \Omega := \{(z, w, t) \in E^3 : h_{\mathcal{B}, E}^*(z) + h_{\mathcal{B}, E}^*(w) + h_{\mathcal{B}, E}^*(t) < 2\}.$$

Since \mathcal{B} is open and therefore locally pluriregular, it can be proved using Lemma 5 in [2] that Ω is a domain. Moreover it can be easily checked that $T \subset \Omega$ using (3.6) and (3.7).

We now prove that $\Omega \subsetneq E^3$. Indeed, since $h_{\mathcal{B}, E}^*(0) > 0$, by Lemma 3.3 there are $z, w \in E$ such that $h_{\mathcal{B}, E}^*(z) > 2/3$ and $h_{\mathcal{B}, E}^*(w) > 2/3$. Then the fiber

$$\{t \in E : (z, w, t) \in \Omega\} \subset \{t \in E : h_{\mathcal{B}, E}^*(t) < 2/3\}.$$

Another application of Lemma 3.3 shows that the latter set is strictly contained in E . This proves our assertion.

We can now complete the proof. By Lemma 3.4, there is a holomorphic function f in Ω which cannot be meromorphically extended to E^n . Therefore, there is no meromorphic function $\hat{f} \in \mathcal{M}(E^n)$ such that $\hat{f} = f$ on $T \setminus S$, a set of unicity for meromorphic functions. Hence, the proof is finished. ■

4. Auxiliary results. Let S be a subset of an open set $D \subset \mathbb{C}^n$. Then S is said to be of *Baire category I* if S is contained in a countable union of relatively closed sets in D with empty interior. Otherwise, S is said to be of *Baire category II*.

The following lemma is very useful.

LEMMA 4.1. For $j \in \{1, \dots, M\}$ and $M \geq 2$, let Ω_j be a domain in \mathbb{C}^{m_j} and let S be a relatively closed subset of $\Omega_1 \times \dots \times \Omega_M$ with $\text{int } S = \emptyset$. For $a_j \in \Omega_j$, $j \in \{3, \dots, M\}$, let $S(a_3, \dots, a_M)$ denote the set of all $a_2 \in \Omega_2$ such that $\text{int}_{\mathbb{C}^{m_1}} S(\cdot, a_2, a_3, \dots, a_M) = \emptyset$. For $j \in \{4, \dots, M\}$, let $S(a_j, \dots, a_M)$ denote the set of all $a_{j-1} \in \Omega_{j-1}$ such that $\Omega_{j-2} \setminus S(a_{j-1}, a_j, \dots, a_M)$ is of Baire category I, and finally let \mathcal{S} denote the set of all $a_M \in \Omega_M$ such that $\Omega_{M-1} \setminus S(a_M)$ is of Baire category I. Then $\Omega_M \setminus \mathcal{S}$ is of Baire category I.

Proof. For $j \in \{1, \dots, M\}$ let $(\mathbb{Q} + i\mathbb{Q})^{m_j} = \{q_1^j, \dots, q_n^j, \dots\}$ and $\delta_n := 1/n$, $n \in \mathbb{N}$. For $q \in \Omega_j$ and $r > 0$, let $\Delta_q(r)$ denote the polydisc in \mathbb{C}^{m_j} with center q and multi-radius (r, \dots, r) .

Suppose to get a contradiction that $\Omega_M \setminus \mathcal{S}$ is of Baire category II. Then for all $a_M \in \Omega_M \setminus \mathcal{S}$, $\Omega_{M-1} \setminus S(a_M)$ is of Baire category II. Therefore, for $j = M-1, \dots, 3$ and any $a_j \in \Omega_j \setminus S(a_{j+1}, \dots, a_M)$, the set $\Omega_{j-1} \setminus S(a_j, \dots, a_M)$ is of Baire category II. Put

$$S_n(a_3, \dots, a_M) := \{a_2 \in \Omega_2 : S(\cdot, a_2, a_3, \dots, a_M) \supset \Delta_{q_n^1}(\delta_n)\}.$$

Since S is relatively closed, $S_n(a_3, \dots, a_M)$ is also relatively closed in Ω_2 . Moreover, from the definition of $S(a_3, \dots, a_M)$, we have the identity

$$\Omega_2 \setminus S(a_3, \dots, a_M) = \bigcup_{n=1}^{\infty} S_n(a_3, \dots, a_M).$$

Since it is shown in the above discussion that $\Omega_2 \setminus S(a_3, \dots, a_M)$ is of Baire category II in Ω_2 , we can apply the Baire Theorem to the right side of the latter identity. Consequently, there exist $n_1, n_2 \in \mathbb{N}$ such that $S_{n_1}(a_3, \dots, a_M) \supset \Delta_{q_{n_2}^2}(\delta_{n_2})$. This implies that $S(\cdot, \cdot, a_3, \dots, a_M) \supset \Delta_{q_{n_1}^1}(\delta_{n_1}) \times \Delta_{q_{n_2}^2}(\delta_{n_2})$.

