Proper holomorphic mappings vs. peak points and Shilov boundary

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Abstract. We present a result on the existence of some kind of peak functions for $\mathbb{C}$-convex domains and for the symmetrized polydisc. Then we apply the latter result to show the equivariance of the set of peak points for $A(D)$ under proper holomorphic mappings. Additionally, we present a description of the set of peak points in the class of bounded pseudoconvex Reinhardt domains.

1. Introduction. The aim of the paper is to present some results on different kinds of holomorphic peak points—for basic definitions and relevant properties see the survey article [14]. First we show the existence of a special kind of weak peak functions for the class of $\mathbb{C}$-convex domains (see Theorem 2.2). We also show the existence of peak functions for the symmetrized polydisc (see Theorem 2.1). The latter fact is crucial in the proof of the main result of the paper, the equivariance of the set of peak points under proper holomorphic mappings (see Theorem 3.1). It is interesting that the symmetrized polydisc, a domain that has recently been extensively studied, is also an important tool in the proof of the invariance of c-finite compactness (a notion slightly stronger than Carathéodory completeness) under proper holomorphic mappings (see Theorem 4.1). Then we give a description of the set of peak points for the class of pseudoconvex Reinhardt domains (see Theorem 5.2). Finally we deal with the claim of Bremermann on the form of the Shilov boundary of the schlicht envelope of holomorphy—although we do not know whether Bremermann's claim is correct in general, we can prove it in the class of Reinhardt domains (see Corollary 6.2).

2. Peak functions in the symmetrized polydisc and in $\mathbb{C}$-convex domains. We start by recalling a description of the symmetrized polydisc which is basic in many proofs that we present in the paper.

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Let us recall that the *symmetrized polydisc* is a domain denoted by $\mathbb{G}_n$ and given by the formula $\mathbb{G}_n = \pi_n(\mathbb{D}^n)$, where $\pi_n = (\pi_{n,1}, \ldots, \pi_{n,n})$ is defined as follows:

$$\pi_{n,k}(\lambda) = \sum_{1 \leq j_1 < \ldots < j_k \leq n} \lambda_{j_1} \cdots \lambda_{j_k}, \quad \lambda \in \mathbb{C}^n, \, k = 1, \ldots, n.$$

Here $\mathbb{D}$ denotes the unit disc in $\mathbb{C}$. We also put $\mathbb{T} := \partial\mathbb{D}$.

Following [6], for $n \geq 2$ we define

$$p_{n,\lambda}(z) = \tilde{z}(\lambda) = (\tilde{z}_1(\lambda), \ldots, \tilde{z}_{n-1}(\lambda)) \in \mathbb{C}^{n-1}, \quad z \in \mathbb{C}^n, \, \lambda \in \mathbb{C}, \, n + \lambda z_1 \neq 0,$$

where

$$\tilde{z}_j(\lambda) = \frac{(n-j)z_j + \lambda(j+1)z_{j+1}}{n + \lambda z_1}, \quad 1 \leq j \leq n-1.$$

It is known (see [6]) that $z \in \mathbb{G}_n$ if and only if $\tilde{z}(\lambda) \in \mathbb{G}_{n-1}$ and $n + \lambda z_1 \neq 0$ for any $\lambda \in \overline{\mathbb{D}}$. Recall that the symmetrized polydisc is a domain that came up naturally a few years ago in the problem of μ-synthesis and turned out to have extremely interesting function-geometric properties (see e.g. [1], [5], [7] and many others).

We shall need the following result which is interesting in its own right.

THEOREM 2.1. *For any point a in the topological boundary of $\mathbb{G}_n$ there is a neighborhood U of a and a mapping φ holomorphic on $\mathbb{G}_n \cup U$ such that $|\varphi| < 1$ on $\mathbb{G}_n$ and $\varphi(a) = 1$.*

Proof. We proceed inductively. For $n = 1$ the statement is clear. Let us take $n > 1$. If $|a_1| = n$ then it suffices to define $\varphi(z) := z_1|a_1|/na_1$. In the other case it follows from the above description of $\mathbb{G}_n$ that there is a $\lambda \in \overline{\mathbb{D}}$ such that $p_{n,\lambda}(a) \in \mathbb{C}^{n-1} \setminus \mathbb{G}_{n-1}$. By a continuity argument $p_{n,\lambda}(a)$ lies in $\overline{\mathbb{G}}_{n-1}$, whence $p_{n,\lambda}(a)$ lies in the topological boundary of $\mathbb{G}_{n-1}$. Applying the inductive assumption we find a mapping $\tilde{\varphi}$ peaking at $p_{n,\lambda}(a)$ for $\mathbb{G}_{n-1}$. Composing it with $p_{n,\lambda}$ we get a desired mapping. ∎

Although it is the above result that we need in the proofs of the main results of the paper, we present below another result on the existence of a (weaker) peak function in $\mathbb{C}$-convex domains which we find interesting too. To the authors' knowledge the existence of functions presented in the theorem below has not been proved before.

