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Uniqueness and weighted sharing of meromorphic functions

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Abstract. We study the uniqueness of meromorphic functions using nonlinear differential polynomials and the weighted value sharing method. Though the main concern of the paper is to improve a recent result of L. Liu [Comput. Math. Appl. 56 (2008), 3236–3245], as a consequence of the main result we also improve and generalize some former results of T. Zhang and W. Lu [Comput. Math. Appl. 55 (2008), 2981–2992], A. Banerjee [Int. J. Pure Appl. Math. 48 (2008), 41–56] and a recent result of the present author [Mat. Vesnik 62 (2010), 169–182].

1. Introduction, definitions and results. Let f and g be two non-constant meromorphic functions defined in the open complex plane $\mathbb{C}$. For $a \in \mathbb{C} \cup \{\infty\}$ we say that f and g *share the value a CM* (counting multiplicities) if $f-a$ and $g-a$ have the same set of zeros with the same multiplicities, and we say that f and g *share the value a IM* (ignoring multiplicities) if we do not consider the multiplicities.

It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a non-constant meromorphic function h, we denote by $T(r,h)$ the Nevanlinna characteristic of h and by $S(r,h)$ any quantity satisfying $S(r,h) = o\{T(r,h)\}$ $(r \to \infty,\ r \notin E)$. We denote by $T(r)$ the maximum of $T(r,f)$ and $T(r,g)$. The symbol $S(r)$ denotes any quantity satisfying $S(r) = o\{T(r)\}$ $(r \to \infty,$ $r \notin E)$.

Throughout this paper, we need the following definition:

$$\Theta(a,f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,a;f)}{T(r,f)},$$

where a is a value in the extended complex plane.

In 1959, Hayman [7] proved the following theorem.

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Theorem A. *Let f be a transcendental meromorphic function and n (≥ 3) be an integer. Then $f^n f' = 1$ has infinitely many solutions.*

To establish the corresponding uniqueness theorem, Fang and Hua [6] proved the following theorem.

Theorem B. *Let $f(z)$ and $g(z)$ be two non-constant entire functions, $n \geq 6$ be an integer. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$, or $f(z) \equiv t g(z)$ for a constant t such that $t^{n+1} = 1$.*

In 1997 Yang and Hua [16] generalized the above result and proved the following theorem.

Theorem C. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions, $n \geq 11$ an integer and $a \in \mathbb{C} - \{0\}$. If $f^n f'$ and $g^n g'$ share the value a CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(c_1 c_2)^{n+1} c^2 = -a^2$, or $f(z) \equiv t g(z)$ for some $(n+1)$-th root of unity t.*

A considerable amount of research work has been devoted to the value sharing of the particular type of differential polynomial as in Theorem C (see [2], [11], [13]). In the meantime Fang [5] investigated the uniqueness of entire functions corresponding to more general differential polynomials and obtained the following results.

Theorem D. *Let $f(z)$ and $g(z)$ be two non-constant entire functions, and let n, k be two positive integers with $n > 2k + 4$. If $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k (c_1 c_2)^n (nc)^{2k} = 1$, or $f(z) \equiv t g(z)$ for a constant t such that $t^n = 1$.*

Theorem E. *Let $f(z)$ and $g(z)$ be two non-constant entire functions, and let n, k be two positive integers with $n \geq 2k + 8$. If $[f^n(f-1)]^{(k)}$ and $[g^n(g-1)]^{(k)}$ share 1 CM, then $f(z) \equiv g(z)$.*

Recently Bhoosnurmath and Dyavanal [4] extended Theorem D to meromorphic functions and proved the following.

Theorem F. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions, and let n, k be two positive integers with $n > 3k + 8$. If $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share 1 CM, then the conclusion of Theorem D holds.*

A natural question arises: Is it possible to relax in any way the nature of sharing the value 1 in the above results?

It is worth mentioning that some investigations in this area have already been carried out by Zhang and Lü [19]. To state the result we require the following notion known as *weighted sharing of values*, introduced by I. Lahiri

[10, 11], which measures how close a shared value is to being shared CM or to being shared IM.

DEFINITION 1.1. Let k be a non-negative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g *share the value* a *with weight* k.

The definition implies that if f, g share a value a with weight k, then z_0 is an a-point of f with multiplicity m $(\leq k)$ if and only if it is an a-point of g with multiplicity m $(\leq k)$, and z_0 is an a-point of f with multiplicity m $(> k)$ if and only if it is an a-point of g with multiplicity n $(> k)$, where m is not necessarily equal to n.

We write f, g *share* (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ and (a, ∞) respectively.

Zhang and Lü [19] proved the following theorem.

THEOREM G. *Let $f(z)$ and $g(z)$ be two non-constant transcendental meromorphic functions, and let n (≥ 1), k (≥ 1), l (≥ 0) be three integers. Suppose that $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share $(1, l)$. If $l \geq 2$ and $n > 3k + 8$, or $l = 1$ and $n > 5k + 11$, or $l = 0$ and $n > 9k + 14$, then the conclusion of Theorem D holds.*

Recently Banerjee [3] improved the above result of Zhang–Lü [19] by reducing the lower bound of n. He proved the following theorem.

THEOREM H. *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions and n (≥ 1), k (≥ 1), l (≥ 0) be three integers. Suppose that $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share (b, l) for a non-zero constant b. If $l \geq 2$ and $n > 3k + 8$, or $l = 1$ and $n > 4k + 9$, or $l = 0$ and $n > 9k + 14$, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k (c_1 c_2)^n (nc)^{2k} = b^2$, or $f(z) \equiv t g(z)$ for some nth root of unity t.*

Regarding the above theorems it is natural to ask the following questions.