Now define inductively, for $j = 2, \dots, M-1$ and $n_1, \dots, n_j \in \mathbb{N}$,

$$\begin{aligned} & S_{n_1, \dots, n_j}(a_{j+2}, \dots, a_M) \\ & := \{a_{j+1} \in \Omega_{j+1} : S(\dots, a_{j+1}, \dots, a_M) \supset \Delta_{q_{n_1}^1}(\delta_{n_1}) \times \dots \times \Delta_{q_{n_j}^j}(\delta_{n_j})\}. \end{aligned}$$

Since S is relatively closed, $S_{n_1, \dots, n_j}(a_{j+2}, \dots, a_M)$ is also relatively closed. Moreover, it can be checked that

$$\Omega_{j+1} \setminus S(a_{j+2}, \dots, a_M) \subset \bigcup_{n_1, \dots, n_j=1}^{\infty} S_{n_1, \dots, n_j}(a_{j+2}, \dots, a_M).$$

By the Baire Theorem again, it follows that there are $n_1, \dots, n_{j+1} \in \mathbb{N}$ such that $S_{n_1, \dots, n_j}(a_{j+2}, \dots, a_M) \supset \Delta_{q_{n_{j+1}}^{j+1}}(\delta_{n_{j+1}})$, and hence

$$S(\dots, a_{j+2}, \dots, a_M) \supset \Delta_{q_{n_1}^1}(\delta_{n_1}) \times \dots \times \Delta_{q_{n_{j+1}}^{j+1}}(\delta_{n_{j+1}}).$$

Finally, we deduce for $j = M - 1$ that $\text{int } S \neq \emptyset$, which contradicts the hypothesis. Hence, the proof is complete. ■

REMARK 4.2. If we apply Lemma 4.1 to the case where $\Omega_1 := D_j$ and $\Omega_2 := (D_1 \times \dots \times D_{j-1}) \times (D_{j+1} \times \dots \times D_N)$ then, for each $j \in \{1, \dots, N\}$, the set S_j in the statement of Theorem D is of Baire category I. In particular, the set $\Omega \setminus ((T \setminus S) \cap \Omega)$ is of Baire category I for all open sets $\Omega \subset D$.

LEMMA 4.3. *Let $U \subset \mathbb{C}^p$ and $V \subset \mathbb{C}^q$ be two pseudoconvex domains. Consider four sets $C \subset A \subset U$ and $D \subset B \subset V$ such that $\overline{C} = \overline{A}$, $\overline{D} = \overline{B}$ and $\overline{A}, \overline{B}$ are locally pluriregular. Put $X := \mathbb{X}(A, B; U, V)$ and $\widehat{X} := \widehat{\mathbb{X}}(\overline{A}, \overline{B}; U, V)$. Assume $f \in \mathcal{O}_s(X)$ and there is a finite constant K such that for all $c \in C$ and $d \in D$,*

$$\sup_V |f(c, \cdot)| < K \quad \text{and} \quad \sup_U |f(\cdot, d)| < K.$$

Then there exists a unique function $\widehat{f} \in \mathcal{O}(\widehat{X})$ such that $\widehat{f} = f$ on X .

Proof. From the hypothesis on the boundedness of f , it follows that the two families $\{f(c, \cdot) : c \in C\}$ and $\{f(\cdot, d) : d \in D\}$ are normal. We now define two functions f_1 on $\overline{A} \times V$ and f_2 on $U \times \overline{B}$ as follows.