First recall that a domain $D \subset \mathbb{C}^n$ is *$\mathbb{C}$-convex* if $l \cap D$ is connected and simply connected for any complex line l intersecting D (for basic properties of $\mathbb{C}$-convex domains see e.g. [2]).

THEOREM 2.2. *Let D be a $\mathbb{C}$-convex domain in $\mathbb{C}^n$. Then for any $a \in \partial D$ there is a $\varphi \in \mathcal{O}(D) \cup \mathcal{C}(D \cup \{a\})$ such that $|\varphi| < 1$ on D and $\varphi(a) = 1$.*

For the case of a one-dimensional bounded domain a stronger result has been obtained in [3] under the additional assumption that every boundary

point of D is linearly accessible from the interior. In the proof below we shall show that this assumption is superfluous. It is possible that this one-dimensional result is known. However, as we could not find a reference, we present the proof.

Proof of Theorem 2.2. First we focus on the case when D is a bounded domain of a complex plane. Let $d = \operatorname{diam} D$. Since D is simply connected, $f(x) = \log((x-a)d^{-1})$, $x \in D$, is a well-defined holomorphic function on D. Then $\varphi = \exp(1/f)$ has the desired properties.

Now we show that *for any domain $D \subset \mathbb{C}$ and for any $a \in \partial D$ such that the connected component of ∂D containing the point a contains at least two elements there is a holomorphic mapping φ on D continuous on $D \cup \{a\}$ such that $|\varphi| < 1$ on D and $\varphi(a) = 1$.* (Clearly, this statement implies the assertion for an arbitrary simply-connected proper domain of the complex plane.) Indeed, let I be the connected component of ∂D containing a and take any nonempty closed connected subset J of I not containing a and such that J is not a point. Let C be the connected component of $\overline{\mathbb{C}} \setminus J$ containing a. It follows from the Riemann mapping theorem that there is a conformal mapping F between C and the unit disc. Applying the first step to $F(\tilde{D})$, where $\tilde{D}$ is a connected component of $\overline{\mathbb{C}} \setminus I$ containing D, we get a function $\tilde{\varphi}$ weakly-peaking at $F(a)$. Then the composition $\varphi = \tilde{\varphi} \circ F$ satisfies the assertion.

We are left with the case $n > 1$. Consider a complex hyperplane H through a such that $D \cap H = \emptyset$ (see Theorem 2.3.9 in [2]) and let l be a complex line orthogonal to H. Denote by $\tilde{D}$ (respectively $\tilde{a}$) the projection of D (resp. a) onto l in direction H. Then $\tilde{D}$ is a simply-connected domain (see [2, Theorem 2.3.6]) and $\tilde{a}$ is its boundary point. It remains to apply the one-dimensional case and compose the peak function with the projection in direction H. ∎

REMARK 2.3. The above theorem needs some comment. First, note that the peak functions appearing in Theorem 2.2 are weaker than the ones existing in $\mathbb{G}_n$ (see Theorem 2.1). Secondly, one cannot apply the above theorem to $\mathbb{G}_n$ since the symmetrized polydisc $\mathbb{G}_n$ is $\mathbb{C}$-convex iff $n \leq 2$ (see [13]).

Motivated by the last result we introduce the following definition. We shall say that a domain D in $\mathbb{C}^n$ has *property* (†) if for any $a \in \partial D$ there is a $\varphi \in \mathcal{O}(D, \mathbb{D}) \cup \mathcal{C}(D \cup \{a\})$ such that $\varphi(a) = 1$. We call such a function a *weak peak function at a*.

The example of pointed disc shows that being linearly convex is not sufficient for a domain to have property (†).

It is clear that any bounded domain with property (†) is c-finitely compact (for definition see e.g. [10]) and (hence) Carathéodory complete and

hyperconvex. It also follows from Theorem 2.1 that the symmetrized poly-disc $\mathbb{G}_n$ is c-finitely compact, which was first proven in [13].

It seems very probable that property (†) is very closely related to the Carathéodory completeness of a domain. Recall that any bounded complete Reinhardt domain is c-finitely compact (see e.g. [10]). Therefore, it would be interesting to know whether all bounded complete pseudoconvex Reinhardt domains have property (†).

3. Peak points under proper holomorphic mappings. For a bounded domain $D \subset \mathbb{C}^n$ we denote by $A(D)$ the algebra of all functions holomorphic on D and continuous on $\overline{D}$. The set of peak points for $A(D)$, i.e. points $z \in \overline{D}$ such that there is an $f \in A(D)$ with $|f(z)| > |f(w)|$ for any $w \in \overline{D} \setminus \{z\}$, is denoted by $P(D)$. Our main result in this section and in the whole paper is the following.