QUESTION 1.2. *What can be said about the relation between two non-constant meromorphic functions f and g if $\{f^n(\mu f^m + \lambda)\}^{(k)}$ and $\{g^n(\mu g^m + \lambda)\}^{(k)}$ share $(1, l)$ where λ and μ are two constants such that $|\lambda| + |\mu| \neq 0$?*

QUESTION 1.3. *What can be said about the relation between two non-constant meromorphic functions f and g if $\{f^n(f-1)^m\}^{(k)}$ and $\{g^n(g-1)^m\}^{(k)}$ share $(1, l)$?*

Regarding the above-mentioned questions, most recently Liu [12] and the present author [14] proved the following theorems respectively. For a positive integer m and a number μ, let $m^* = \chi_\mu m$, where $\chi_\mu = 0$ if $\mu = 0$ and $\chi_\mu = 1$ if $\mu \neq 0$.

THEOREM I ([12]). *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions, let n, m and k be three positive integers, and let λ, μ be two constants such that $|\lambda| + |\mu| \neq 0$. Let $[f^n(\mu f^m + \lambda)]^{(k)}$ and $[g^n(\mu g^m + \lambda)]^{(k)}$ share $(1, l)$, and one of the following conditions holds:*

(a) $l \geq 2$ *and* $n > 3m^* + 3k + 8$;
(b) $l = 1$ *and* $n > 4m^* + 5k + 10$;
(c) $l = 0$ *and* $n > 6m^* + 9k + 14$.

Then

(i) *when $\lambda\mu \neq 0$, if $m \geq 2$ and $\delta(\infty, f) > 3/(m+n)$, or $m = 1$ and $\Theta(\infty, f) > 3/(n+1)$, then $f(z) \equiv g(z)$;*
(ii) *when $\lambda\mu = 0$, if $f(z) \neq \infty$ and $g(z) \neq \infty$, then either $f(z) \equiv tg(z)$, where t is a constant satisfying $t^{n+m^*} = 1$, or $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying*

$$(-1)^k \lambda^2 (c_1 c_2)^{n+m^*} [(n+m^*)c]^{2k} = 1$$

or

$$(-1)^k \mu^2 (c_1 c_2)^{n+m^*} [(n+m^*)c]^{2k} = 1.$$

THEOREM J ([14]). *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions, and let n (≥ 1), $k(\geq 1)$ and m (≥ 0) be three integers. Let $[f^n(f-1)^m]^{(k)}$ and $[g^n(g-1)^m]^{(k)}$ share $(1,0)$. Then one of the following holds:*

(i) *when $m = 0$, if $f(z) \neq \infty$, $g(z) \neq \infty$ and $n > 9k + 14$, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k (c_1 c_2)^n (nc)^{2k} = 1$, or $f \equiv tg$ for a constant t such that $t^n = 1$;*
(ii) *when $m = 1$, $n > 9k + 20$ and $\Theta(\infty, f) > 2/n$, then either*

$$[f^n(f-1)^m]^{(k)} [g^n(g-1)^m]^{(k)} \equiv 1,$$

or $f \equiv g$;
(iii) *when $m \geq 2$ and $n > 9k + 4m + 16$, then either*

$$[f^n(f-1)^m]^{(k)} [g^n(g-1)^m]^{(k)} \equiv 1,$$

or $f \equiv g$, or f and g satisfy the algebraic equation $R(f, g) = 0$, where $R(x, y) = x^n(x-1)^m - y^n(y-1)^m$.

The possibility $[f^n(f-1)^m]^{(k)} [g^n(g-1)^m]^{(k)} \equiv 1$ does not arise for $k = 1$.

In the paper, we will prove the following theorem which will not only improve Theorem I by reducing the lower bound of n, but also improve and supplement Theorems G, H and J. This is the main result of the paper.

Theorem 1.4. *Let $f(z)$ and $g(z)$ be two non-constant transcendental meromorphic functions, and let n (≥ 1), k (≥ 1), m (≥ 1) and l (≥ 0) be four integers. Let $P(z) = a_m z^m + \cdots + a_1 z + a_0$, where a_0 $(\neq 0)$, $a_1, \ldots, a_m$ $(\neq 0)$ are complex constants. Let $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share $(1, l)$ and one of the following conditions hold:*

(a) $l \geq 2$ *and* $n > 3k + m + 8$;
(b) $l = 1$ *and* $n > 4k + 3m/2 + 9$;
(c) $l = 0$ *and* $n > 9k + 4m + 14$.

Then either $[f^n P(f)]^{(k)}[g^n P(g)]^{(k)} \equiv 1$, or $f(z) \equiv tg(z)$ for a constant t such that $t^d = 1$, where $d = \gcd\{n+m, \ldots, n+m-i, \ldots, n+1, n\}$, $a_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$, or f and g satisfy the algebraic equation $R(f, g) = 0$, where $R(x, y) = x^m(a_m x^m + \cdots + a_1 x + a_0) - y^m(a_m y^m + \cdots + a_1 y + a_0)$.

The possibility $[f^n P(f)]^{(k)}[g^n P(g)]^{(k)} \equiv 1$ does not occur for $k = 1$.

Corollary 1.5. *Under the same condition of Theorem 1.4, we set $P(z) = \mu z^m + \lambda$, where λ and μ are two constants such that $|\lambda| + |\mu| \neq 0$. If*

(a) $l \geq 2$ *and* $n > 3k + m^* + 8$,
(b) $l = 1$ *and* $n > 4k + 3m^*/2 + 9$, *or*
(c) $l = 0$ *and* $n > 9k + 4m^* + 14$,

then the following statements are valid:

(i) *when $\lambda\mu \neq 0$ and $\Theta(\infty, f) > 2/(n + m - 1)$, then either*

$$[f^n(\mu f^m + \lambda)]^{(k)}[g^n(\mu g^m + \lambda)]^{(k)} \equiv 1$$

or $f(z) \equiv g(z)$ or $f(z) \equiv -g(z)$;

(ii) *when $\lambda\mu = 0$, if $f(z) \neq \infty$ and $g(z) \neq \infty$, then either $f(z) \equiv tg(z)$, where t is a constant satisfying $t^{n+m^*} = 1$, or $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying*

$$(-1)^k \lambda^2 (c_1 c_2)^{n+m^*}[(n + m^*)c]^{2k} = 1$$

or

$$(-1)^k \mu^2 (c_1 c_2)^{n+m^*}[(n + m^*)c]^{2k} = 1.$$

The possibility $[f^n(\mu f^m + \lambda)]^{(k)}[g^n(\mu g^m + \lambda)]^{(k)} \equiv 1$ does not arise for $k = 1$ and the possibility $f(z) \equiv -g(z)$ may arise only when both m and n are even.

