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Distributional chaos of time-varying discrete dynamical systems

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Abstract. This paper is concerned with distributional chaos of time-varying discrete systems in metric spaces. Some basic concepts are introduced for general time-varying systems, including sequentially distributive chaos, weak mixing, and mixing. We give an example of sequentially distributive chaos of finite-dimensional linear time-varying dynamical systems, which is not distributively chaotic of type i (DCi for short, $i = 1, 2$). We also prove that two uniformly topological equiconjugate time-varying systems have simultaneously sequentially distributive chaos and weak topological mixing.

1. Introduction. Since Li and Yorke first gave the definition of chaos with strict mathematical language in 1975 [9], the research of chaos has had a great influence on modern science. Various extensions of the definition have been given, e.g. Devaney chaos [2], Wiggins chaos [20], dense chaos [14, 15], generic chaos [16], distributional chaos [19] and sequentially distributive chaos [21]. Chaos of the time-invariant discrete system $x_{n+1} = f(x_n)$ $(n \geq 0)$ has been studied, where $f : D \subset X \to X$ is a map and (X, d) is a metric space. Many significant results have been obtained [2, 3]. Moreover, for high-dimensional and infinite-dimensional maps, some significant progress has been made [10, 11, 17]. At the same time general time-varying discrete system (TVDS) have been studied in a large number of publications [5, 6, 7, 12]. A TVDS can be written in the form

$$x_{n+1} = f_n(x_n), \quad n \geq 0, \tag{1.1}$$

where $f_n : D_n \to D_{n+1}$ is a map and D_n is a subset of a metric space (X, d); f_n is not required to be invertible, and only the positive orbits of system (1.1) are considered. Y. M. Shi and G. R. Chen [18] have studied chaos of finite-dimensional linear time-varying dynamical systems and showed that

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topological conjugacy alone cannot guarantee two topologically conjugate time-varying systems to have the same topological properties in general.

In this paper, we study sequentially distributive chaos and weak topological mixing of two uniformly topologically equiconjugate time-varying systems (see Definition 2.6). We also exhibit a system which is sequentially distributively chaotic, but is neither DC1 nor DC2. Furthermore, we prove that being sequentially distributively chaotic and DC2 are not equivalent.

2. Basic definitions and preparations. Throughout this paper, (X,d) will denote a metric space with metric d, $f_n : D_n \to D_{n+1}$ a map, and D_n a subset of the metric space (X,d). Let S be a subset of D_0 containing at least two distinct points. Let $x_0, y_0 \in S$, $x_0 \neq y_0$, and $\{p_k\}$ be a sequence of positive integers. For any $\delta > 0$, put

$$F_{x_0y_0}(\delta,\{p_k\}) = \liminf_{n\to\infty} \frac{1}{n} \sharp\{k \mid d(x_{p_k}, y_{p_k}) < \delta,\ 1 \le k \le n\},$$
$$F^*_{x_0y_0}(\delta,\{p_k\}) = \limsup_{n\to\infty} \frac{1}{n} \sharp\{k \mid d(x_{p_k}, y_{p_k}) < \delta,\ 1 \le k \le n\},$$

where $\sharp A$ is the number of elements in A.

Definition 2.1. A set S is called *distributively scrambled for system* (1.1) *along a sequence* $\{p_k\}$ of positive integers if for any distinct points $x_0, y_0 \in S$,

(1) $F_{x_0y_0}(\varepsilon,\{p_k\}) = 0$ for some $\varepsilon > 0$,
(2) $F^*_{x_0y_0}(\delta,\{p_k\}) = 1$ for all $\delta > 0$.

If system (1.1) has an uncountable set distributively scrambled along a sequence, the system is said to be *sequentially distributively chaotic* (briefly *sd-chaotic*).

A system distributively chaotic along the sequence of all positive integers is also said to be *distributively chaotic* (briefly DC1). Suppose (1.1) is sd-chaotic and let S be a set distributively scrambled along a sequence for (1.1). If there exists $\varepsilon > 0$ such that $F_{xy}(\varepsilon) = 0$ for any distinct $x, y \in S$, then the sequentially distributive chaos is said to be *uniform*. If condition (1) is replaced by $F_{xy} < F^*_{xy}$, then we obtain the definition of DC2.