THEOREM 3.1. *Let D, G be bounded domains in $\mathbb{C}^n$. Let $F : D \to G$ be a proper holomorphic mapping which extends to a proper holomorphic mapping $U \to V$, $\overline{D} \subset U$, $\overline{G} \subset V$, with the same multiplicity. Then $a \in \partial D$ is a peak point for $A(D)$ if and only if $F(a)$ is a peak point for $A(G)$.*

The example of $\{z \in \mathbb{D} : \operatorname{Re} z > 0\} \ni z \mapsto z^2 \in \mathbb{D} \setminus (-1, 0]$ shows that some additional assumption (besides the continuous extension of F onto $\mathbb{D}$) is needed in Theorem 3.1.

It is known that the Shilov boundary of a bounded domain equals the closure of the set of its peak points (see [8]). Thus Theorem 3.1 has the following interesting corollary. We denote by $\partial_s D$ the Shilov boundary of a bounded domain D in $\mathbb{C}^n$.

COROLLARY 3.2. *Let F be as above. Then $F^{-1}(\partial_s G) = \partial_s D$.*

The above result generalizes a result from [12] on equivariance of the Shilov boundary under proper holomorphic mappings.

Theorem 3.1 is a consequence of the following two more general lemmas.

LEMMA 3.3. *Let D, G be bounded domains in $\mathbb{C}^n$, $a \in \partial D$. Let $F : D \to G$ be a proper holomorphic mapping of multiplicity m extending continuously onto $\bar{D}$ and such that the following condition is satisfied:*

For any $b = F(a) \in \partial G$, there is a set $\{a_1, \ldots, a_l\} \subset \partial D$ of pairwise different points and a sequence of positive integers $n_1, \ldots, n_l$ such that for any sequence $G \ni b_k \to b$ such that $F^{-1}(b_k) = \{a_{k,1}, \ldots, a_{k,m}\}$ after a permutation (if necessary) the sequences $(a_{k,j})_j$ are convergent to elements $a_1, \ldots, a_l$ and for any $k = 1, \ldots, l$ the number of sequences $(a_{k,j})_j$ tending to a_k is equal to n_k.

If a is a peak point for $A(D)$, then $F(a)$ is a peak point for $A(G)$.

Proof. Let $\{a_1, \ldots, a_l\}$ be as above with $a_1 = a$. Let $\phi \in A(D)$ be such that $\phi(a) = 1$, $|\phi(z)| < 1$ for any $z \in \bar{D} \setminus \{a\}$. Consider the mapping $g := \pi_m \circ (\phi \times \ldots \times \phi) \circ F^{-1}$. Then $g \in \mathcal{O}(G, \mathbb{G}_m)$ and because of properties imposed on F we find that g extends continuously to $\bar{G}$, $g(\bar{G} \setminus \{b\}) \subset \mathbb{G}_m$ and $g(b) \in \partial \mathbb{G}_m$. Now let $\varphi : \mathbb{G}_m \cup \{g(b)\} \to \mathbb{D} \cup \{1\}$ be a continuous function, holomorphic on $\mathbb{G}_m$, such that $\varphi(g(b)) = 1$ and $|\varphi(z)| < 1$, $z \in \mathbb{G}_m$ (such a function exists by Theorem 2.1). Then $\varphi \circ g$ is an $A(G)$-peak function for b. ∎

REMARK 3.4. Using a similar reasoning one may show that F maps peak points for $A^k(D)$ ($k \in \mathbb{N} \cup \{\infty, \omega\}$) to peak points for $A^k(D)$ provided that F extends to a proper and holomorphic mapping $U \to V$, where $U \supset \bar{D}$, $V \supset \bar{G}$ and $D = f^{-1}(G)$. Actually, it suffices to make use of the fact that Theorem 2.1 guarantees that the mapping $\varphi \circ g$ occurring in the proof of the above lemma is of class A^k provided that ϕ is.