Corollary 1.6. *Under the same condition of Theorem 1.4, if $P(z) = (z - 1)^m$ $(m \geq 0)$, then the following statements are valid:*

(i) *when $m = 0$, if $f(z) \neq \infty$, $g(z) \neq \infty$, and either $l \geq 2$, $n > 3k+8$, or $l = 1$, $n > 4k + 9$, or $l = 0$, $n > 9k + 14$, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying*

$(-1)^k(c_1c_2)^n(nc)^{2k} = 1$, or $f(z) \equiv tg(z)$ for a constant t such that $t^n = 1$;

(ii) *when $m = 1$ and $\Theta(\infty, f) > 2/n$, then either*

$$[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)} \equiv 1$$

or $f(z) \equiv g(z)$ provided one of $l \geq 2$, $n > 3k+9$ or $l = 1$, $n > 4k+21/2$ or $l = 0$, $n > 9k+18$ holds;

(iii) *when $m \geq 2$ and either $l \geq 2$, $n > 3k+m+8$ or $l = 1$, $n > 4k+3m/2+9$ or $l = 0$, $n > 9k+4m+14$, then either*

$$[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)} \equiv 1,$$

or $f(z) \equiv tg(z)$ for a constant t such that $t^d = 1$, where $d = \gcd\{n+m, \dots, n+m-i, \dots, n+1, n\}$, or $f(z)$ and $g(z)$ satisfy the algebraic equation $R(f,g) = 0$, where $R(x,y) = x^n(x-1)^m - y^n(y-1)^m$.

The possibility $[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)} \equiv 1$ does not arise for $k = 1$.

Remark 1.7. Corollary 1.6 is an improvement of Theorem G for $l = 1$ and $m = 0$.

Remark 1.8. Corollary 1.6 improves and supplements Theorem H.

Remark 1.9. Corollary 1.5 is an improvement of Theorem I.

Remark 1.10. Corollary 1.6 improves and supplements Theorem J.

Though the standard definitions and notations of value distribution theory are available in [8], we explain some definitions and notations which are used in the paper.

Definition 1.11 ([9]). For $b \in \mathbb{C} \cup \{\infty\}$ we denote by $N(r, b; f \mid =1)$ the counting function of simple b-points of f. For a positive integer p we denote by $N(r, b; f \mid \leq p)$ the counting function of those b-points of f (counted with multiplicities) whose multiplicities are not greater than p. By $\overline{N}(r, b; f \mid \leq p)$ we denote the corresponding reduced counting function. In an analogous manner we define $N(r, b; f \mid \geq p)$ and $\overline{N}(r, b; f \mid \geq p)$.

Definition 1.12 ([11]). Let k be a positive integer or infinity. We denote by $N_k(r, b; f)$ the counting function of b-points of f, where a b-point of multiplicity m is counted m times if $m \leq k$ and k times if $m > k$. Then

$$N_k(r, b; f) = \overline{N}(r, b; f) + \overline{N}(r, b; f \mid \geq 2) + \cdots + \overline{N}(r, b; f \mid \geq k).$$

Clearly $N_1(r, b; f) = \overline{N}(r, b; f)$.

Definition 1.13 ([3]). Let $a, b \in \mathbb{C} \cup \{\infty\}$ and p be a positive integer. Then we denote by $\overline{N}(r, a; f \mid \geq p \mid g = b)$ (resp. $\overline{N}(r, a; f \mid \geq p \mid g \neq b)$)

the reduced counting function of those a-points of f with multiplicities $\geq p$ which are b-points (resp. not b-points) of g.

DEFINITION 1.14 ([1, 2]). Let f and g be two non-constant meromorphic functions such that f and g share the value 1 IM. Let z_0 be a 1-point of f with multiplicity p and also a 1-point of g with multiplicity q. We denote by $\overline{N}_L(r,1;f)$ the counting function of those 1-points of f and g where $p > q$, by $N_E^{1)}(r,1;f)$ the counting function of those 1-points of f and g where $p = q = 1$, and by $N_E^{(2}(r,1;f)$ the counting function of those 1-points of f and g where $p = q \geq 2$; each point in these counting functions is counted only once. In the same manner we can define $\overline{N}_L(r,1;g)$, $N_E^{1)}(r,1;g)$ and $N_E^{(2}(r,1;g)$.

DEFINITION 1.15 ([1, 2]). Let f and g be two non-constant meromorphic functions such that f and g share the value 1 IM. Let z_0 be a 1-point of f with multiplicity p and also a 1-point of g with multiplicity q. For a positive integer k, we denote by $\overline{N}_{f>k}(r,1;g)$ the reduced counting function of those 1-points of f and g such that $p > q = k$. In an analogous way we can define $\overline{N}_{g>k}(r,1;f)$.

2. Lemmas. In this section we present some lemmas which will be needed later.

LEMMA 2.1 ([15]). *Let f be a non-constant meromorphic function and $P(f) = a_0 + a_1 f + a_2 f^2 + \cdots + a_n f^n$, where $a_0, a_1, a_2, \ldots, a_n$ are constants and $a_n \neq 0$. Then*

$$T(r, P(f)) = nT(r,f) + S(r,f).$$

LEMMA 2.2 ([8]). *Let f be a non-constant meromorphic function, k be a positive integer, and let c be a non-zero finite complex number. Then*

$$\begin{aligned} T(r,f) &\leq \overline{N}(r,\infty;f) + N(r,0;f) + N(r,c;f^{(k)}) - N(r,0;f^{(k+1)}) + S(r,f) \\ &\leq \overline{N}(r,\infty;f) + N_{k+1}(r,0;f) + \overline{N}(r,c;f^{(k)}) - N_0(r,0;f^{(k+1)}) \\ &\quad + S(r,f), \end{aligned}$$

where $N_0(r,0;f^{(k+1)})$ denotes the counting function which only counts those points such that $f^{(k+1)} = 0$ but $f(f^{(k)} - c) \neq 0$.

