

## Flatness testing over singular bases

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*Dedicated to Edward Bierstone and Pierre D. Milman  
for their 65th birthdays*

**Abstract.** We show that non-flatness of a morphism  $\varphi : X \rightarrow Y$  of complex-analytic spaces with a locally irreducible target of dimension  $n$  manifests in the existence of vertical components in the  $n$ -fold fibred power of the pull-back of  $\varphi$  to the desingularization of  $Y$ . An algebraic analogue follows: Let  $R$  be a locally (analytically) irreducible finite type  $\mathbb{C}$ -algebra and an integral domain of Krull dimension  $n$ , and let  $S$  be a regular  $n$ -dimensional algebra of finite type over  $R$  (but not necessarily a finite  $R$ -module), such that  $\text{Spec } S \rightarrow \text{Spec } R$  is dominant. Then a finite type  $R$ -algebra  $A$  is  $R$ -flat if and only if  $(A^{\otimes_R^n}) \otimes_R S$  is a torsion-free  $R$ -module.

**1. Introduction and main results.** This note is concerned with the algebraic notion of flatness. The past few years have seen a significant progress in the attempts to understand flatness of an (analytic or algebraic) morphism in explicit geometric terms. Following the work of Galligo and Kwieciński [8], Adamus, Bierstone and Milman [1] characterized flatness of a morphism  $\varphi : X \rightarrow Y$  of complex-analytic spaces in terms of the so-called vertical components (see 1.1) in fibred powers of  $\varphi$ . An analogous result in the algebraic category was later obtained by Avramov and Iyengar [4]. Both theorems assume smoothness of the target  $Y$ , as the arguments, inspired by Auslander's [3], rely on homological properties of modules over regular local rings.

On the other hand, very little is known in this context in the case of singular targets. The best result we know of works only for finite maps over irreducible plane curves (see [13]).

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2010 *Mathematics Subject Classification*: Primary 13C11, 32B99; Secondary 13H99, 13B10, 13P99, 32S45.

*Key words and phrases*: flat, torsion-free, fibred power, tensor power, vertical component, vertical element, analytic tensor product.

Our goal in the present paper is to generalize the results of [1] to the singular setting. The following is a special case of our main result (Theorem 1.6 and Corollary 1.8 below).

**THEOREM 1.1.** *Let  $\varphi : X \rightarrow Y$  be a morphism of complex-analytic spaces, where  $Y$  is locally irreducible. Let  $\xi \in X$ . Let  $X^{\{n\}}$  denote the  $n$ -fold fibred power of  $X$  over  $Y$ , where  $n = \dim Y$ , and let  $\xi^{\{n\}} \in X^{\{n\}}$  be the diagonal point corresponding to  $\xi$ . Finally, let  $\sigma_\zeta : Z_\zeta \rightarrow Y_\eta$  be any dominant morphism of complex-analytic space germs with  $Z_\zeta$  smooth of dimension  $n$  (e.g.,  $Z_\zeta$  can be a desingularization of  $Y_\eta$ ), where  $\eta = \varphi(\xi)$ . Then  $\varphi$  is not flat at  $\xi$  if and only if the fibred product  $X^{\{n\}} \times_Y Z$  has a vertical component at  $(\xi^{\{n\}}, \zeta)$ , i.e., a local irreducible component (perhaps embedded) of  $X^{\{n\}} \times_Y Z$  at  $(\xi^{\{n\}}, \zeta)$  whose image under the canonical map  $X^{\{n\}} \times_Y Z \rightarrow Y$  is nowhere-dense in  $Y$ .*

Throughout the paper, by a *dominant morphism*  $\sigma_\zeta : Z_\zeta \rightarrow Y_\eta$  of complex-analytic space germs we mean a germ at  $\zeta$  of a holomorphic mapping  $\sigma : Z \rightarrow Y$  such that  $\sigma(\zeta) = \eta$  and, for an arbitrarily small open neighbourhood  $U$  of  $\zeta$  in  $Z$ , we have  $\dim \sigma(U) = \dim Y_\eta$ .