LEMMA 3.5. *Let D, G be bounded domains in $\mathbb{C}^n$. Let $F : D \to G$ be a proper holomorphic mapping extending continuously to $\bar{D}$ and such that the fibers $F^{-1}(y)$ are finite for any $y \in \bar{G}$. If y is a peak point for $A(G)$ then any point of $F^{-1}(y)$ is a peak point for $A(D)$.*

Proof. The properness of F implies that the fibers $F^{-1}(y)$ are nonempty. Fix $y \in P(G)$ and let $\varphi \in A(G)$ be a function peaking at y. Put $E := F^{-1}(y) = \{x^1, \ldots, x^k\}$. It is clear that E is a peak set and $\psi = \varphi \circ F$ peaks at E. Take any $j = 1, \ldots, k$. For any $l \neq j$ chose $\sigma_l = 1, \ldots, n$ such that $x^j_{\sigma_l} \neq x^l_{\sigma_l}$. Let us define $\tilde{\Phi}(\lambda) = \prod_{l \neq j} (\lambda_{\sigma_l} - x^l_{\sigma_l})$ and $\Phi(\lambda) = \tilde{\Phi}(\lambda)\tilde{\Phi}(x^j)^{-1}$. Now it suffices to apply the 1/4-3/4 method of Bishop (see e.g. [14]) to functions of the form $\psi^l(x)\Phi^m(x)$ with suitably chosen $l, m \in \mathbb{N}$. ∎

REMARK 3.6. Note that Lemma 3.5 remains true if $F : D \to G$ is a holomorphic mapping extending continuously to a surjective mapping $\bar{D} \to \bar{G}$ whose fibers $F^{-1}(y)$ are finite whenever $y \in \partial G$.

What is more, since any peak set contains a peak point (see [8]) we infer that $P(G) \subset F(P(D))$ (where $P(D)$ denotes the set of peak points with respect to $A(D)$) for any holomorphic mapping $F : D \to G$ which extends continuously to a surjective mapping $\bar{D} \to \bar{G}$.

REMARK 3.7. We do not know whether for a proper holomorphic mapping $F : U \supset \bar{D} \to \bar{G} \subset V$ the inclusion $F^{-1}(A^k(G)) \subset A^k(D)$ is true. If for example D is smooth, then the $\bar{\partial}$-problem has a smooth solution on $\bar{D}$ (Kohn's theorem; see [11]). Thus one may repeat the argument of Pflug from [15] (see also [14]) to get an affirmative answer to this problem.

4. Invariance of c-finite compactness under proper holomorphic mappings.
Now we show another result on invariance under proper holomorphic mappings. It turns out that c-finite compactness is invariant with

respect to proper holomorphic mappings (for the definition of Carathéodory (pseudo)-distance and c-finitely compact domain see e.g. [10]) and in the proof of Theorem 4.1 below we use once more the properties of the symmetrized polydisc. The theorem is motivated by the invariance of hyperconvexity and pseudoconvexity under proper holomorphic mappings.

THEOREM 4.1. *For any bounded domains $D, G \subset \mathbb{C}^n$ and for any proper, holomorphic mapping $F : D \to G$, c-finite compactness of D is equivalent to c-finite compactness of G.*

Proof. The sufficiency of c-finite compactness of G for c-finite compactness of D follows from the holomorphic contractibility of c and properness of F. To show sufficiency of c-finite compactness of D for c-finite compactness of G we proceed as follows.

Fix $w_0 = F(z_0) \in G$. For any $w \in G$ we take a point z from $F^{-1}(w)$ and choose a function $f \in \mathcal{O}(D, \mathbb{D})$ with $f(z_0) = 0$ and $|f(z)| = c_D^*(z_0, z)$. Then $g_w := \pi_m \circ (f \times \cdots \times f) \circ F^{-1}$ is a well-defined holomorphic function on G with $g_w(G) \subset \mathbb{G}_m$ such that $g_w(w_0)$ lies in some fixed compact $K \subset \mathbb{G}_m$ and $g_w(w) \to \partial \mathbb{G}_m$ as $w \to \partial G$ (here we use the fact that F is proper and D is c-finitely compact). Since $c_G(w_0, w) \geq c_{\mathbb{G}_m}(g_w(w_0), g_w(w))$, the c-finite compactness of $\mathbb{G}_m$ gives $c_G(w_0, w) \to \infty$. ∎

REMARK 4.2. Recall here that c-finite compactness implies Carathéodory (c-)completeness. Moreover, the question whether c-completeness implies c-finite compactness is not solved yet (see e.g. [10]). It would be interesting to know if the notion of c-completeness is invariant under proper holomorphic mappings. It is also interesting to know whether notions of completeness with respect to other invariant functions (e.g. Kobayashi, Bergman or inner Carathéodory) remain invariant under proper holomorphic mappings. In this context recall that the Kobayashi completeness is invariant under holomorphic coverings (see e.g. [10]).

5. Peak points of Reinhardt domains. Below we denote $M := \{z \in \mathbb{C}^n : z_1 \cdot \ldots \cdot z_n = 0\}$. Recall also that a point x of a convex set $X \subset \mathbb{R}^n$ is called *extremal* if there are no two different points $y, z \in X$ such that $x = (y + z)/2$.

In [9] a description of the set of peak points of compact, Reinhardt sets with respect to $H(K)$ was given, where $H(K)$ is the uniform closure of the space of functions holomorphic in a neighborhood of K.