LEMMA 2.3 ([1]). *Let f and g be two non-constant meromorphic functions that share $(1,1)$. Then*

$$\begin{aligned} 2\overline{N}_L(r,1;f) + 2\overline{N}_L(r,1;g) + N_E^{(2}(r,1;f) - \overline{N}_{f>2}(r,1;g) \\ \leq N(r,1;g) - \overline{N}(r,1;g). \end{aligned}$$

Lemma 2.4 ([2]). *Let f and g share $(1,1)$. Then*

$$\overline{N}_{f>2}(r,1;g) \le \frac{1}{2}\overline{N}(r,0;f) + \frac{1}{2}\overline{N}(r,\infty;f) - \frac{1}{2}N_{\oplus}(r,0;f') + S(r,f),$$

where $N_{\oplus}(r,0;f')$ denotes the counting function of those zeros of f' which are not zeros of $f(f-1)$.

Lemma 2.5 ([2]). *Let f and g be two non-constant meromorphic functions that share $(1,0)$. Then*

$$\overline{N}_L(r,1;f) + 2\overline{N}_L(r,1;g) + N_E^{(2}(r,1;f) - \overline{N}_{f>1}(r,1;g) - \overline{N}_{g>1}(r,1;f) \\ \le N(r,1;g) - \overline{N}(r,1;g).$$

Lemma 2.6 ([17]). *Let f and g be two non-constant meromorphic functions sharing $(1,0)$. Then*

$$\overline{N}_L(r,1;f) \le \overline{N}(r,0;f) + \overline{N}(r,\infty;f) + S(r,f).$$

Lemma 2.7 ([2]). *Let f and g share $(1,0)$. Then*

(i) $\overline{N}_{f>1}(r,1;g) \le \overline{N}(r,0;f) + \overline{N}(r,\infty;f) - N_{\oplus}(r,0;f') + S(r,f)$,
(ii) $\overline{N}_{g>1}(r,1;f) \le \overline{N}(r,0;g) + \overline{N}(r,\infty;g) - N_{\oplus}(r,0;g') + S(r,g)$,

where $N_{\oplus}(r,0;f')$ and $N_{\oplus}(r,0;g')$ are defined as in Lemma 2.4.

Lemma 2.8 ([18]). *Let f and g be two non-constant meromorphic functions, and let p, k be two positive integers. Then*

$$N_p(r,0;f^{(k)}) \le N_{p+k}(r,0;f) + k\overline{N}(r,\infty;f) + S(r,f).$$

Lemma 2.9. *Let f and g be two non-constant meromorphic functions, and let n (≥ 1), m (≥ 1), k (≥ 1) be three integers. Let $P(z)$ be defined as in Theorem 1.4. Then*

$$[f^nP(f)]^{(k)}[g^nP(g)]^{(k)} \not\equiv 1$$

for $k=1$ and $n > 3m+2$.

Proof. If possible, let

$$[f^nP(f)]^{(k)}[g^nP(g)]^{(k)} \equiv 1$$

for $k=1$. That is,

$$f^{n-1}Q(f)f'g^{n-1}Q(g)g' \equiv 1, \tag{2.1}$$

where $Q(z) = b_mz^m + b_{m-1}z^{m-1} + \dots + b_1z + b_0$ with $b_j = (n+j)a_j$, $j = 0,1,\dots,m$. We write $Q(z)$ as

$$Q(z) = b_m(z-d_1)^{l_1}(z-d_2)^{l_2}\cdots(z-d_i)^{l_i}\cdots(z-d_s)^{l_s},$$

where $\sum_{i=1}^{s} l_i = m$, $1 \le s \le m$; $d_i \ne d_j$, $i \ne j$, $1 \le i,j \le s$; d_i's are non-zero constants and l_i's are positive integers, $i = 1,\dots,s$.

Let z_0 be a zero of f with multiplicity p (≥ 1). Then z_0 is a pole of g with multiplicity q (≥ 1), say. Then from (2.1) we obtain

$$np - 1 = (n+m)q + 1,$$

i.e.,

$$mq + 2 = n(p-q). \tag{2.2}$$

From (2.2) we get $q \geq (n-2)/m$ and so we have

$$p \geq \frac{1}{n}\left[\frac{(n+m)(n-2)}{m} + 2\right] = \frac{n+m-2}{m}.$$

Let z_1 be a zero of $Q(f)$ with multiplicity p, and a zero of $f - d_i$ of order q_i for $i = 1, \ldots, s$. Then $p = l_i q_i$ for $i = 1, \ldots, s$. Hence z_1 is a pole of g with multiplicity q, say. So from (2.1) we get

$$q_i l_i + q_i - 1 = (n+m)q + 1 \geq n + m + 1,$$

i.e.,

$$q_i \geq \frac{n+m+2}{l_i + 1}$$

for $i = 1, \ldots, s$. Since a pole of f is either a zero of $g^{n-1}Q(g)$ or a zero of g', we have

$$\begin{aligned}\overline{N}(r,\infty;f) &\leq \overline{N}(r,0;g) + \sum_{i=1}^{s}\overline{N}(r,d_i;g) + \overline{N}_0(r,0;g') + S(r,f) + S(r,g)\\ &\leq \left(\frac{m}{n+m-2} + \frac{m+s}{n+m+2}\right)T(r,g) + \overline{N}_0(r,0;g')\\ &\quad + S(r,f) + S(r,g),\end{aligned}$$

where $\overline{N}_0(r,0;g')$ denotes the reduced counting function of those zeros of g' which are not zeros of $gQ(g)$.