For any $z \in \overline{A}$, choose a sequence $(c_n)_{n=1}^\infty \subset C$ such that $\lim_{n \rightarrow \infty} c_n = z$ and the sequence $(f(c_n, \cdot))_{n=1}^\infty$ converges uniformly on compact subsets of V . We let

$$f_1(z, w) := \lim_{n \rightarrow \infty} f(c_n, w) \quad \text{for all } w \in V.$$

Similarly, for any $w \in \overline{B}$, choose a sequence $(d_n)_{n=1}^\infty \subset D$ such that $\lim_{n \rightarrow \infty} d_n = w$ and the sequence $(f(\cdot, d_n))_{n=1}^\infty$ converges uniformly on compact subsets of U . We let

$$f_2(z, w) := \lim_{n \rightarrow \infty} f(z, d_n) \quad \text{for all } z \in U.$$

We first check that f_1 and f_2 are well defined. Indeed, it suffices to verify this for f_1 since the same argument also applies to f_2 . Let $(c'_n)_{n=1}^\infty \subset C$ be another sequence such that $\lim_{n \rightarrow \infty} c'_n = z$ and the sequence $(f(c'_n, \cdot))_{n=1}^\infty$ converges uniformly on compact subsets of V . Since for all $b \in B$,

$$\lim_{n \rightarrow \infty} f(c_n, b) = f(z, b) = \lim_{n \rightarrow \infty} f(c'_n, b),$$

and since B is a set of unicity for holomorphic functions on V , our claim follows.

One next verifies that $f_1 = f_2$ on $\bar{A} \times \bar{B}$. Indeed, let $z \in \bar{A}$, $w \in \bar{B}$ and let $(c_n)_{n=1}^\infty \subset C$, $(d_n)_{n=1}^\infty \subset D$ be as above. Then clearly, we have

$$f_1(z, w) = \lim_{n \rightarrow \infty} f(c_n, d_n) = f_2(z, w).$$

We are now able to define a function \tilde{f} on $\mathbb{X}(\bar{A}, \bar{B}; U, V)$ by $f = f_1$ on $\bar{A} \times V$ and $f = f_2$ on $U \times \bar{B}$. It follows from the construction of f_1 and f_2 that $\tilde{f} \in \mathcal{O}_s(\mathbb{X}(\bar{A}, \bar{B}; U, V))$.

One next checks that $\tilde{f} = f$ on X . Indeed, since for each $a \in A$, $f(a, \cdot)$ and $\tilde{f}(a, \cdot)$ are holomorphic, it suffices to verify that $\tilde{f}(a, d) = f(a, d)$. But the latter equality follows easily from the definition of f_1 and the hypothesis.

Finally, one applies the classical Cross Theorem (see for instance [7], [1]) to $\tilde{f} \in \mathcal{O}_s(\mathbb{X}(\bar{A}, \bar{B}; U, V))$; thus the existence of \hat{f} follows. The unicity of \hat{f} is also clear. ■

LEMMA 4.4 (Rothstein type theorem, cf. [8]). *Let $f \in \mathcal{O}(E^p \times E^q)$. Assume that $A \subset E^p$ is such that for all open subsets $U \subset E^p$, $A \cap U$ is of Baire category II. Let $G \subset \mathbb{C}^q$ be a domain such that $E^q \subset G$ and assume that for all $a \in A$, the function $f(a, \cdot)$ extends meromorphically to $\tilde{f}(a, \cdot) \in \mathcal{M}(G)$. Then for any relatively compact subdomain $\tilde{G} \subset G$, there are an open dense set $\mathcal{A} \subset E^p$ and a function $\tilde{f} \in \mathcal{M}(\Omega)$, where $\Omega := E^p \times E^q \cup \mathcal{A} \times \tilde{G}$, such that $\tilde{f} = f$ on $E^p \times E^q$.*

Sketch of proof. (1) *The case where $G := \Delta_0(R)$ ($R > 1$).* Argue as in the proof of Rothstein’s theorem given in [10].

(2) *The general case.* Fix an $a \in E^p$ and $r > 0$. Let B denote the set of all $b \in G$ such that there exist $0 < r_b < r$, an open dense subset A_b of $\Delta_a(r_b)$ and $f_b \in \mathcal{M}(A_b \times \Delta_b(r_b))$ such that $f_b(\alpha, \cdot) = \tilde{f}(\alpha, \cdot)$ on $\Delta_b(r_b)$ for all $\alpha \in A \cap A_b$.