Definition 2.2. Let A be a nonempty subset of D_0. System (1.1) is said to be *topologically mixing* in A if for any two nonempty relatively open subsets U_0 and V_0 of A, there exists a positive integer N such that $U_n \cap V_0 \neq \emptyset$ when $n \ge N$, where $U_{i+1} = f_i(U_i)$, $0 \le i \le n-1$.

Definition 2.3. Let A be a nonempty subset of D_0. System (1.1) is said to be *weakly mixing* in A if for any four nonempty relatively open subsets U_0^1, V_0^1, U_0^2 and V_0^2 of A, there exists a positive integer n such that $U_n^k \cap V_0^k \neq \emptyset$ $(k = 1, 2)$, where $U_{i+1} = f_i(U_i)$, $0 \le i \le n-1$.

Now, we introduce the concept of topological conjugacy and uniform topological equiconjugacy for TVDSs. Consider the system

$$E_{n+1} = g_n(E_n), \qquad n \geq 0, \tag{2.1}$$

where $g_n : E_n \to E_{n+1}$ and E_n is a subset of a metric space (Y, ρ), $n \geq 0$.

DEFINITION 2.4. System (1.1) is said to be *topologically* $\{h_n\}_{n=0}^{\infty}$ *conjugate* to system (2.1) if for each $n \geq 0$, there exists a homeomorphism $h_n : D_n \to E_n$ such that $h_{n+1} \circ f_n = g_n \circ h_n, n \geq 0$.

DEFINITION 2.5. Assume that $\{D_n\}_{n=0}^{\infty}$ is a sequence of subsets in a metric space (X, d), $\{E_n\}_{n=0}^{\infty}$ is a sequence of subsets in a metric space (Y, ρ) and $h_n : D_n \to E_n$ is a uniformly continuous map for each $n \geq 0$. The sequence $\{h_n\}_{n=0}^{\infty}$ is said to be *uniformly equicontinuous* in $\{D_n\}_{n=0}^{\infty}$ if for any $\varepsilon > 0$, there exists a positive constant δ such that $\rho(h_n(x), h_n(y)) < \varepsilon$ for all $n \geq 0$ and $x, y \in D_n$ with $d(x, y) < \delta$.

DEFINITION 2.6. Assume that system (1.1) is topologically $\{h_n\}_{n=0}^{\infty}$ conjugate to system (2.1). System (1.1) is said to be *uniformly topologically* $\{h_n\}_{n=0}^{\infty}$ *conjugate* (resp. *equiconjugate*) to system (2.1) if $\{h_n\}_{n=0}^{\infty}$ and $\{h_n^{-1}\}_{n=0}^{\infty}$ are uniformly continuous (resp. equicontinuous) in $\{D_n\}_{n=0}^{\infty}$ and $\{E_n\}_{n=0}^{\infty}$ respectively.

DEFINITION 2.7. Let S be a subset of D_0 containing at least two distinct points. Then S is called a *scrambled set* of system (1.1) if the orbits of any two distinct points $x_0, y_0 \in S$ satisfy

$$\text{(i)} \ \liminf_{n\to\infty} d(x_n, y_n) = 0, \qquad \text{(ii)} \ \limsup_{n\to\infty} d(x_n, y_n) > 0.$$

Further, S is called a *δ-scrambled set* for some positive constant δ if for any two distinct $x_0, y_0 \in S$, (i) holds, and instead of (ii),

$$\text{(iii)} \ \limsup_{n\to\infty} d(x_n, y_n) > \delta.$$

DEFINITION 2.8. System (1.1) is said to be *chaotic in the strong sense of Li–Yorke* if it has an uncountable δ-scrambled set S such that all the orbits starting from the points in S are bounded.

DEFINITION 2.9. Let f be sd-chaotic and let D_0 be a set distributively scrambled along a sequence for f. The sequentially distributive chaos is said to be *uniform* if there exists $\varepsilon > 0$ such that $F_{xy}(\varepsilon, \{p_k\}) = 0$ for any distinct $x, y \in D_0$ (see [13]).

DEFINITION 2.10. Let $S = \{0, 1\}$, $\Sigma = \{s = s_0 s_1 \cdots \mid s_i \in S, \forall i \geq 0\}$, and define $\rho : \Sigma \times \Sigma \to \mathbb{R}$ by setting, for any $s = s_0 s_1 \cdots, t = t_0 t_1 \cdots \in \Sigma$,

$$\rho(s, t) = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i}.$$

It is not difficult to check that ρ is a metric on Σ; (Σ, ρ) is compact and called the *one-sided symbolic space* (with two symbols).