Then by the second fundamental theorem of Nevanlinna we get

$$\begin{aligned}sT(r,f) &\leq \overline{N}(r,\infty;f) + \overline{N}(r,0;f)\\ &\quad + \sum_{i=1}^{s}\overline{N}(r,d_i;f) - \overline{N}_0(r,0;f') + S(r,f)\\ &\leq \left(\frac{m}{n+m-2} + \frac{m+s}{n+m+2}\right)\{T(r,f) + T(r,g)\}\\ &\quad + \overline{N}_0(r,0;g') - \overline{N}_0(r,0;f') + S(r,f) + S(r,g).\end{aligned} \tag{2.3}$$

Similarly

$$\begin{aligned}sT(r,g) &\leq \left(\frac{m}{n+m-2} + \frac{m+s}{n+m+2}\right)\{T(r,f) + T(r,g)\}\\ &\quad + \overline{N}_0(r,0;f') - \overline{N}_0(r,0;g') + S(r,f) + S(r,g).\end{aligned} \tag{2.4}$$

Adding (2.3) and (2.4) we obtain

$$\left(s-\frac{2m}{n+m-2}-\frac{2(m+s)}{n+m+2}\right)\{T(r,f)+T(r,g)\}\le S(r,f)+S(r,g),$$

which contradicts the fact that $n>3m+2$. This proves the lemma. ∎

Lemma 2.10 ([14]). *Let f and g be two non-constant meromorphic functions, and let n (≥ 1) and m (≥ 1) be two integers. If $n\ge m+3$, then*

$$[f^n(f-1)^m]'[g^n(g-1)^m]'\not\equiv 1.$$

Lemma 2.11. *Let f and g be two non-constant meromorphic functions, and let n (≥ 1) and m (≥ 1) be two integers. If $\lambda\mu\ne 0$ and $n+m>6$, then*

$$[f^n(\mu f^m+\lambda)]'[g^n(\mu g^m+\lambda)]'\not\equiv 1.$$

Proof. The lemma can be proved similarly to Lemma 8 in [14]. ∎

Lemma 2.12 ([3]). *Let f, g be two non-constant meromorphic functions and let $k\ge 1$ and $n>3k+8$ be two integers. If $[f^n]^{(k)}[g^n]^{(k)}\equiv b^2$, where b $(\ne 0)$ is a constant, then $f(z)=c_1e^{cz}$, $g(z)=c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k}=b^2$.*

Lemma 2.13. *Let f, g be two non-constant meromorphic functions, and let $P(z)$ be defined as in Theorem 1.4. If $F=f^nP(f)$ and $G=g^nP(g)$, then $S(r,F)$ and $S(r,G)$ are replaceable by $S(r,f)$ and $S(r,g)$ respectively.*

Proof. By Lemma 2.1, we have

$$T(r,F)=T(r,f^nP(f))\le (n+m)T(r,f)+S(r,f).$$

Similarly

$$T(r,G)\le (n+m)T(r,g)+S(r,g).$$

This proves the lemma. ∎

3. Proof of the main results

Proof of Theorem 1.4. We consider $F(z)=f^nP(f)$ and $G(z)=g^nP(g)$. Then $F^{(k)}$ and $G^{(k)}$ share $(1,l)$. Let

$$H=\left(\frac{F^{(k+2)}}{F^{(k+1)}}-\frac{2F^{(k+1)}}{F^{(k)}-1}\right)-\left(\frac{G^{(k+2)}}{G^{(k+1)}}-\frac{2G^{(k+1)}}{G^{(k)}-1}\right). \tag{3.1}$$

We assume that $H\not\equiv 0$. Let $l\ge 1$. Suppose that z_0 be a simple 1-point of $F^{(k)}$. Then z_0 is a simple 1-point of $G^{(k)}$. So by a simple computation on local expansions we see that z_0 is a zero of H. Thus

$$\begin{aligned}N(r,1;F^{(k)}\mid{=1})&\le N(r,0;H)\le T(r,H)+O(1)\\&\le N(r,\infty;H)+S(r,F)+S(r,G).\end{aligned} \tag{3.2}$$

From (3.1) we know that poles of H possibly result from those zeros of $F^{(k+1)}$ and $G^{(k+1)}$ which are not common 1-points of $F^{(k)}$ and $G^{(k)}$, from poles of F and G, and from those common 1-points of $F^{(k)}$ and $G^{(k)}$ such that each such point has different multiplicity relative to $F^{(k)}$ and $G^{(k)}$. Thus

$$\begin{aligned}(3.3)\qquad & N(r,\infty;H)\\ &\leq \overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}(r,0;F^{(k)}\mid\geq 2)\\ &\quad+\overline{N}(r,0;G^{(k)}\mid\geq 2)+\overline{N}_L(r,1;F^{(k)})+\overline{N}_L(r,1;G^{(k)})\\ &\quad+\overline{N}_{\otimes}(r,0;F^{(k+1)})+\overline{N}_{\otimes}(r,0;G^{(k+1)}),\end{aligned}$$

where $\overline{N}_{\otimes}(r,0;F^{(k+1)})$ denotes the reduced counting function of those zeros of $F^{(k+1)}$ which are not zeros of $F^{(k)}(F^{(k)}-1)$. Now we consider the following three cases.

Case 3.1. Let $l\geq 2$. By (3.2) and (3.3) we obtain

$$\begin{aligned}(3.4)\qquad & \overline{N}(r,1;F^{(k)})\\ &\leq N(r,1;F^{(k)}\mid =1)+\overline{N}(r,1;F^{(k)}\mid\geq 2)\\ &\leq \overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}(r,0;F^{(k)}\mid\geq 2)\\ &\quad+\overline{N}(r,0;G^{(k)}\mid\geq 2)+\overline{N}_L(r,1;F^{(k)})+\overline{N}_L(r,1;G^{(k)})\\ &\quad+\overline{N}(r,1;F^{(k)}\mid\geq 2)+\overline{N}_{\otimes}(r,0;F^{(k+1)})\\ &\quad+\overline{N}_{\otimes}(r,0;G^{(k+1)})+S(r,F)+S(r,G).\end{aligned}$$