Obviously, B is open. Using case (1) and the hypothesis on A , one can show that B is closed in G . Thus $B = G$. Moreover, one can also show that if $A_b \cap A_{b'} \neq \emptyset$ and $\Delta_b(r_b) \cap \Delta_{b'}(r_{b'}) \neq \emptyset$, then $f_b = f_{b'}$ on $(A_b \cap A_{b'}) \times (\Delta_b(r_b) \cap \Delta_{b'}(r_{b'}))$. Therefore, using the hypothesis that \tilde{G} is relatively compact, we see that for any $a \in E^p$ and any $r > 0$, there is an open set $\mathcal{A}_{a,r} \subset \Delta_a(r)$ and $f_{a,r} \in \mathcal{M}(\mathcal{A}_{a,r} \times \tilde{G})$ such that $f_{a,r}(\alpha, \cdot) = \tilde{f}(\alpha, \cdot)$ on \tilde{G} for all $\alpha \in A \cap \mathcal{A}_{a,r}$.

Finally, let $\mathcal{A} := \bigcup_{a \in E^p, r > 0} \mathcal{A}_{a,r}$. This open set is clearly dense in E^p . By gluing the functions $f_{a,r}$ together, we obtain the desired meromorphic extension $\tilde{f} \in \mathcal{M}(\Omega)$; so the proof of the lemma is complete. ■

5. Proof of Theorem D. We will only give the proof of Theorem D for the case where f is separately meromorphic. Since the case where f is separately holomorphic is quite similar and in some sense simpler, it is left to the reader.

Proof of (ii). Put

$$(5.1) \quad \mathcal{A}_j := \{z_j \in D_j : \text{int}_{\mathbb{C}^{n_j}} S(z_j, \cdot) = \emptyset\} \quad \text{for } j = 1, 2.$$

By Lemma 4.1, $\widetilde{D_j} \setminus \mathcal{A}_j$ is of Baire category I. For $a_j \in \mathcal{A}_j$ ($j = 1, 2$), let $f(a_1, \cdot)$ (resp. $f(\cdot, a_2)$) denote the meromorphic extension of $f(a_1, \cdot)$ (resp. $f(\cdot, a_2)$) to D_2 (resp. to D_1).

Let $\mathcal{U} \subset D_1$, $\mathcal{V} \subset D_2$ be arbitrary open sets. For a relatively compact pseudoconvex subdomain V of \mathcal{V} and for a positive number K , let $Q_{V,K}^1$ denote the set of $a_1 \in \mathcal{A}_1 \cap \mathcal{U}$ such that $\sup_V |\widetilde{f(a_1, \cdot)}| \leq K$ (and thus $\widetilde{f(a_1, \cdot)} \in \mathcal{O}(V)$). By (5.1) and the hypothesis, a countable number of the $Q_{V,K}^1$ cover $\mathcal{A}_1 \cap \mathcal{U}$. Since the latter set is of Baire category II, we can choose V, K_1 such that the closure $\overline{Q_{V,K_1}^1}$ contains a polydisc $\Delta_1 \subset \mathcal{U}$ and $Q_{V,K_1}^1 \cap \Delta_1$ is of Baire category II in Δ_1 .

For a relatively compact pseudoconvex subdomain U of Δ_1 and for a positive number K , we denote by $Q_{U,K}^2$ the set of $a_2 \in \mathcal{A}_2 \cap V$ such that $\sup_U |\widetilde{f(\cdot, a_2)}| \leq K$ (and thus $\widetilde{f(\cdot, a_2)} \in \mathcal{O}(U)$). By (5.1) and the hypothesis, a countable number of the $Q_{U,K}^2$ cover $\mathcal{A}_2 \cap V$. Since the latter set is of Baire category II, we can choose U, K_2 such that $\overline{Q_{U,K_2}^2}$ contains a polydisc $\Delta_2 \subset V$ and $Q_{U,K_2}^2 \cap \Delta_2$ is of Baire category II in Δ_2 .

Now let $K := \max\{K_1, K_2\}$, $A := \mathcal{A}_1 \cap U$, $C := Q_{V,K}^1 \cap U$, $B := \mathcal{A}_2 \cap \Delta_2$, $D := Q_{V,K}^2 \cap \Delta_2$. Then it is easy to see that $\overline{A} = \overline{C} = U$ and $\overline{B} = \overline{D} = \Delta_2$. Moreover, all other hypotheses of Lemma 4.3 are satisfied. Consequently, an application of this lemma gives the following.

Let $\mathcal{U} \subset D_1$, $\mathcal{V} \subset D_2$ be arbitrary open sets. Then there are a polydisc $\Delta_a(r) \subset \mathcal{U} \times \mathcal{V}$ and a function $\widehat{f} \in \mathcal{O}(\Delta_a(r))$ such that $\widehat{f} = f$ on $(T \setminus S) \cap \Delta_a(r)$.