**REMARK 1.2.** Note that, according to Theorem 1.1,  $(X^{\{n\}} \times_Y Z)_{(\xi^{\{n\}}, \zeta)}$  has no vertical components for *some* map germ  $\sigma_\zeta : Z_\zeta \rightarrow Y_\eta$  satisfying the hypotheses of the theorem if and only if it has no vertical components for *all* such map germs. In particular, if  $Y$  is smooth then, to investigate flatness of  $\varphi_\xi$ , we can choose  $Z = Y$  and  $\sigma = \text{id}_Y$ . Then  $X^{\{n\}} \times_Y Z$  can be identified with  $X^{\{n\}}$ , and so in this case Theorem 1.1 specializes to [1, Thm. 1.1].

Theorem 1.6 and Corollary 1.8 provide a more general criterion for  $\mathcal{O}_{Y,\eta}$ -flatness of a finitely generated  $\mathcal{O}_{X,\xi}$ -module. Before formulating these results, we will recall the notion of verticality in complex-analytic modules.

### 1.1. Background: vertical elements and analytic tensor product.

Let  $\varphi : X \rightarrow Y$  be a holomorphic mapping of complex-analytic spaces. Let  $\xi \in X$ ,  $\eta = \varphi(\xi) \in Y$ , and let  $\varphi_\xi : X_\xi \rightarrow Y_\eta$  be the induced morphism of germs. Let  $\Sigma_\xi$  be an irreducible component (isolated or embedded) of  $X_\xi$ , i.e., the zero-set germ of an associated prime (minimal or not) of the zero ideal in the local ring  $\mathcal{O}_{X,\xi}$ . We will say that  $\Sigma_\xi$  is a *geometric vertical component* with respect to  $\varphi_\xi$  when a sufficiently small representative of  $\Sigma_\xi$  is mapped by a representative of  $\varphi_\xi$  into a nowhere-dense (with respect to the strong topology) subset of a representative of  $Y_\eta$ .

**REMARK 1.3.** One also defines another kind of vertical components, called *algebraic vertical* (see, e.g., [1]). We will however not consider them at all in the present paper, and hence will simply use the term *vertical components* when referring to the geometric vertical ones defined above.

Since the set of zero-divisors of the ring  $\mathcal{O}_{X,\xi}$  coincides with the union of its associated primes, it follows that  $\varphi_\xi : X_\xi \rightarrow Y_\eta$  has a vertical component if and only if there exists a non-zero element  $m \in \mathcal{O}_{X,\xi}$  such that (a sufficiently small representative of) the zero-set germ  $\mathcal{V}(\text{Ann}_{\mathcal{O}_{X,\xi}}(m))$  of the annihilator of  $m$  in  $\mathcal{O}_{X,\xi}$  is mapped by (a representative of)  $\varphi_\xi$  to a nowhere-dense subset of (a representative of)  $Y_\eta$ . We can thus extend the notion of vertical component to a finitely generated  $\mathcal{O}_{X,\xi}$ -module  $F$ :

DEFINITION 1.4. Let  $\mathfrak{p}_1, \dots, \mathfrak{p}_s$  be the associated primes of  $F$  in  $\mathcal{O}_{X,\xi}$  and, for  $j = 1, \dots, s$ , let  $(Z_j)_\xi$  be the germ of a complex-analytic subspace of  $X$ , defined by  $\mathcal{O}_{Z_j,\xi} := \mathcal{O}_{X,\xi}/\mathfrak{p}_j$ . We say that  $F$  has a *vertical component* over  $Y_\eta$  (or over  $\mathcal{O}_{Y,\eta}$ ) when, for some  $j$ ,  $(Z_j)_\xi$  is vertical over  $Y_\eta$  in the above sense. Equivalently, there exists a non-zero  $m \in F$  such that  $\mathcal{V}(\text{Ann}_{\mathcal{O}_{X,\xi}}(m))$  is mapped to a nowhere-dense subgerm of  $Y_\eta$ . We will call such  $m$  a *geometric vertical element* (or simply a *vertical element*) of  $F$  over  $Y_\eta$  (or over  $\mathcal{O}_{Y,\eta}$ ).