In this section we shall describe the set of peak points of bounded Reinhardt domains of holomorphy with respect to $A(D)$. In Section 6 this result will be extended to arbitrary bounded Reinhardt domains. We introduce some notation. For a point $z = (z_1, \ldots, z_n) \in D \subset \mathbb{C}^n$ with $z_j \neq 0$ we denote $\log z := (\log |z_1|, \ldots, \log |z_n|) \in \mathbb{R}^n$ and $\exp(z) := (e^{z_1}, \ldots, e^{z_n})$. For

a Reinhardt domain $D \subset \mathbb{C}^n$ we denote $\log D := \{\log z : z \in D, z_j \neq 0, j = 1, \ldots, n\}$. Moreover, for a set $A \in \mathbb{R}^n$ we denote $\exp(A) := \{z \in \mathbb{C}^n : \log |z| \in A\}$. For $z = (z_1, \ldots, z_n)$, $w = (w_1, \ldots, w_n) \in \mathbb{C}^n$, we put $z \cdot w = (z_1 w_1, \ldots, z_n w_n)$.

LEMMA 5.1. *Let D be a bounded pseudoconvex Reinhardt domain in $\mathbb{C}^n$. Let $z_0 \in \partial D \setminus M$ be such that $\log z_0$ is an extremal point of $\overline{\log D}$. Then there is a sequence $(f_\mu) \subset A(D)$ of Laurent monomials such that $f_\mu(z_0) \to 1$, $\|f_\mu\|_D \to 1$ and $\|f_\mu\|_{D \setminus U} \to 0$ as $\mu \to \infty$, for any Reinhardt neighborhood U of z_0.*

Proof. We proceed inductively. For $n = 1$ the assertion is trivial. Let $n > 1$. Fix $\epsilon > 0$, $N > 0$ and let U be a Reinhardt neighborhood of z_0. We are looking for a Laurent monomial f in $A(D)$ such that $\|f\|_D < e^\epsilon$, $\|f\|_{D \setminus U} < e^{-N}$ and $f(z_0) > e^{-\epsilon}$. Put $x_0 := \log z_0$, $G := \log D$, $V = \log U$ and let $\{L = l_0\}$, $Lx = \langle l, x \rangle$, $x \in \mathbb{R}^n$, $l = (l_1, \ldots, l_n) \in \mathbb{R}^n$, $l \neq 0$, be a supporting hyperplane to G at x_0, $G \subset \{L < l_0\}$. We lose no generality assuming that $l_n \neq 0$ and that V is bounded. Dividing l and l_0 by l_n we may assume that $l_n = 1$. Clearly $\bar{G} \cap \{L = l_0\}$ is convex and x_0 is its extremal point. Let $G' = \mathrm{proj}_{n-1}(\bar{G} \cap \{L = l_0\})$ be the projection of $\bar{G} \cap \{L = l_0\}$ onto the first $n - 1$ variables. G' is a convex subset and x_0 is its extreme point. Note that its interior may be empty (viewed as a subset of $\mathbb{R}^{n-1} = \mathbb{R}^{n-1} \times \{0\}$). However we may increase it, if necessary, to the closure of a convex domain such that $x_0' = \mathrm{proj}_{n-1}(x_0)$ is its extreme point and $\mathrm{int}(\exp(G'))$ is a bounded Reinhardt domain of $\mathbb{C}^{n-1}$. To increase it we may proceed as follows. Take a supporting hyperplane at x_0' to G'. This hyperplane divides $\mathbb{R}^{n-1}$ into two sets. Choose the one containing G' and take from its interior any $n - 1$ vectors $y_1, \ldots, y_{n-1}$ such that $y_1 - \tilde{y}, \ldots, y_{n-1} - \tilde{y}$ are linearly independent for some $\tilde{y} \in G'$. Direct calculations show that the convex hull of $G' \cup \{y_1, \ldots, y_{n-1}\}$ has the desired property.

This allows us to apply the inductive assumption to the Reinhardt domain $\mathrm{int}(\exp(G'))$. Then, passing to the logarithmic image for any $\epsilon > 0$ we find a linear mapping T, $Tx' = \langle t', x' \rangle$, $x' \in \mathbb{R}^{n-1}$, $t' \in \mathbb{Z}^{n-1}$ and $t_0 \in \mathbb{R}$ such that $T < t_0 + \epsilon/2$ on G', $|Tx_0' - t_0| < \epsilon/2$ and $T < -2N$ on $G' \setminus V'$, where $V' = \mathrm{proj}_{n-1}(V \cap \{L = l_0\})$.

Observe that for M large enough the mapping

$$T_M : x \mapsto M(Lx - l_0) + Tx' - t_0$$

is bounded from above on G by ϵ and less than $-N$ on $G \setminus V$.