LEMMA 2.11. *There is an uncountable subset E in Σ such that for any distinct points $s = s_0 s_1 \cdots, t = t_0 t_1 \cdots \in E$, $s_n = t_n$ for infinitely many n and $s_m \neq t_m$ for infinitely many m.*

Proof. For a proof, see [8, 21, 22]. ∎

THEOREM 2.12. *Assume that systems* (1.1) *and* (2.1) *are uniformly topologically equiconjugate. If system* (2.1) *is sd-chaotic, then so is* (1.1).

THEOREM 2.13. *Assume that system* (1.1) *is topologically $\{h_n\}_{n=0}^{\infty}$ conjugate to system* (2.1) *and $h_i^{-1}(u) = h_j^{-1}(u)$ for all $u \in E_i \cap E_j$ whenever $E_i \cap E_j \neq \emptyset$ for $i \neq j$, $i, j \geq 0$. If system* (2.1) *is weakly topologically mixing in E_0, then the same is true for* (1.1) *in D_0.*

THEOREM 2.14. *Let S be a subset of $D_0 \subset X$, $a, b \in S$ with $a \neq b$ and let $p_k \to \infty$ be a sequence of positive integers. If for any sequence $C = C_1 C_2 \cdots$ with $C_k \in \{\overline{B(a, 1/k)}, \overline{B(b, 1/k)}\}$, $k = 1, 2, \ldots$, where $B(a, 1/k) = \{x \mid d(a, x) < 1/k\}$, there exists $x^C \in C_k$ such that $x^C_{p_k} \in C_k$ for each $k \geq 1$, then:*

1. *System* (1.1) *is sd-chaotic and the chaos is uniform.*
2. *If (X, d) is a compact metric space, then* (1.1) *is chaotic in the strong sense of Li–Yorke.*

3. Proof of main theorems

Proofs of Theorem 2.12. Suppose that system (2.1) has an uncountable set distributively scrambled along a sequence $S \subset E_0$. Since h_0 is a homeomorphism, $T = h_0^{-1}(S)$ is also uncountable. We will show that T is distributively scrambled along a sequence for system (1.1). Let x_0 and y_0 be distinct points in T, $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ be the orbits of (1.1) starting from x_0 and y_0, respectively, and $u_n = h_n(x_n)$ and $v_n = h_n(y_n)$ for $n \geq 0$. Then $u_0, v_0 \in S, u_0 \neq v_0$, and $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ are the orbits of (2.1) starting from u_0 and v_0, respectively. By assumption, there exists a sequence $\{p_k\}$ of positive integers such that

$$F_{u_0 v_0}(\varepsilon_0, \{p_k\}) = \liminf_{n\to\infty} \frac{1}{n} \sharp\{k \mid \rho(u_{p_k}, v_{p_k}) < \varepsilon_0,\ 1 \leq k \leq n\} = 0 \tag{3.1}$$

for some $\varepsilon_0 > 0$, and

$$F^*_{u_0 v_0}(\delta, \{p_k\}) = \limsup_{n\to\infty} \frac{1}{n} \sharp\{k \mid \rho(u_{p_k}, v_{p_k}) < \delta,\ 1 \leq k \leq n\} = 1 \tag{3.2}$$

for all $\delta > 0$. We will prove

$$F_{x_0 y_0}(r, \{p_k\}) = \liminf_{n\to\infty} \frac{1}{n} \sharp\{k \mid d(x_{p_k}, y_{p_k}) < r,\ 1 \leq k \leq n\} = 0,$$

for some $r > 0$, and

$$F^*_{x_0y_0}(\delta, \{p_k\}) = \limsup_{n\to\infty} \frac{1}{n}\sharp\{k \mid d(x_{p_k}, y_{p_k}) < \delta,\ 1 \le k \le n\} = 1$$

for all $\delta > 0$.