Vertical elements are geometric analogues of zero-divisors in commutative algebra (see [1, §1] for details).

REMARK 1.5. In the special case that  $F = \mathcal{O}_{X,\xi}$ ,  $X_\xi$  has no vertical components over  $Y_\eta$  if and only if  $\mathcal{O}_{X,\xi}$  (as an  $\mathcal{O}_{X,\xi}$ -module) has no vertical elements over  $\mathcal{O}_{Y,\eta}$ .

Now let  $R$  denote a local analytic  $\mathbb{C}$ -algebra, that is, a quotient ring  $\mathbb{C}\{y\}/J = \mathbb{C}\{y_1, \dots, y_n\}/J$  for some  $n \in \mathbb{N}$  and an ideal  $J$  in  $\mathbb{C}\{y\}$  (the ring of convergent power series in  $n$  complex variables). By a *local analytic  $R$ -algebra* we mean a ring of the form  $R\{x\}/I := \mathbb{C}\{y, x\}/I$ , with the canonical homomorphism  $R \rightarrow R\{x\}/I$ , where  $I$  is an ideal in  $\mathbb{C}\{y, x\} = \mathbb{C}\{y_1, \dots, y_n, x_1, \dots, x_m\}$  containing  $J \cdot \mathbb{C}\{x, y\}$ . Let  $F$  denote an  $R$ -module. Following [9], we call  $F$  an *analytic  $R$ -module* <sup>(1)</sup> when  $F$  is a finitely generated  $A$ -module for some local analytic  $R$ -algebra  $A$ . In this case, there is a morphism of germs of complex-analytic spaces  $\varphi_\xi : X_\xi \rightarrow Y_\eta$  such that  $R \cong \mathcal{O}_{Y,\eta}$ ,  $A \cong \mathcal{O}_{X,\xi}$ ,  $R \rightarrow A$  is the induced pull-back homomorphism  $\varphi_\xi^* : \mathcal{O}_{Y,\eta} \rightarrow \mathcal{O}_{X,\xi}$ , and  $F$  is a finitely generated  $\mathcal{O}_{X,\xi}$ -module. We say that a non-zero element  $m \in F$  is *vertical* over  $R$  if  $m$  is vertical over  $\mathcal{O}_{Y,\eta}$  in the sense of Definition 1.4. It is easy to see that the notion of vertical element is well-defined, i.e., independent of the choice of a witness  $A$  for  $F$  (cf. [8]).

In the category of local analytic  $R$ -algebras with local  $R$ -algebra homomorphisms, the coproduct of two objects  $A = R\{x\}/I$ ,  $B = R\{x'\}/I'$  exists, called the *analytic tensor product* over  $R$ , and is denoted by  $A \otimes_R B$ . This can be shown to be isomorphic to  $R\{x, x'\}/(I + I')$  (cf. [9]). Given two analytic  $R$ -modules  $F_1$  and  $F_2$ , witnessed by  $A_1$  and  $A_2$  respectively, one defines

<sup>(1)</sup> Such an  $F$  is also called an *almost finitely generated  $R$ -module*, after [8].

their analytic tensor product over  $R$  as

$$F_1 \tilde{\otimes}_R F_2 := ((F_1 \otimes_{A_1} (A_1 \tilde{\otimes}_R A_2)) \otimes_{A_1 \tilde{\otimes}_R A_2} ((A_1 \tilde{\otimes}_R A_2) \otimes_{A_2} F_2)).$$