From (3.4) and Lemma 2.2 we obtain

$$\begin{aligned}(3.5)\qquad & T(r,F)+T(r,G)\\ &\leq 2\overline{N}(r,\infty;F)+2\overline{N}(r,\infty;G)+N_{k+1}(r,0;F)\\ &\quad+N_{k+1}(r,0;G)+\overline{N}(r,0;F^{(k)}\mid\geq 2)+\overline{N}(r,0;G^{(k)}\mid\geq 2)\\ &\quad+\overline{N}_L(r,1;F^{(k)})+\overline{N}_L(r,1;G^{(k)})+\overline{N}(r,1;F^{(k)}\mid\geq 2)\\ &\quad+\overline{N}(r,1;G^{(k)})+\overline{N}_{\otimes}(r,0;F^{(k+1)})+\overline{N}_{\otimes}(r,0;G^{(k+1)})\\ &\quad-N_0(r,0;F^{(k+1)})-N_0(r,0;G^{(k+1)})+S(r,F)+S(r,G).\end{aligned}$$

It is clear that

$$\begin{aligned}(3.6)\qquad & N_{k+1}(r,0;F)+\overline{N}(r,0;F^{(k)}\mid\geq 2)+\overline{N}_{\otimes}(r,0;F^{(k+1)})\\ &\leq N_{k+1}(r,0;F)+\overline{N}(r,0;F^{(k)}\mid\geq 2\mid F=0)\\ &\quad+\overline{N}(r,0;F^{(k)}\mid\geq 2\mid F\neq 0)+\overline{N}_{\otimes}(r,0;F^{(k+1)})\\ &\leq N_{k+1}(r,0;F)+\overline{N}(r,0;F\mid\geq k+2)+N_0(r,0;F^{(k+1)})\\ &\leq N_{k+2}(r,0;F)+N_0(r,0;F^{(k+1)}).\end{aligned}$$

A similar result holds for G also. Again

$$\begin{aligned}(3.7)\quad \overline{N}(r,1;F^{(k)}\mid\geq 2)&+\overline{N}_L(r,1;F^{(k)})+\overline{N}_L(r,1;G^{(k)})+\overline{N}(r,1;G^{(k)})\\ &\leq \overline{N}(r,1;G^{(k)}\mid =2)+2\overline{N}_L(r,1;F^{(k)})+2\overline{N}_L(r,1;G^{(k)})\\ &\quad+\overline{N}_E^{(3}(r,1;G^{(k)})+\overline{N}(r,1;G^{(k)})\\ &\leq N(r,1;G^{(k)})\leq T(r,G^{(k)})+O(1)\\ &\leq T(r,G)+k\overline{N}(r,\infty;G)+S(r,G).\end{aligned}$$

So from (3.5)–(3.7) we obtain

$$\begin{aligned}T(r,F)\leq N_{k+2}(r,0;F)&+N_{k+2}(r,0;G)+2\overline{N}(r,\infty;F)\\ &+(k+2)\overline{N}(r,\infty;G)+S(r,F)+S(r,G).\end{aligned}$$

From this and using Lemmas 2.1 and 2.13 we obtain

$$(3.8)\qquad (n+m)T(r,f)\leq (3k+2m+8)T(r)+S(r).$$

Similarly

$$(3.9)\qquad (n+m)T(r,g)\leq (3k+2m+8)T(r)+S(r).$$

From (3.8) and (3.9) we get

$$(n-3k-m-8)T(r)\leq S(r),$$

which is a contradiction as $n>3k+m+8$.

Case 3.2. Let $l=1$. In view of Lemmas 2.3, 2.4, 2.8 and estimates (3.2) and (3.3) we obtain

$$\begin{aligned}(3.10)\quad &\overline{N}(r,1;F^{(k)})+\overline{N}(r,1;G^{(k)})\\ &\leq N(r,1;F^{(k)}\mid =1)+\overline{N}_L(r,1;F^{(k)})+\overline{N}_L(r,1;G^{(k)})\\ &\quad+\overline{N}_E^{(2}(r,1;F^{(k)})+\overline{N}(r,1;G^{(k)})\\ &\leq N(r,1;F^{(k)}\mid =1)+N(r,1;G^{(k)})-\overline{N}_L(r,1;F^{(k)})\\ &\quad-\overline{N}_L(r,1;G^{(k)})+\overline{N}_{F^{(k)}>2}(r,1;G^{(k)})\\ &\leq \overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}(r,0;F^{(k)}\mid\geq 2)+\overline{N}(r,0;G^{(k)}\mid\geq 2)\\ &\quad+\frac{1}{2}\overline{N}(r,0;F^{(k)})+\frac{1}{2}\overline{N}(r,\infty;F^{(k)})+T(r,G^{(k)})+\overline{N}_{\otimes}(r,0;F^{(k+1)})\\ &\quad+\overline{N}_{\otimes}(r,0;G^{(k+1)})+S(r,F)+S(r,G)\\ &\leq\left(\frac{k}{2}+\frac{3}{2}\right)\overline{N}(r,\infty;F)+(k+1)\overline{N}(r,\infty;G)+\overline{N}(r,0;F^{(k)}\mid\geq 2)\\ &\quad+\overline{N}(r,0;G^{(k)}\mid\geq 2)+\frac{1}{2}N_{k+1}(r,0;F)+T(r,G)+\overline{N}_{\otimes}(r,0;F^{(k+1)})\\ &\quad+\overline{N}_{\otimes}(r,0;G^{(k+1)})+S(r,F)+S(r,G).\end{aligned}$$

Using (3.6) and (3.10) we deduce from Lemma 2.2 that

$$T(r,F) \le N_{k+2}(r,0;F) + N_{k+2}(r,0;G) + \left(\frac{k}{2}+\frac{5}{2}\right)\overline{N}(r,\infty;F)$$
$$+ (k+2)\overline{N}(r,\infty;G) + \frac{1}{2}N_{k+1}(r,0;F) + S(r,F) + S(r,G).$$

From this and using Lemmas 2.1 and 2.13 we obtain

$$(n+m)T(r,f) \le (4k+5m/2+9)T(r) + S(r). \tag{3.11}$$

Similarly

$$(n+m)T(r,g) \le (4k+5m/2+9)T(r) + S(r). \tag{3.12}$$

From (3.11) and (3.12) we get

$$(n-4k-3m/2-9)T(r) \le S(r),$$

which contradicts our assumption that $n > 4k+3m/2+9$.