Since $\{h_n^{-1}\}_{n=0}^{\infty}$ is uniformly equicontinuous in $\{E_n\}_{n=0}^{\infty}$, for any $\varepsilon > 0$ there exists $\delta_1 > 0$ such that $d(h_n^{-1}(u), h_n^{-1}(v)) < \varepsilon$ for all $n \ge 0$ and $u, v \in E_n$ with $\rho(u, v) < \delta_1$. From (3.2) there exists a subsequence $n_i \to \infty$ as $i \to \infty$ such that

$$\lim_{n_i\to\infty} \frac{1}{n_i}\sharp\{k \mid d(x_{p_k}, y_{p_k}) < \varepsilon,\ 1 \le k \le n_i\} = 1,$$

$$F^*_{x_0y_0}(\varepsilon, \{p_k\}) = \lim_{n\to\infty} \sup \frac{1}{n}\sharp\{k \mid d(x_{p_k}, y_{p_k}) < \varepsilon,\ 1 \le k \le n\} = 1,$$

for all $\varepsilon > 0$. Because $\{h_n\}_{n=0}^{\infty}$ is uniformly equicontinuous in $\{D_n\}_{n=0}^{\infty}$, there exists a positive $r > 0$ such that $\rho(h_n(x), h_n(y)) < \varepsilon_0/2$ for all $n \ge 0$ and $x, y \in D_n$ with $d(x, y) < r$. Now, we show that

$$F_{x_0y_0}(r, \{p_k\}) = \lim_{n\to\infty} \inf \frac{1}{n}\sharp\{k \mid d(x_{p_k}, y_{p_k}) < r,\ 1 \le k \le n\} = 0 \tag{3.3}$$

for some $r > 0$. If not, there exists $\{n_i\}_{i=1}^{\infty}$ such that

$$\lim_{i\to\infty} \frac{1}{n_i}\sharp\{k \mid d(x_{p_k}, y_{p_k}) < r,\ 1 \le k \le n_i\} = t > 0;$$

this yields

$$\lim_{i\to\infty} \frac{1}{n_i}\sharp\{k \mid \rho(u_{p_k}, v_{p_k}) < \varepsilon_0/2 < \varepsilon_0,\ 1 \le k \le n_i\} > t > 0,$$

which contradicts (3.1). So (3.3) holds and consequently, T is distributively scrambled along a sequence for system (1.1).

Proof of Theorem 2.13. Suppose that system (2.1) is weakly topological mixing in E_0. Let A_1^0, A_2^0, B_1^0 and B_2^0 be any four nonempty open subsets of D_0. Since h_0 is a homeomorphism, $U_1^0 = h_0(A_1^0)$, $U_2^0 = h_0(A_2^0)$, $V_1^0 = h_0(B_1^0)$ and $V_2^0 = h_0(B_2^0)$ are also nonempty open subsets of E_0. By assumption, there exists $n \ge 1$ such that $U_1^n \cap U_2^0 \ne \emptyset$, $V_1^n \cap V_2^0 \ne \emptyset$, where $U_1^{i+1} = g_i(U_1^i), V_1^{i+1} = g_i(V_1^i), 0 \le i \le n-1$. Hence, there exist $u_0 \in U_1^0$ and $v_0 \in V_1^0$ such that $u_n \in U_1^n \cap U_2^0 \subset E_n \cap E_0$ and $v_n \in V_1^n \cap V_2^0 \subset E_n \cap E_0$, where $\{u_i\}, \{v_i\}$ are the orbits of system (2.1) starting from u_0 and v_0, respectively. Let $x_i = h_i^{-1}(u_i)$ and $y_i = h_i^{-1}(v_i)$ $(0 \le i \le n)$. It is clear that $x_0 \in A_1^0$ and $x_n \in A_1^n$, $y_0 \in B_1^0$ and $y_n \in B_1^n$, where $A_1^{i+1} = f_i(A_1^i)$, $B_1^{i+1} = f_i(B_1^i)$, $0 \le i \le n-1$. In addition, $x_n = h_n^{-1}(u_n) = h_0^{-1}(u_n) \in h_0^{-1}(U_2^0) = A_2^0$ and $y_n = h_n^{-1}(v_n) = h_0^{-1}(v_n) \in h_0^{-1}(V_2^0) = B_2^0$, which implies that $x_n \in A_1^n \cap A_2^0$ and $y_n \in B_1^n \cap B_2^0$. Consequently, $A_1^n \cap A_2^0 \ne \emptyset$ and $B_1^n \cap B_2^0 \ne \emptyset$. Therefore, system (1.1) is weakly topologically mixing in D_0.