Recall that every local analytic  $\mathbb{C}$ -algebra  $R$  corresponds to a unique complex-analytic space germ, denoted  $\text{Specan } R$ . The duality between the categories of complex-analytic germs and local analytic  $\mathbb{C}$ -algebras (see [7]) implies the following important isomorphism. Suppose that  $\varphi_1 : X_1 \rightarrow Y$  and  $\varphi_2 : X_2 \rightarrow Y$  are holomorphic mappings of analytic spaces, with  $\varphi_1(\xi_1) = \varphi_2(\xi_2) = \eta$ . Then the local rings  $\mathcal{O}_{X_i, \xi_i}$  ( $i = 1, 2$ ) are  $\mathcal{O}_{Y, \eta}$ -modules and, by the uniqueness of fibred product and of analytic tensor product, the local ring  $\mathcal{O}_{Z, (\xi_1, \xi_2)}$  of the fibred product  $Z = X_1 \times_Y X_2$  at  $(\xi_1, \xi_2)$  is canonically isomorphic to  $\mathcal{O}_{X_1, \xi_1} \tilde{\otimes}_{\mathcal{O}_{Y, \eta}} \mathcal{O}_{X_2, \xi_2}$ . Therefore, given a holomorphic germ  $\varphi_\xi : X_\xi \rightarrow Y_\eta$ , we can identify the  $d$ -fold analytic tensor power  $\mathcal{O}_{X, \xi}^{\tilde{\otimes}_{\mathcal{O}_{Y, \eta}} d} = \mathcal{O}_{X, \xi} \tilde{\otimes}_{\mathcal{O}_{Y, \eta}} \cdots \tilde{\otimes}_{\mathcal{O}_{Y, \eta}} \mathcal{O}_{X, \xi}$  with the local ring of the  $d$ -fold fibred power  $\mathcal{O}_{X^{\{d\}}, \xi^{\{d\}}}$  for  $d \geq 1$ .

**1.2. Main results.** Our main theorem is the following flatness criterion.

**THEOREM 1.6.** *Let  $R$  be a local analytic  $\mathbb{C}$ -algebra and an integral domain of dimension  $n$ . Let  $F$  be an analytic  $R$ -module, and let  $S$  be any local analytic  $R$ -algebra which is regular,  $n$ -dimensional, and such that the induced morphism of complex-analytic space germs  $\text{Specan } S \rightarrow \text{Specan } R$  is dominant. Then  $F$  is  $R$ -flat if and only if the analytic tensor product*

$$\underbrace{F \tilde{\otimes}_R \cdots \tilde{\otimes}_R F \tilde{\otimes}_R S}_{n \text{ times}}$$

has no vertical elements over  $R$  (equivalently, over  $S$ ).

**REMARK 1.7.** Note that an  $R$ -algebra  $S$  with the above properties always exists. One can, for instance, take  $S$  to be the local ring of a desingularization of (a sufficiently small representative of)  $\text{Specan } R$  (cf. [6]).

**COROLLARY 1.8.** *Let  $\varphi : X \rightarrow Y$  be a morphism of complex-analytic spaces, where  $Y$  is locally irreducible, and let  $\mathcal{F}$  be a coherent  $\mathcal{O}_X$ -module. Let  $\xi \in X$ . Let  $X^{\{n\}}$  denote the  $n$ -fold fibred power of  $X$  over  $Y$ , where  $n = \dim Y$ , and let  $\xi^{\{n\}} \in X^{\{n\}}$  be the diagonal point corresponding to  $\xi$ . Finally, let  $\sigma_\zeta : Z_\zeta \rightarrow Y_\eta$  be any dominant morphism of complex-analytic space germs with  $Z_\zeta$  smooth of dimension  $n$ , where  $\eta = \varphi(\xi)$ . Then  $\mathcal{F}_\xi$  is a flat  $\mathcal{O}_{Y, \eta}$ -module if and only if the analytic tensor product*

$$\underbrace{\mathcal{F}_\xi \tilde{\otimes}_{\mathcal{O}_{Y, \eta}} \cdots \tilde{\otimes}_{\mathcal{O}_{Y, \eta}} \mathcal{F}_\xi \tilde{\otimes}_{\mathcal{O}_{Y, \eta}} \mathcal{O}_{Z, \zeta}}_{n \text{ times}}$$

has no vertical elements over  $\mathcal{O}_{Y, \eta}$  (equivalently, over  $\mathcal{O}_{Z, \zeta}$ ).