Case 3.3. Let $l = 0$. In this case (3.2) becomes

$$\begin{aligned} N_E^{1)}(r,1;F^{(k)}) &\le N(r,0;H) \le T(r,H) + O(1) \\ &\le N(r,\infty;H) + S(r,F) + S(r,G). \end{aligned} \tag{3.13}$$

Using Lemmas 2.5–2.8 and estimates (3.3), (3.6) and (3.13) we obtain

$$\begin{aligned} &\overline{N}(r,1;F^{(k)}) + \overline{N}(r,1;G^{(k)}) \\ &\le N_E^{1)}(r,1;F^{(k)}) + \overline{N}_L(r,1;F^{(k)}) + \overline{N}_L(r,1;G^{(k)}) \\ &\quad + \overline{N}_E^{(2}(r,1;F^{(k)}) + \overline{N}(r,1;G^{(k)}) \\ &\le \overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + \overline{N}(r,0;F^{(k)} \mid \ge 2) + \overline{N}(r,0;G^{(k)} \mid \ge 2) \\ &\quad + \overline{N}_L(r,1;F^{(k)}) + T(r,G^{(k)}) + \overline{N}_{F^{(k)}>1}(r,1;G^{(k)}) + \overline{N}_{G^{(k)}>1}(r,1;F^{(k)}) \\ &\quad + \overline{N}_{\otimes}(r,0;F^{(k+1)}) + \overline{N}_{\otimes}(r,0;G^{(k+1)}) + S(r,F) + S(r,G) \\ &\le (2k+3)\overline{N}(r,\infty;F) + (2k+2)\overline{N}(r,\infty;G) + \overline{N}(r,0;F^{(k)} \mid \ge 2) \\ &\quad + \overline{N}(r,0;G^{(k)} \mid \ge 2) + 2N_{k+1}(r,0;F) + N_{k+1}(r,0;G) + T(r,G) \\ &\quad + \overline{N}_{\otimes}(r,0;F^{(k+1)}) + \overline{N}_{\otimes}(r,0;G^{(k+1)}) + S(r,F) + S(r,G) \\ &\le (2k+3)\overline{N}(r,\infty;F) + (2k+2)\overline{N}(r,\infty;G) + N_{k+1}(r,0;F) \\ &\quad + N_{k+2}(r,0;F) + N_{k+2}(r,0;G) + T(r,G) + N_0(r,0;F^{(k+1)}) \\ &\quad + N_0(r,0;G^{(k+1)}) + S(r,F) + S(r,G). \end{aligned} \tag{3.14}$$

Using Lemma 2.2 we get

$$\begin{aligned} T(r,F) \le{}& (2k+4)\overline{N}(r,\infty;F) + (2k+3)\overline{N}(r,\infty;G) + 2N_{k+1}(r,0;F) \\ &+ N_{k+2}(r,0;F) + N_{k+1}(r,0;G) \\ &+ N_{k+2}(r,0;G) + S(r,F) + S(r,G). \end{aligned}$$

In view of Lemmas 2.1 and 2.13 we obtain

$$(n+m)T(r,f) \le (9k+5m+14)T(r)+S(r). \tag{3.15}$$

Similarly

$$(n+m)T(r,g) \le (9k+5m+14)T(r)+S(r). \tag{3.16}$$

Combining (3.15) and (3.16) we get

$$(n-9k-4m-14)T(r) \le S(r),$$

a contradiction since $n > 9k+4m+14$.

We now assume that $H \equiv 0$. That is,

$$\frac{F^{(k+2)}}{F^{(k+1)}} - \frac{2F^{(k+1)}}{F^{(k)}-1} \equiv \frac{G^{(k+2)}}{G^{(k+1)}} - \frac{2G^{(k+1)}}{G^{(k)}-1}.$$

Integrating both sides of the above equality twice we get

$$\frac{1}{F^{(k)}-1} \equiv \frac{BG^{(k)}+A-B}{G^{(k)}-1}, \tag{3.17}$$

where A ($\neq 0$) and B are constants. From (3.17) it is clear that $F^{(k)}$ and $G^{(k)}$ share 1 CM and hence $F^{(k)}$ and $G^{(k)}$ share $(1,2)$. Thus $n > 3k+m+8$. Now we consider the following three cases.

Case I. Let $B \neq 0$ and $A = B$.

If $B = -1$, from (3.17) we obtain $F^{(k)}G^{(k)} \equiv 1$. That is,

$$[f^nP(f)]^{(k)}[g^nP(g)]^{(k)} \equiv 1. \tag{3.18}$$

Also, by Lemma 2.9, (3.18) does not occur for $k=1$.

Let $B \neq -1$. Then from (3.17) we get

$$\frac{1}{F^{(k)}} \equiv \frac{BG^{(k)}}{(1+B)G^{(k)}-1}.$$

This together with Lemma 2.8 gives

$$\overline{N}\left(r,\frac{1}{1+B};G^{(k)}\right) \le k\overline{N}(r,\infty;f)+N_{k+1}(r,0;F). \tag{3.19}$$

Using Lemmas 2.2 and 2.13 we deduce from (3.19) that

$$\begin{aligned}
T(r,G) &\le \overline{N}(r,\infty;G)+N_{k+1}(r,0;G)+\overline{N}\left(r,\frac{1}{1+B};G^{(k)}\right)\\
&\quad - N_0(r,0;G^{(k+1)})+S(r,G)\\
&\le \overline{N}(r,\infty;g)+N_{k+1}(r,0;G)+k\overline{N}(r,\infty;f)+N_{k+1}(r,0;F)\\
&\quad +S(r,F)+S(r,G)\\
&\le (2k+m+1)T(r,f)+(k+m+2)T(r,g)+S(r,f)+S(r,g).
\end{aligned}$$

Thus by Lemma 2.1 we obtain

$$(n-3k-m-3)T(r,g) \le S(r,g),$$

a contradiction.

CASE II. Let $B \ne 0$ and $A \ne B$.