Indeed, assume that this is not true. Then there exists a sequence $(M_n) \subset \mathbb{R}_{>0}$ converging to ∞ such that one of the two possibilities holds: either there is a sequence $(y_n) \subset G$ converging to y_0 such that $T_{M_n}(y_n) \geq \epsilon$ if $y_0 \in V$ and

$T_{M_n}(y_n) > -N$ if $y_0 \notin V$, or there is a sequence (y_n) with no limit point in $\bar{G}$ such that $T_{M_n}(y_n) > -N$. If the first possibility holds we immediately find that $L(y_0) = l_0$, so in this case a contradiction follows from the properties of the mapping T.

In the second case one may easily find a sequence (p_n) of positive numbers such that $(p_n y_n)$ converges to $\tilde{y} \in (\mathbb{R}^n)_*$. Let $t > 0$. Clearly, the set $\bar{G} \cap \{T_{M_n} > -N\}$ is convex and contains $tp_n y_n + (1 - tp_n)x_0$ for sufficiently large values of n. Making use of the inequalities $T_{M_n}(tp_n y_n + (1 - tp_n)x_0) > -N$ and $L(tp_n y_n + (1 - tp_n)x_0) \leq l_0$ we easily find that $L(t\tilde{y} + x_0) = l_0$. What is more, the inequality $T_{M_n}(tp_n y_n + (1 - tp_n)x_0) > -N$ implies that $T(t\tilde{y}' + x_0') - t_0 \geq -N$. Collecting the above-mentioned facts and making use of the behavior of T on the hyperplane $\{L = l_0\}$ we get a contradiction for $t > 0$ large enough (i.e. for t such that $x_0 + t\tilde{y} \notin V$).

In the same way one may show the existence of $\delta > 0$ such that $x \mapsto T_M x + \langle \alpha, x \rangle$ has the same properties provided that $\|\alpha\| < \delta$.

It follows from the multidimensional Dirichlet Theorem that for any $\mu \in \mathbb{N}$ there are integers $\alpha_{\mu,1}, \ldots, \alpha_{\mu,n}$ and k_μ such that $|l_j - \alpha_{\mu,j}/k_\mu| \leq 1/\mu k_\mu$, $j = 1, \ldots, n$. Losing no generality we may assume that $(k_\mu)_\mu$ converges to ∞.

Finally it suffices to observe that the functions defined by the formula

$$F_\mu(z) = e^{i\theta_\mu - l_0 k_\mu - t_0} z^{t + \alpha_\mu},$$

where $t = (t', 0) \in \mathbb{C}^n$, have the desired property for some $\theta_\mu \in \mathbb{R}$, $\mu \gg 1$. Indeed, let us compute

$$\log F_\mu(e^x) = k_\mu(Lx - l_0) + Tx - t_0 + \sum_{j=1}^{n} x_j(\alpha_{\mu,j} - k_\mu l_j).$$

Thus it is enough to make use of the properties of the functions constructed. ∎

THEOREM 5.2. *Let D be a bounded pseudoconvex Reinhardt domain in $\mathbb{C}^n$. Then $z \notin M$ is a peak point with respect to $A(D)$ iff $\log z$ is an extremal point of the closure of the logarithmic image of D.*

Moreover, 0 is not a peak point for $A(D)$ and $z = (z', 0) \in \mathbb{C}^{n-1} \times \{0\}$ is a peak point for $A(D)$ if and only if z' is a peak point of $\operatorname{proj}_{n-1}(D)$ and $(\{z'\} \times \mathbb{C}) \cap \bar{D} = \{z\}$.

Note that Lemma 5.1 provides a tool which allows us to use Bishop's method of constructing peak functions. Below we present a slightly different proof relying upon the methods developed by Gamelin and involving the concept of representing measures (see [8]).

Proof of Theorem 5.2. Some ideas of the proof come from [9]. It is clear that any $z \in \partial D \setminus M$ such that $\log z$ is not an extremal point of $\overline{\log D}$ lies in an analytic disc contained in $\bar{D}$, hence it is not a peak point.

We shall show that every $z \in \partial D \setminus M$ such that $\log z$ is extremal in $\overline{\log D}$ is a peak point for $A(D)$. Let ν be a representing measure for $A(D)$. Applying Lemma 5.1 to the point z we get a sequence (f_μ) of Laurent monomials with properties as in that lemma. In particular, $f_\mu(z_0) = \int_D f_\mu \, d\nu$ and f_μ is a bounded sequence in $L^\infty(d\nu)$. Let f be a weak-star limit point of f_μ. It follows from the choice of f_μ that $\|f\|_D \leq 1$ and $1 = |\int_D f d\nu|$. Thus $|f| = 1$ ν-almost everywhere and it follows that the support of the measure ν is contained in the torus $z_0 \cdot \mathbb{T}^n$.