Now, Theorem 1.1 follows from Corollary 1.8 with  $\mathcal{F} = \mathcal{O}_X$ , according to Remark 1.5 and the canonical isomorphism

$$\underbrace{\mathcal{O}_{X,\xi} \tilde{\otimes}_{\mathcal{O}_{Y,\eta}} \cdots \tilde{\otimes}_{\mathcal{O}_{Y,\eta}} \mathcal{O}_{X,\xi}}_{n \text{ times}} \cong \mathcal{O}_{X^{\{n\}}, \xi^{\{n\}}}.$$

In the case when the source  $X$  is smooth, of the same dimension as  $Y$ , and the map germ  $\varphi_\xi$  is dominant, Theorem 1.1 has a particularly pleasing form:

**COROLLARY 1.9.** *Let  $\varphi_\xi : X_\xi \rightarrow Y_\eta$  be a dominant morphism of complex-analytic space germs, where  $Y_\eta$  is irreducible,  $X_\xi$  is smooth, and  $\dim X_\xi = \dim Y_\eta = n$ . Then  $\varphi_\xi$  is flat if and only if the induced morphism*

$$\varphi_{\xi^{\{n+1\}}} : X_{\xi^{\{n+1\}}} \rightarrow Y_\eta$$

*has no vertical components over  $Y_\eta$ .*

*Proof.* In Theorem 1.1, take  $Z_\zeta := X_\xi$  and  $\sigma_\zeta := \varphi_\xi$ . ■

We derive Theorem 1.6 from [1, Thm. 1.9] in Section 3, via an analytic flatness descent (see below). Section 4 contains an algebraic version of our flatness criterion (Theorem 4.1) as well as Example 4.5 proving sharpness of Theorem 1.6.

**2. Analytic flatness descent.** By definition, a ring homomorphism  $R \rightarrow S$  descends flatness if, for any  $R$ -module  $F$ , flatness of  $F \otimes_R S$  (as an  $S$ -module) implies flatness of  $F$  (as an  $R$ -module). In general, flatness descent is a rare luxury. However, as we show below, it does hold for analytic modules over integral domains and local analytic  $\mathbb{C}$ -algebra homomorphisms inducing dominant morphisms of analytic space germs.

We will say that a homomorphism  $R \rightarrow S$  of local analytic  $\mathbb{C}$ -algebras *analytically descends flatness* if  $S$ -flatness of  $F \tilde{\otimes}_R S$  implies  $R$ -flatness of  $F$  for every analytic  $R$ -module  $F$ .

**PROPOSITION 2.1.** *Let  $\kappa : R \rightarrow S$  be a homomorphism of local analytic  $\mathbb{C}$ -algebras, where  $R$  is an integral domain. If the induced morphism  $\text{Specan } S \rightarrow \text{Specan } R$  of complex-analytic space germs is dominant, then  $\kappa$  analytically descends flatness.*

*Proof.* For a proof by contradiction, suppose the morphism  $\text{Specan } S \rightarrow \text{Specan } R$  is dominant and there exists a non-flat analytic  $R$ -module  $F$  such that  $F \tilde{\otimes}_R S$  is  $S$ -flat. By Hironaka's local flattener theorem (see, e.g., [5, Thm. 7.12]), there exists a unique, non-zero (!), ideal  $P$  in  $R$  such that  $F \tilde{\otimes}_R R/P$  is  $R/P$ -flat and, for every morphism of analytic space germs

$\varphi : T \rightarrow \text{Specan } R$ , if  $\mathcal{O}_T \tilde{\otimes}_R F$  is  $\mathcal{O}_T$ -flat then  $\varphi$  factors as

$$T \rightarrow \text{Specan } R/P \hookrightarrow \text{Specan } R.$$

Since  $R$  is an integral domain, it follows that  $\text{Specan } R/P$  (hence also the image of  $T$  via  $\varphi$ ) is nowhere-dense in  $\text{Specan } R$ . Taking  $T := \text{Specan } S$ , we get a contradiction. ■

**3. Proof of the main theorem.** We begin with a simple observation regarding commutativity of the analytic tensor product. Let  $R$  be a local analytic  $\mathbb{C}$ -algebra, let  $A$  and  $S$  be local analytic  $R$ -algebras, and let  $F$  be a finitely generated  $A$ -module. Set  $A' := A \tilde{\otimes}_R S$  and  $F' := F \tilde{\otimes}_R S$ .