Write $a = (a_1, a_2) \in D_1 \times D_2$. Since the set $\mathcal{A}_j \cap \Delta_{a_j}(r)$ is of Baire category II, by replacing $\Delta_{a_j}(r)$ with a smaller polydisc we see that this set satisfies the hypothesis of Lemma 4.4. Consequently, an application of that lemma gives $f_a^1 \in \mathcal{M}(\Delta_{a_1}(r) \times \Omega_2)$ and $f_a^2 \in \mathcal{M}(\Omega_1 \times \Delta_{a_2}(r))$ which coincide with f on $(T \setminus S) \cap \Delta_a(r)$. Moreover, one sees that the function $f_{\mathcal{U},\mathcal{V}}$ given by

$$f_{\mathcal{U},\mathcal{V}} := f_a^1 \quad \text{on } \Delta_{a_1}(r) \times \Omega_2, \quad f_{\mathcal{U},\mathcal{V}} := f_a^2 \quad \text{on } \Omega_1 \times \Delta_{a_2}(r),$$

is well defined, meromorphic on the cross $X_{\mathcal{U},\mathcal{V}} = \mathbb{X}(\Delta_{a_1}(r), \Delta_{a_2}(r); \Omega_1, \Omega_2)$, and $f_{\mathcal{U},\mathcal{V}} = f$ on $(T \setminus S) \cap X_{\mathcal{U},\mathcal{V}}$.

Using Remark 4.2, one can also prove the following. If $\mathcal{U}' \subset D_1$, $\mathcal{V}' \subset D_2$ are arbitrary open sets and $f_{\mathcal{U}',\mathcal{V}'}$ is the corresponding meromorphic function defined on the corresponding cross $X_{\mathcal{U}',\mathcal{V}'}$, then $f_{\mathcal{U},\mathcal{V}} = f_{\mathcal{U}',\mathcal{V}'}$ on $X_{\mathcal{U},\mathcal{V}} \cap X_{\mathcal{U}',\mathcal{V}'}$.

Let $A_j := \bigcup_{\mathcal{U} \subset \Omega_1, \mathcal{V} \subset \Omega_2} \Delta_{a_j}(r)$ for $j = 1, 2$. It is clear that A_j is an open dense set in Ω_j . Then gluing all $f_{\mathcal{U}, \mathcal{V}}$ together, we obtain a function \widehat{f} meromorphic on $X := \mathbb{X}(A_1, A_2; \Omega_1, \Omega_2)$ satisfying $\widehat{f} = f$ on $(T \setminus S) \cap X$. Finally, one applies Theorem 1.3 of [4] to \widehat{f} , and the conclusion of (ii) follows.

Proof of (i). In what follows, Σ_M will denote the group of permutations of $\{1, \dots, M\}$. Moreover, for any $\sigma \in \Sigma_M$ and under the hypothesis and in the notation of Lemma 4.1, we define

$$S^\sigma := \{z^\sigma : z \in S\} \quad \text{and} \quad \Omega^\sigma := \Omega_{\sigma(1)} \times \dots \times \Omega_{\sigma(M)},$$

where

$$z^\sigma := (z_{\sigma(1)}, \dots, z_{\sigma(M)}), \quad z \in \Omega = \Omega_1 \times \dots \times \Omega_M.$$

If in the statement of Lemma 4.1, one replaces S by S^σ and Ω by Ω^σ , then one obtains $\mathcal{S}^\sigma, \mathcal{S}^\sigma(a_{\sigma(N)}), \dots, \mathcal{S}^\sigma(a_{\sigma(3)}, \dots, a_{\sigma(N)})$. The proof will be divided into three steps.

STEP 1: $N = 2$. By (ii), for each pair of relatively compact pseudoconvex subdomains $\Omega_j \subset D_j$ ($j = 1, 2$) we obtain a polydisc $\Delta_{\Omega_1, \Omega_2} \subset \Omega_1 \times \Omega_2$ and a function $f_{\Omega_1, \Omega_2} \in \mathcal{O}(\Delta_{\Omega_1, \Omega_2})$ such that $f = f_{\Omega_1, \Omega_2}$ on $(\Delta_{\Omega_1, \Omega_2} \cap T) \setminus S$. A routine identity argument shows that any two functions f_{Ω_1, Ω_2} coincide on the intersection of their domains of definition. Gluing f_{Ω_1, Ω_2} , we obtain the desired function $\widehat{f} \in \mathcal{O}(\bigcup \Delta_{\Omega_1, \Omega_2})$.