Proof of Theorem 2.14. (1) Let E be an uncountable subset of Σ as in Lemma 2.11. For each $s = s_0 s_1 \cdots \in E$, by hypothesis, we can choose $x^s \in \Sigma$ such that for $n! < k \leq (n+1)!$, we have

$$x^s_{p_k} \in \begin{cases} \overline{B(a, 1/k)} & \text{if } s_n = 0, \\ \overline{B(b, 1/k)} & \text{if } s_n = 1. \end{cases}$$

Put $D = \{x^s \mid s \in E\}$. Clearly, if $s \neq t$, then $x^s \neq x^t$. Because E is uncountable, D is uncountable. Let $x^s, x^t \in D$ be any different points, where $s = s_0 s_1 \cdots, t = t_0 t_1 \cdots \in E$. By the property of E, there exist sequences of positive integers $n_i \to \infty$ and $m_i \to \infty$ such that $s_{n_i} = t_{n_i}$ and $s_{m_i} \neq t_{m_i}$ for all i.

Firstly, for given $\delta > 0$, we have $1/n_i < \delta/2$ provided that n_i is large enough, and by definition, if $n_i! < k \leq (n_i+1)!$, then $x^s_{p_k}$ and $x^t_{p_k}$ lie in the same ball of diameter less than δ; thus

$$\frac{1}{(n_i+1)!} \sharp\{k \mid d(x^s_{p_k}, x^t_{p_k}) < \delta, 1 \leq k \leq (n_i+1)!\} \geq \frac{(n_i+1)! - n_i!}{(n_i+1)!}$$
$$= 1 - \frac{1}{n_i+1} \to 1 \quad (n_i \to \infty).$$

That is, $F^*_{x^s x^t}(\delta, \{p_k\}) = 1$.

Secondly, let $\varepsilon = d(a,b)/2$. For m_i large enough, we have $1/m_i < d(a,b)/4$. Then for $m_i! < k \leq (m_i+1)!$, we have $d(x^s_{p_k}, x^t_{p_k}) > \varepsilon$. Thus

$$\frac{1}{(m_i+1)!} \sharp\{k \mid d(x^s_{p_k}, x^t_{p_k}) < \varepsilon, 1 \leq k \leq (m_i+1)!\} \leq \frac{m_i!}{(m_i+1)!} = \frac{1}{m_i+1} \to 0$$

$(m_i \to \infty)$. There exists $\varepsilon > 0$ such that $F_{x^s x^t}(\varepsilon, \{p_k\}) = 0$ for any distinct $x^s, x^t \in D$. So system (1.1) is sd-chaotic and the chaos is uniform.

(2) Let $x^s, x^t \in D$ be different, where $s = s_0 s_1 \cdots s_i \cdots, t = t_0 t_1 \cdots t_i \cdots \in E$. By the property of E, there exist sequences of positive integers $m_i, n_i \to \infty$ such that $s_{m_i} \neq t_{m_i}$ and $s_{n_i} = t_{n_i}$ for all i, and for i so large that $1/i < d(a,b)/4 = \delta$, we have $d(x^s_{m_i}, x^t_{m_i}) > \delta$. Thus

$$\lim_{i\to\infty} d(x^s_{m_i}, x^t_{m_i}) \geq \delta.$$

This shows

$$\limsup_{n\to\infty} d(x^s_n, x^t_n) \geq \delta.$$

At the same time, for n_i large enough, $x^s_{n_i}$ and $x^t_{n_i}$ lie in the same ball of diameter less than $2/n_i$. Thus $d(x^s_{n_i}, x^t_{n_i}) < 2/n_i$, so

$$\lim_{i\to\infty} d(x^s_{n_i}, x^t_{n_i}) = 0.$$

This shows

$$\liminf_{n\to\infty} d(x^s_n, x^t_n) = 0. \ \blacksquare$$

4. Example. It is well known that a finite-dimensional linear time-invariant system cannot be chaotic. But an infinite-dimensional linear time-invariant system may be chaotic [1, 4]. This subsection shows that a finite-dimensional LTVDS can be sd-chaotic, but cannot be DC1 or DC2. This means that for a finite-dimensional LTVDS, DC2 and sd-chaotic are not equivalent, and shows from another angle that finite-dimensional LTVDSs can have complex dynamical behaviors, but the complexity is not too strong.