LEMMA 3.1. *With the above notation, for every  $i \geq 1$ , we have:*

- (a)  $A^{\tilde{\otimes}_R i} \tilde{\otimes}_R S \cong A'^{\tilde{\otimes}_S i}$  (as  $R$ -algebras).
- (b)  $F^{\tilde{\otimes}_R i} \tilde{\otimes}_R S \cong F'^{\tilde{\otimes}_S i}$  (as  $R$ -modules).

*Proof.* We will prove the isomorphism (a) for  $i = 2$ . The general case follows easily by induction. Write  $S = R\{u\}/L$ ,  $A = R\{x\}/I$ , and  $R\{x'\}/I'$  for another copy of  $A$ . Then, by 1.1, we have

$$\begin{aligned} (A^{\tilde{\otimes}_R 2}) \tilde{\otimes}_R S &= \left( \frac{R\{x\}}{I} \tilde{\otimes}_R \frac{R\{x'\}}{I'} \right) \tilde{\otimes}_R S \cong \frac{R\{x, x'\}}{I + I'} \tilde{\otimes}_R \frac{R\{u\}}{L} \\ &\cong \frac{R\{x, x', u\}}{I + I' + L} \cong \frac{R\{x, u\}}{I + L} \tilde{\otimes}_{R\{u\}/L} \frac{R\{x', u\}}{I' + L} = A'^{\tilde{\otimes}_S 2}. \end{aligned}$$

For the proof of part (b), apply (a) to a presentation of  $F$  as a finite  $A$ -module. ■

For the reader's convenience, we recall the analytic flatness criterion of [1], on which the proof of Theorem 1.6 is based.

THEOREM 3.2 ([1, Thm. 1.9]). *Let  $S$  be a regular local analytic  $\mathbb{C}$ -algebra and let  $M$  denote an analytic  $S$ -module. Let  $n = \dim S$ . Then  $M$  is  $S$ -flat if and only if the  $n$ -fold analytic tensor power  $M^{\tilde{\otimes}_S n}$  has no vertical elements over  $S$ .*

**3.1. Proof of Theorem 1.6.** Let  $S$  be an arbitrary local analytic  $R$ -algebra, which is a regular ring of dimension  $\dim S = n = \dim R$ , and such that the induced morphism  $\text{Specan } S \rightarrow \text{Specan } R$  is dominant. Let  $F$  be an arbitrary analytic  $R$ -module.

By Proposition 2.1 and since flatness is preserved by base change ([12, Prop. 6.8]), it follows that  $F$  is a flat  $R$ -module if and only if  $F \tilde{\otimes}_R S$  is a flat  $S$ -module. Hence, by Theorem 3.2,  $F$  is  $R$ -flat if and only if the  $n$ -fold tensor power  $(F \tilde{\otimes}_R S)^{\tilde{\otimes}_S n}$  has no vertical elements over  $S$ . By Lemma 3.1, this is equivalent to saying that  $F^{\tilde{\otimes}_R n} \tilde{\otimes}_R S$  has no vertical elements over  $S$ .

To complete the proof, it remains to show that the latter is equivalent to the lack of vertical elements in  $F^{\tilde{\otimes}_R^n} \tilde{\otimes}_R S$  over  $R$ . Let then  $A$  denote a local analytic  $R$ -algebra for which  $F$  is a finitely generated  $A$ -module, and let  $\varphi_\xi : X_\xi \rightarrow Y_\eta$  denote a morphism of complex-analytic space germs such that  $R = \mathcal{O}_{Y,\eta}$ ,  $A = \mathcal{O}_{X,\xi}$ , and  $\varphi_\xi^* : \mathcal{O}_{Y,\eta} \rightarrow \mathcal{O}_{X,\xi}$  gives the  $R$ -algebra structure of  $A$ . Further, let  $\sigma_\zeta : Z_\zeta \rightarrow Y_\eta$  denote a morphism of space germs such that  $S = \mathcal{O}_{Z,\zeta}$  and  $\sigma_\zeta^* : \mathcal{O}_{Y,\eta} \rightarrow \mathcal{O}_{Z,\zeta}$  gives the  $R$ -algebra structure of  $S$ . Consider the following fibred product square:

$$\begin{array}{ccc} X^{\{n\}} \times_Y Z & \longrightarrow & X^{\{n\}} \\ \lambda \downarrow & & \downarrow \varphi^{\{n\}} \\ Z & \xrightarrow{\sigma} & Y \end{array}$$