STEP 2: $N = 3$. Consider the following elements of Σ_3 :

$$\begin{aligned} \sigma_1 &:= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, & \sigma_2 &:= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \\ \sigma_3 &:= \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, & \sigma_4 &:= \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}. \end{aligned}$$

Fix any subdomain $\Omega_1 \times \Omega_2 \times \Omega_3 \subset D$ and pick any $a_3 \in \mathcal{S}^{\sigma_1} \cap \mathcal{S}^{\sigma_2}$. Then by the definition, $\Omega_1 \setminus \mathcal{S}^{\sigma_1}(a_3)$ (resp. $\Omega_2 \setminus \mathcal{S}^{\sigma_2}(a_3)$) is of Baire category I in Ω_1 (resp. Ω_2).

Also, for any $a_1 \in \mathcal{S}^{\sigma_1}(a_3) \cap \mathcal{S}^{\sigma_3}$, we have $\text{int } S(a_1, \cdot, a_3) = \emptyset$ and the set $\Omega_2 \setminus \{a_2 \in \Omega_2 : \text{int } S(a_1, a_2, \cdot) = \emptyset\}$ is of Baire category I.

Similarly, for any $a_2 \in \mathcal{S}^{\sigma_2}(a_3) \cap \mathcal{S}^{\sigma_4}$, we have $\text{int } S(\cdot, a_2, a_3) = \emptyset$ and the set $\Omega_1 \setminus \{a_1 \in \Omega_1 : \text{int } S(a_1, a_2, \cdot) = \emptyset\}$ is of Baire category I.

Thus f is well defined on the union X of the following two subsets of $\Omega_1 \times \Omega_2 \times \{a_3\}$:

$$(5.2) \quad \{(z_1, z_2, a_3) : z_1 \in \mathcal{S}^{\sigma_1}(a_3) \cap \mathcal{S}^{\sigma_3}, z_2 \in \mathcal{S}^{\sigma_3}(z_1) \cap \mathcal{S}^{\sigma_2}(a_3)\},$$

$$(5.3) \quad \{(z_1, z_2, a_3) : z_2 \in \mathcal{S}^{\sigma_2}(a_3) \cap \mathcal{S}^{\sigma_4}, z_1 \in \mathcal{S}^{\sigma_4}(z_2) \cap \mathcal{S}^{\sigma_1}(a_3)\}.$$

Observe that by the definition in Lemma 4.1, $\Omega_1 \setminus (S^{\sigma_4}(z_2) \cap S^{\sigma_1}(a_3))$ (resp. $\Omega_2 \setminus (S^{\sigma_3}(z_1) \cap S^{\sigma_2}(a_3))$) is of Baire category I in Ω_1 (resp. Ω_2). By virtue of (5.2)–(5.3), the same conclusion also holds for the fibers $X(z_1, \cdot, a_3)$ and $X(\cdot, z_2, a_3)$, $z_1 \in S^{\sigma_1}(a_3) \cap S^{\sigma_3}$ (resp. $z_2 \in S^{\sigma_2}(a_3) \cap S^{\sigma_4}$).

Let $\mathcal{U}_j \subset \Omega_j$ ($j = 1, 2$) be an arbitrary open subset. If $\Delta := \Delta_q(r)$ is a polydisc, then we denote by $\lambda\Delta$ the polydisc $\Delta_q(\lambda r)$ for all $\lambda > 0$. Repeating the Baire category argument already used in the proof of part (ii), one can show that there are a positive number K , polydiscs $\Delta_j \subset \mathcal{U}_j$, and subsets $Q_{\mathcal{U}_1, \mathcal{U}_2}^1$ of $S^{\sigma_1}(a_3) \cap S^{\sigma_3}$ (resp. $Q_{\mathcal{U}_1, \mathcal{U}_2}^2$ of $S^{\sigma_2}(a_3) \cap S^{\sigma_4}$) such that $\overline{Q_{\mathcal{U}_1, \mathcal{U}_2}^j} = \Delta_j$, $Q_{\mathcal{U}_1, \mathcal{U}_2}^j$ is of Baire category II, and $\sup_{2\Delta_2} |f(\widetilde{a_1, \cdot, a_3})| \leq K$ (resp. $\sup_{2\Delta_1} |f(\cdot, a_2, a_3)| \leq K$) for $a_j \in Q_{\mathcal{U}_1, \mathcal{U}_2}^j$.