Applying the formula $f(z_0) = \int f d\nu$ to every component of $g(z) = e^{i\theta} \cdot z$, where $\theta \in \mathbb{R}^n$ is such that $|z_0| = e^{i\theta} \cdot z_0$, we easily find that μ is the Dirac measure supported at z_0.

Now suppose that $0 \in \overline{D}$. Then there are $\alpha_1, \ldots, \alpha_n \in \mathbb{R}_{\leq 0}$ not vanishing simultaneously such that $x_0 + E(\alpha) := x_0 + \{(\alpha_1 t, \ldots, \alpha_n t) : t > 0\} \subset \log D$ for some (so by a simple convexity argument for any) $x_0 \in \log D$. To simplify the notation assume that $x_0 = 0$. Seeking a contradiction suppose that $f \in A(D)$ peaks at 0. Observe that $v(\lambda) = (\sup |f(\lambda^\alpha)|)^*$, where the supremum is taken over all branches of $\lambda^\alpha = (\lambda^{\alpha_1}, \ldots, \lambda^{\alpha_n})$, is a subharmonic function on the unit disc of the complex plane attaining its maximum at 0. Thus v is constant. From this we easily get a contradiction.

The remaining part of the theorem is self-evident. ∎

6. Some remarks on Bremermann's claim. Bremermann claimed in [4] that the Shilov boundary of any domain with schlicht envelope of holomorphy coincides with the Shilov boundary of its envelope of holomorphy. Note that $A(\widehat{D}) \subset A(D)$ so we immediately see that $\partial_s \widehat{D} \subset \partial_s D$. Bremermann stated that the inclusion $\partial_s D \subset \partial_s \widehat{D}$ is trivial, however we do not know whether this statement is true. That would follow from the inclusion $A(D) \subset A(\widehat{D})$, which however does not hold in general as the following example (in the class of bounded Reinhardt domains) shows.

EXAMPLE 6.1. Consider the Reinhardt domain $D \subset \mathbb{C}_* \times \mathbb{C}$ such that $\mathbb{D}_* \times \{0\} \subset D$ and

$$\log D = \{(x, y) \in \mathbb{R}_{<1} \times \mathbb{R}_{<0} : y < 0 \text{ if } x \in (0, 1) \text{ and}$$
$$y < -n^2 - n \text{ if } x \in (-n^2, -(n-1)^2], n \in \mathbb{N}\}.$$

Observe that $\widehat{D} = \{(z, w) \in \mathbb{C}^2 : |w| < |z| < 1 \text{ or } 1 \leq |z| < e, |w| < 1\}$. Moreover, one may check that the function $f(z, w) := w/z$ is an element of the algebra $A(D)$ obviously not belonging to $A(\widehat{D})$.

Although we do not know whether the claim of Bremermann holds in general, we know that it does hold in the class of Reinhardt domains.

Recall that if D is a Reinhardt domain, then the (schlicht) envelope of holomorphy $\widehat{D}$ exists and is a Reinhardt domain.

COROLLARY 6.2. *Let D be a bounded Reinhardt domain in $\mathbb{C}^n$. Then its set of peak points coincides with the set of peak points of the envelope of holomorphy of D.*

Proof. It suffices to show that if $z_0 \in \partial D$ is a peak point for $A(D)$, then it is also a peak point for $A(\widehat{D})$. Put $G = \log D$.

First suppose that z_0 omits M. We aim to show that $\log z_0$ is extremal in $\overline{\text{conv}(G)}$ (here $\text{conv}(G)$ stands for the convex hull of G, i.e. $\overline{\text{conv}(G)} = \log \widehat{D}$). To get a contradiction assume that $\log z_0$ is not an extreme point of $\overline{\text{conv}(G)}$.

We shall make use of a few definitions. Following [16] for a convex domain U of $\mathbb{R}^n$ we define

$$\mathfrak{C} = \mathfrak{C}(U) = \{\alpha \in (\mathbb{R}^n)_* : x + \alpha \mathbb{R}_+ \subset U \text{ for some (any) } x\}.$$

Boundedness of D implies that $\mathfrak{C} \subset (\mathbb{R}_{\leq 0})^n$. Note that for any $x_0 \in \overline{U}$ there are $m \geq 1$, $x_1, \ldots, x_m$ extremal in $\overline{U}$, $p_1, \ldots, p_m > 0$, $\sum_{j=1}^m p_j = 1$, $\alpha \in \mathfrak{C}$ and $t_0 \geq 0$ such that

$$(6.1) \qquad\qquad x_0 = \sum_{j=1}^m p_j x_j + \alpha t_0.$$

To prove this we proceed inductively. Let $n > 1$ (for $n = 1$ there is nothing to prove). The statement follows immediately from the Krein–Milman theorem provided that U is bounded, so suppose that U is unbounded and take any $\alpha \in \mathfrak{C}$ (unboundedness of U implies that $\mathfrak{C}(U) \neq \emptyset$). Then there is $t_0 \geq 0$ such that $y_0 = x_0 - \alpha t_0 \in \partial U$. Let H be a supporting hyperplane to U at y_0. Applying the inductive assumption to y_0 and $H \cap \overline{U}$ (increase it to a convex domain of $\mathbb{R}^{n-1}$ if necessary) and using the fact that $\mathfrak{C}$ is a cone we easily obtain the formula (6.1).