Then, by Definition 1.4,  $F^{\tilde{\otimes}_R^n} \tilde{\otimes}_R S$  has a vertical element over  $R$  if and only if there is a non-zero  $m \in F^{\tilde{\otimes}_R^n} \tilde{\otimes}_R S$  such that the zero-set germ  $\mathcal{V}(\text{Ann}_{\mathcal{O}_{X^{\{n\}} \times_Y Z, (\xi^{\{n\}}, \zeta)}}(m))$  is mapped by  $(\sigma \circ \lambda)_{(\xi^{\{n\}}, \zeta)}$  to a nowhere-dense subgerm of  $Y_\eta$ . But  $\sigma_\zeta$  is a dominant morphism of irreducible germs of the same dimension, and hence  $\sigma_\zeta(W_\zeta)$  is nowhere-dense if and only if  $W_\zeta$  is so. Therefore verticality of  $m$  over  $R$  is equivalent to saying that  $\lambda_{(\xi^{\{n\}}, \zeta)}$  maps  $\mathcal{V}(\text{Ann}_{\mathcal{O}_{X^{\{n\}} \times_Y Z, (\xi^{\{n\}}, \zeta)}}(m))$  to a nowhere-dense subgerm of  $Z_\zeta$ , i.e.,  $m$  is vertical over  $S$ . ■

**4. Algebraic case.** The following is an algebraic version of our flatness criterion.

**THEOREM 4.1.** *Let  $R$  be an  $n$ -dimensional finite type  $\mathbb{C}$ -algebra, which is locally analytically irreducible. Let  $A$  denote an  $R$ -algebra essentially of finite type, and let  $F$  denote a finitely generated  $A$ -module. Let  $S$  be any  $n$ -dimensional regular  $R$ -algebra of finite type such that the induced morphism  $\text{Spec } S \rightarrow \text{Spec } R$  is dominant. Then  $F$  is  $R$ -flat if and only if the tensor product*

$$\underbrace{F \otimes_R \cdots \otimes_R F}_{n \text{ times}} \otimes_R S$$

*is a torsion-free  $R$ -module (equivalently, a torsion-free  $S$ -module).*

Here, by a *locally analytically irreducible* algebra we mean a finite type  $\mathbb{C}$ -algebra  $R$  such that the complex-analytic space canonically associated with  $\text{Spec } R$  is locally irreducible. This is the case, for example, if  $R$  is normal. An  *$R$ -algebra essentially of finite type* means a localization of a finite type  $R$ -algebra.

By analogy with Corollary 1.9, if  $F = A$  is itself regular  $n$ -dimensional, we get the following:

**COROLLARY 4.2.** *Let  $R$  be an  $n$ -dimensional finite type  $\mathbb{C}$ -algebra which is locally analytically irreducible, and let  $A$  be a regular  $n$ -dimensional  $R$ -algebra of finite type such that the induced morphism  $\text{Spec } A \rightarrow \text{Spec } R$  is dominant. Then  $A$  is  $R$ -flat if and only if the  $(n + 1)$ -fold tensor power  $A^{\otimes_R^{n+1}}$  is a torsion-free  $R$ -module.*

**REMARK 4.3.** By Theorem 4.1, in order to verify that  $F$  is not  $R$ -flat, it suffices to find an associated prime  $\mathfrak{p}$  of  $F^{\otimes_R^n} \otimes_R S$  in  $A^{\otimes_R^n} \otimes_R S$  for which  $\mathfrak{p} \cap R \neq (0)$ . Thus our criterion paired with computer algorithms for primary decomposition and desingularization (see, e.g., [10]) provides a tool for checking flatness by effective computation, over an arbitrary complex affine locally analytically irreducible domain.