Therefore, by applying Lemma 4.2, we obtain a function $f_{a_3} = f_{\mathcal{U}_1, \mathcal{U}_2, a_3} \in \mathcal{O}(\Delta_1 \times \Delta_2)$ which extends $f(\cdot, \cdot, a_3)$ to $\Delta_1 \times \Delta_2 \times \{a_3\}$ for all $a_3 \in S^{\sigma_1} \cap S^{\sigma_2}$.

Now let $\mathcal{U}_j \subset \Omega_j$ ($j = 1, 2, 3$) be an arbitrary open subset. Since the set $\Omega_3 \setminus (S^{\sigma_1} \cap S^{\sigma_2})$ is of Baire category I, by using the previous discussion we can employ the Baire category argument already used in the proof of (ii). Consequently, there are a positive number K , polydiscs $\Delta_j \subset \mathcal{U}_j$, and subsets $Q_{\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3}^3$ of $S^{\sigma_1} \cap S^{\sigma_2}$ such that $\overline{Q_{\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3}^3} = \Delta_3$, $Q_{\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3}^3$ is of Baire category II, and $\sup_{2\Delta_1 \times 2\Delta_2} |f_{a_3}(\cdot, \cdot)| \leq K$ for $a_3 \in Q_{\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3}^3$.

By changing the role of 1, 2, 3 and by taking smaller polydiscs, we obtain in the same way subsets $Q_{\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3}^j \subset \Delta_j$ ($j = 1, 2$) with similar properties.

For $j \in \{1, 2, 3\}$ consider the following subsets of T :

$$T_j := \{a = (a_1, a_2, a_3) :$$

$$a_k \in \Delta_k, a_l \in \Delta_l, a_j \in Q_{\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3}^j, \{k, l, j\} = \{1, 2, 3\} \text{ and}$$

$$\text{either } \text{int}_{\mathbb{C}^{n_l}} S(\cdot, a_k, a_j) = \emptyset \text{ or } \text{int}_{\mathbb{C}^{n_k}} S(a_l, \cdot, a_j) = \emptyset\}.$$

One next proves that

$$(5.4) \quad f(a) = f_{a_j}(a_k, a_l), \quad a \in T_1 \cup T_2 \cup T_3.$$

Indeed, let $a = (a_1, a_2, a_3) \in T_3$ with $\text{int } S(\cdot, a_2, a_3) = \emptyset$. In view of (5.2), we can choose a sequence $(z_1^n)_{n=1}^\infty \rightarrow a_1$ and for every $n \geq 1$ a sequence $(z_2^{n,m})_{m=1}^\infty \rightarrow a_2$ such that $(z_1^n, z_2^{n,m}, a_3) \in X$ for all $n, m \in \mathbb{N}$. Clearly, $f(z_1^n, z_2^{n,m}, a_3) = f_{a_3}(z_1^n, z_2^{n,m})$. Therefore,

$$f(a) = \lim_{n \rightarrow \infty} f(z_1^n, a_2, a_3) = \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} f(z_1^n, z_2^{n,m}, a_3) = f_{a_3}(a_1, a_2).$$

Now, we wish to glue the three functions f_{a_j} ($j = 1, 2, 3$) together. Since the family $\{f_{a_j} : a_j \in Q_{\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3}^j\}$ is normal, we define an extension f_j of f_{a_j} ($j = 1, 2, 3$) to $\Delta := \Delta_1 \times \Delta_2 \times \Delta_3$ as follows.

Let $\{j, k, l\} \in \{1, 2, 3\}$ and for $z = (z_1, z_2, z_3) \in \Delta$, choose a sequence $(a_j^n)_{n=1}^\infty \subset Q_{\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3}^j$ such that $\lim_{n \rightarrow \infty} a_j^n = z_j$ and the sequence $(f_{a_j^n})_{n=1}^\infty$

converges uniformly on compact subsets of $\Delta_k \times \Delta_l$. We let

$$(5.5) \quad f_j(z) := \lim_{n \rightarrow \infty} f_{a_j^n}(a_k^n, a_l^n),$$

for any sequence $((a_k^n, a_l^n, a_j^n))_{n=1}^\infty \subset T_j \rightarrow z$ as $n \rightarrow \infty$.