We first consider the following simple one-dimensional LTVDS:

$$x_{n+1} = a_n x_n, \qquad n \geq 0, \tag{4.1}$$

where a_n is a real number.

EXAMPLE 4.1. Consider the above system with $a_{2k} = 2^{2k}$ and $a_{2k+1} = 2^{-(2k+1)}$, $k \geq 0$. For any $x_0 \in \mathbb{R}$, one has $x_{2k-1} = 2^{k-1}x_0$ and $x_{2k} = 2^{-k}x_0$, $k \geq 1$, which implies that for any two different $x_0, y_0 \in [0,1]$,

$$d(x_{2k-1}, y_{2k-1}) = 2^{k-1}d(x_0,y_0), d(x_{2k}, y_{2k}) = 2^{-k}d(x_0,y_0).$$

Firstly, for given $\delta > 0$, we have $d(x_0,y_0)/n_i < \delta/2$ provided that n_i is large enough. Let $\{p_n\} \subset \mathbb{N}$ be an increasing sequence defined as follows:

$$\begin{aligned} n_i! < k \leq (n_i+1)! \ &\Rightarrow\ p_k = 2k, \\ (n_i+1)! < k \leq (n_i+2)! \ &\Rightarrow\ p_k = 2k-1, \\ (n_i+2)! < k \leq (n_i+3)! \ &\Rightarrow\ p_k = 2k, \end{aligned}$$

and so forth: if $(n_i+j-1)! < k \leq (n_i+j)!$, and j is even, then $p_k = 2k-1$; when j is odd, $p_k = 2k$. Thus we have

$$\begin{aligned} &\frac{1}{(n_i+j)!}\sharp\{k \mid d(x_{p_k}, y_{p_k}) < \delta,\ 1 \leq k \leq (n_i+j)!\} \\ &\qquad \geq \frac{(n_i+j)! - (n_i+j-1)!}{(n_i+j)!} = 1 - \frac{1}{n_i+j} \to 1, \qquad n_i \to \infty, \end{aligned}$$

when j is odd, that is,

$$F^*_{x_0y_0}(\delta, \{p_k\}) = 1. \tag{4.2}$$

And

$$\begin{aligned} &\frac{1}{(n_i+j)!}\sharp\{k \mid d(x_{p_k}, y_{p_k}) < \varepsilon,\ 1 < k < (n_i+j)!\} \\ &\qquad \leq \frac{(n_i+j-1)!}{(n_i+j)!} = \frac{1}{n_i+j} \to 0, \qquad n_i \to \infty, \end{aligned}$$

when j is even, that is,

$$F_{x_0y_0}(\varepsilon, \{p_k\}) = 0. \tag{4.3}$$

Combining (4.2) with (4.3), we see that x_0, y_0 are sd-chaotic, and so f is sd-chaotic.

In the following, we show that

$$x_{n+1} = a_n x_n, \qquad n \geq 1,$$

where $a_{2k} = 2^{2k}$ and $a_{2k+1} = 2^{-(2k+1)}$ $(k \geq 0)$, is neither DC1 nor DC2.

Let $x_0, y_0 \in [0,1]$, $d(x_{2k-1}, y_{2k-1}) = 2^{k-1} d(x_0, y_0)$, and $d(x_{2k}, y_{2k}) = 2^{-k} d(x_0, y_0)$, $k \geq 1$. Thus, for any real $\delta > 0$, there exists $n_0 \in \mathbb{N}$ so that $2^{-k} d(x_0, y_0) < \delta$, $2^{k-1} d(x_0, y_0) \geq \delta$, when $k \geq n_0$. Hence

$$\frac{1}{n} \cdot \frac{n - n_0}{2} \leq \frac{1}{n} \sharp\{i \mid d(x_i, y_i) < \delta,\ 1 \leq i \leq n\} \leq \frac{1}{n} \cdot \frac{n+1}{2},$$

$$\lim_{n\to\infty} \frac{1}{n} \sharp\{i \mid d(x_i, y_i) < \delta,\ 1 \leq i \leq n\} = \frac{1}{2}.$$

This yields

$$F^*_{x_0,y_0}(\delta, \{N\}) = F_{x_0,y_0}(\delta, \{N\}) \qquad \text{for all } \delta > 0.$$

So system (4.1) is neither DC1 nor DC2. This shows that being sd-chaotic and DC2 are not equivalent.

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