On the other hand, the bound  $n = \dim R$  is sharp (see Remark 1.2 and Example 4.5).

The proof of Theorem 4.1 is virtually identical with that of [1, Thm. 1.3] (an algebraic variant of the flatness criterion of [1]). One reduces to the analytic case by considering the complex-analytic spaces canonically associated with the spectra of the given rings, and then uses standard faithful flatness arguments and the fact that vertical elements are precisely the zero-divisors in the algebraic case. For details we refer the reader to [1].

**REMARK 4.4.** It is worth pointing out that, by the Tarski–Lefschetz principle, one can generalize Theorem 4.1 by replacing  $\mathbb{C}$  with any field of characteristic zero (see [2] for details). In this case, “analytically irreducible” should be replaced with “geometrically unibranch” (cf. [11]).

We conclude with an explicit calculation showing Theorem 4.1 at work. This example proves as well that, in general, over a non-regular  $R$ , tensoring with  $S$  is necessary to detect non-flatness of an  $R$ -module  $F$ .

**EXAMPLE 4.5.** The following example of a non-flat module is due to Douady (see, e.g., [7, §3.13]). Let

$$R := \frac{\mathbb{C}[y_1, y_2]}{(4y_1^3 + 27y_2^2)} \quad \text{and} \quad F := \frac{\mathbb{C}[y_1, y_2, x]}{\sqrt{(4y_1^3 + 27y_2^2, x^3 + y_1x + y_2)}},$$

where  $\sqrt{I}$  denotes the radical of  $I$ . We will use Theorem 4.1 to verify that  $F$  is a non-flat  $R$ -module.

Consider the (locally analytically irreducible) plane curve defined by  $R$ , and let  $S = \mathbb{C}[u]$  denote the coordinate ring of its normalization. Then  $S$  is a regular  $R$ -algebra, of dimension  $\dim S = 1 = \dim R$ , and the mapping  $\text{Spec } S \rightarrow \text{Spec } R$  is dominant. Now, since  $\dim R = 1$ , we need to look for  $R$ -torsion in  $F \otimes_R S$ . Note that, as  $R$ -modules,

$$S \cong \mathbb{C}[y_1, y_2, u]/(y_1 + 3u^2, y_2 - 2u^3),$$

and hence

$$F \otimes_R S \cong \frac{\mathbb{C}[y_1, y_2, x, u]}{\sqrt{(4y_1^3 + 27y_2^2, x^3 + y_1x + y_2) + (y_1 + 3u^2, y_2 - 2u^3)}}.$$

One can use a computer algebra software, like Singular (cf. [10]), to verify that the  $S$ -module  $F \otimes_R S$  has an associated prime  $\mathfrak{p}$  such that  $\mathfrak{p} \cap R = (y_1, y_2)$ . Therefore  $F \otimes_R S$  has non-zero  $R$ -torsion, and hence  $F$  is not  $R$ -flat, by Theorem 4.1. On the other hand, notice that  $F$  itself is a torsion-free  $R$ -module. To see this, observe that  $F$  has precisely two associated primes, each of which contracts to zero in  $R$ .

Note also that  $S$  itself is not  $R$ -flat. This can be verified (in light of Theorem 4.1) by looking at the  $R$ -module structure of  $S \otimes_R S$ . One can readily see that the element  $u - t$  is a zero-divisor in

$$S \otimes_R S \cong \frac{\mathbb{C}[y_1, y_2, u, t]}{(y_1 + 3u^2, y_2 - 2u^3, y_1 + 3t^2, y_2 - 2t^3)}.$$

Therefore  $S$  is not  $R$ -flat, by Theorem 4.1 (or Corollary 4.2). On the other hand,  $S$  itself is  $R$ -torsion-free, which proves sharpness of our flatness criterion.

**Acknowledgments.** J. Adamus's research was partially supported by Natural Sciences and Engineering Research Council of Canada Discovery Grant 355418-2008 and Polish Ministry of Science Discovery Grant N N201 540538.

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*Received 14.1.2012  
and in final form 22.4.2012*

(2707)