Let us first check that the functions f_j are well defined. Indeed, this will follow from the estimate

$$(5.6) \quad |f_{a_j}(a_k, a_l) - f_{b_j}(b_k, b_l)| \leq CK|a - b|, \\ a = (a_k, a_l, a_j), b = (b_k, b_l, b_j) \in T_j.$$

Here C is a constant that depends only on Δ . It now remains to prove (5.6) for example in the case $j = 3$. To do this, let $z = (z_1, z_2, z_3), w = (w_1, w_2, w_3) \in T_3$. Then by (5.2) and (5.3), one can choose $a_1, a'_1 \in \Delta_1$ and $a_2 \in \Delta_2$ such that

$$(5.7) \quad (a_1, a_2, z_3), (a'_1, a_2, z_3) \in T_2 \cap T_3, \\ |z - (a_1, a_2, z_3)| \leq 2|z - w|, \quad |w - (a'_1, a_2, z_3)| \leq 2|z - w|.$$

Write

$$|f_3(z) - f_3(w)| \leq |f_3(z) - f_3(a_1, a_2, z_3)| \\ + |f_3(a'_1, a_2, z_3) - f_3(w)| + |f_2(a_1, a_2, z_3) - f_2(a'_1, a_2, z_3)|.$$

Since $\sup_{2\Delta_1 \times 2\Delta_2} |f_{z_3}| \leq K, \sup_{2\Delta_1 \times 2\Delta_2} |f_{w_3}| \leq K$ and $\sup_{2\Delta_1 \times 2\Delta_3} |f_{a_2}| \leq K$, applying Schwarz's lemma to each term on the right side of the latter estimate and using (5.7) we obtain the desired estimate (5.6).

From the construction (5.5) above, $f_j(\cdot, \cdot, z_j) \in \mathcal{O}(\Delta_k \times \Delta_l)$. Moreover, a routine identity argument using (5.2) and (5.3) shows that $f_1 = f_2 = f_3$. Finally, define

$$\widehat{f}_{\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3}(z) = f_1(z) = f_2(z) = f_3(z), \quad z \in \Delta;$$

then $\widehat{f}_{\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3}$ extends f holomorphically from $T_1 \cup T_2 \cup T_3$ to Δ . Again an identity argument as in (5.4) shows that $\widehat{f}_{\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3} = f$ on $(T \cap \Delta) \setminus S$. Gluing $\widehat{f}_{\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3}$ for all $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3$ together we obtain the desired extension function \widehat{f} . Hence the proof is complete in this case.

STEP 3: $N \geq 4$. The general case uses induction on N . Since the proof is very similar to the case $N = 3$ making use of Lemmas 4.1 and 4.3 and using the inductive hypothesis for $N - 1$, we leave the details to the reader. ■

References

- [1] M. Jarnicki and P. Pflug, *Extension of Holomorphic Functions*, de Gruyter Exp. Math. 34, Walter de Gruyter, 2000.
- [2] —, —, *An extension theorem for separately holomorphic functions with analytic singularities*, Ann. Polon. Math. 80 (2003), 143–161.

- [3] M. Jarnicki and P. Pflug, *An extension theorem for separately holomorphic functions with pluripolar singularities*, Trans. Amer. Math. Soc. 355 (2003), 1251–1267.
- [4] —, —, *An extension theorem for separately meromorphic functions with pluripolar singularities*, Kyushu J. Math. (2003), to appear; arXiv:math.CV/0209207 v2, 1–11.
- [5] M. Klimek, *Pluripotential Theory*, London Math. Soc. Monogr. 6, Oxford Univ. Press, 1991.
- [6] Nguyễn Thanh Vân et J. Siciak, *Fonctions plurisousharmoniques extrémales et systèmes doublement orthogonaux de fonctions analytiques*, Bull. Sci. Math. 115 (1991), 235–244.
- [7] Nguyễn Thanh Vân et A. Zeriahi, *Une extension du théorème de Hartogs sur les fonctions séparément analytiques*, in: *Analyse Complexe Multivariable : Récents Développements* (Guadeloupe, 1988), A. Meril (ed.), EditEl, Rende, 1991, 183–194.
- [8] W. Rothstein, *Ein neuer Beweis des Hartogsschen Hauptsatzes und seine Ausdehnung auf meromorphe Funktionen*, Math. Z. 53 (1950), 84–95.
- [9] E. Sakai, *A note on meromorphic functions in several complex variables*, Mem. Fac. Sci. Kyusyu Univ. 11 (1957), 75–80.
- [10] Y. T. Siu, *Techniques of Extension of Analytic Objects*, Lecture Notes in Pure and Appl. Math. 8, Marcel Dekker, 1974.

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