Now we come back to the proof of the corollary. Let $f \in A(D)$ be a peak function at z_0 and denote by $\hat{f}$ its extension to $\widehat{D}$. Applying a decomposition (6.1) to $x_0 = \log z_0$ and $U = \text{conv}(G)$ we obtain $x_1, \ldots, x_m$ extremal in $\overline{\text{conv}(G)}$ and $t_0 \geq 0$ satisfying (6.1). Every point $z_j \in \mathbb{C}^n$ whose logarithmic image equals x_j lies in the Shilov boundary of $\widehat{D}$, $j = 1, \ldots, m$. Since $\partial_s \widehat{D} \subset \partial_s D$, we find that $x_1, \ldots, x_m \in \overline{G}$.

First consider the case $t_0 = 0$. Since $\log z_0$ is not extremal in $\overline{\text{conv}(G)}$, $\log z_0 \neq x_j$, $j = 1, \ldots, m$. Take any $y_j \in G$ close to x_j, $j = 1, \ldots, m$, and consider the closed simplex $\mathfrak{S}$ with vertices $y_1, \ldots, y_m$. Clearly $\hat{f}$ is holomorphic in a neighborhood of $\exp(\mathfrak{S})$. Let $\mathfrak{S}_\epsilon = \{x \in \mathbb{R}^n : \text{dist}(x, \mathfrak{S}) < \epsilon\}$ be the ϵ-hull of $\mathfrak{S}$. Then $S_\epsilon := \exp(\mathfrak{S}_\epsilon)$ is a Reinhardt domain and its Shilov boundary is close to $\{x_1, \ldots, x_m\}$ (in the sense of the Hausdorff distance)

provided that ϵ is close to 0 and y_j is close to x_j, $j = 1, \ldots, m$. Using Theorem 5.2 applied to S_ϵ one can find that $|f|$ attains values close to 1 on an arbitrarily small neighborhood of $\{z \in \mathbb{C}^n : \log |z| = y_j$ for some $j\}$. This gives an immediate contradiction with the fact that f peaks at z_0 (as $|f|$ is small far away from the point z_0).

If $t_0 > 0$ then again we take $y_j \in G$ sufficiently close to x_j. Put $w_0 := \exp(\sum_{j=1}^m p_j y_j) \in G$. Note that $w_0 \cdot \lambda^\alpha \in \widehat{D}$ for any $\lambda \in \mathbb{D}_*$. Therefore $u(\lambda) = (\sup |\hat{f}(w_0 \cdot \lambda^\alpha)|)^*$, where the supremum is taken over all branches of $\lambda^\alpha = (\lambda^{\alpha_1}, \ldots, \lambda^{\alpha_n})$, is subharmonic on a neighborhood of $\overline{\mathbb{D}}$ (it may be extended through 0 because it is bounded). By the maximum principle applied to the subharmonic function u we get $\max_{e^{-t_0}\partial\mathbb{D}} u \leq \max_{\partial\mathbb{D}} u$. Thus, as in the case $t_0 = 0$, we find that there is $z \in \bar{D}$ close to e^{y_j} for some $j = 1, \ldots, m$, such that $f(z)$ is close to 1. This contradicts the fact that f peaks at z_0.

In the case when $z_0 \in M$ we proceed similarly. One may easily see that $0 \notin \partial_s D$. Suppose that $z_0 \in M \setminus \{0\}$. Losing no generality assume that $z_0 = (z_0', 0) \in \mathbb{C}_*^k \times \mathbb{C}^{n-k}$. Note that $\bar{D} \cap (\{z_0'\} \times \mathbb{C}^{n-k}) = (z_0', 0)$ (use the fact that z_0 peaks for $A(D)$). If z_0' is not a peak point of the projection of $\widehat{D}$ onto $\mathbb{C}^k$, then to get a contradiction one may use the reasoning from the first part of the proof. ∎

COROLLARY 6.3. *Let D be a bounded Reinhardt domain in $\mathbb{C}^n$. Then its Shilov boundary coincides with the Shilov boundary of its envelope of holomorphy.*

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