If $B=-1$, from (3.17) we have

$$F^{(k)} = \frac{A}{-G^{(k)}+A+1}.$$

Therefore

$$\overline{N}\left(r,\infty;\frac{A}{-G^{(k)}+A+1}\right) = \overline{N}(r,\infty;f).$$

By Lemmas 2.2, 2.13 and using the same argument as in Case I, we arrive at a contradiction.

If $B \ne -1$, from (3.17) we have

$$F^{(k)} - \left(1+\frac{1}{B}\right) \equiv \frac{-A}{B^2\left(G^{(k)}+(A-B)/B\right)}.$$

Therefore

$$\overline{N}\left(r,0;G^{(k)}+\frac{A-B}{B}\right) = \overline{N}(r,\infty;f).$$

Again by Lemmas 2.2, 2.13 and proceeding as above we reach a contradiction.

CASE III. Now we assume that $B=0$. Then (3.17) becomes

$$F^{(k)} = \frac{1}{A}G^{(k)} + 1 - \frac{1}{A}, \tag{3.20}$$

i.e.,

$$F = \frac{1}{A}G + \varphi(z), \tag{3.21}$$

where $\varphi(z)$ is a polynomial of degree $\le k$. Let $\varphi(z) \not\equiv 0$. Then in view of Lemmas 2.1, 2.13 and the fact that f is transcendental, by the second fundamental theorem of Nevanlinna we get

$$\begin{aligned} (n+m)T(r,f)+O(1) &= T(r,F) \\ &\le \overline{N}(r,\infty;F)+\overline{N}(r,0;F)+\overline{N}(r,\varphi(z);F)+S(r,F) \\ &\le \overline{N}(r,\infty;f)+\overline{N}(r,0;F)+\overline{N}(r,0;G)+S(r,F) \\ &\le (m+2)T(r,f)+(m+1)T(r,g)+S(r,f). \end{aligned} \tag{3.22}$$

It is clear from (3.21) that

$$T(r,f) = T(r,g)+S(r,f).$$

This together with (3.22) gives

$$(n-m-3)T(r,f) \le S(r,f),$$

which is impossible. Hence $\varphi(z) \equiv 0$. From equations (3.20) and (3.21) together with the fact that $F^{(k)}$ and $G^{(k)}$ share 1 CM it follows that $A = 1$ and so $F \equiv G$. That is,

$$f^n P(f) \equiv g^n P(g). \tag{3.23}$$

This gives

$$\begin{aligned} f^n(a_m f^m + a_{m-1} f^{m-1} + \cdots + a_1 f + a_0) \\ = g^n(a_m g^m + a_{m-1} g^{m-1} + \cdots + a_1 g + a_0). \end{aligned} \tag{3.24}$$

Let $h = f/g$. If h is a constant, then by putting $f = gh$ in (3.24) we get

$$\begin{aligned} a_m g^m(h^{n+m} - 1) + a_{m-1} g^{m-1}(h^{n+m-1} - 1) + \cdots \\ + a_1 g(h^{n+1} - 1) + a_0(h^n - 1) = 0, \end{aligned}$$

which implies $h^d = 1$, where $d = \gcd\{n+m, \ldots, n+m-i, \ldots, n\}$, $a_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$. Thus $f \equiv tg$ for a constant t such that $t^d = 1$, $d = \gcd\{n+m, \ldots, n+m-i, \ldots, n\}$, $a_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$.

If h is not a constant, then from (3.24) we can say that f and g satisfy the algebraic equation $R(f,g) = 0$, where

$$\begin{aligned} R(x,y) = x^n(a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0) \\ - y^n(a_m y^m + a_{m-1} y^{m-1} + \cdots + a_1 y + a_0). \end{aligned}$$

This completes the proof of Theorem 1.4. ∎

Proof of Corollary 1.5. By (3.18) we have

$$[f^n(\mu f^m + \lambda)]^{(k)} [g^n(\mu g^m + \lambda)]^{(k)} \equiv 1. \tag{3.25}$$

We consider the following subcases.

Subcase (i). We assume that $\lambda = 0$ and $\mu \neq 0$. Since $f(z) \neq \infty$ and $g(z) \neq \infty$, by (3.25) and Lemma 2.12 we obtain $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are constants satisfying $(-1)^k \mu^2 (c_1 c_2)^{n+m} [(n+m)c]^{2k} = 1$. A similar result holds for $\lambda \neq 0$ and $\mu = 0$.

Subcase (ii). Let $\lambda\mu \neq 0$. Then by Lemma 2.11 we see that (3.25) does not arise for $m \geq 1$ and $k = 1$.

Again from (3.23) we have

$$f^n(\mu f^m + \lambda) \equiv g^n(\mu g^m + \lambda). \tag{3.26}$$

If $\lambda\mu = 0$, then from $|\lambda| + |\mu| \neq 0$ we get $f(z) = tg(z)$, where t is a constant such that $t^{n+m^*} = 1$.

If $\lambda\mu \neq 0$, then we suppose that $h = f/g$. Let $h \not\equiv 1$ and $h \not\equiv -1$. Then from (3.26) we obtain

$$g^m = -\frac{\lambda}{\mu}\frac{1-h^n}{1-h^{n+m}}.$$

Hence

$$f^m = -\frac{\lambda}{\mu}\frac{1+h+\cdots+h^{n-1}}{1+h+\cdots+h^{n+m-1}}h^m.$$

If $m \geq 1$, then from the above it follows that

$$T(r,f) = \frac{n+m-1}{m}T(r,h) + S(r,f).$$

Since a pole of f with multiplicity p must be a zero of the equation $h^{n+m}-1 = 0$ with multiplicity mp, by the second fundamental theorem of Nevanlinna we deduce that

$$\overline{N}(r,\infty;f) = \frac{1}{m}\sum_{j=1}^{n+m-1}\overline{N}(r,\alpha_j;h) \geq \frac{n+m-3}{m}T(r,h) + S(r,f),$$

where α_j $(\neq 1)$ $(j = 1,\ldots,n+m-1)$ are distinct roots of $h^{n+m}-1 = 0$. So

$$\Theta(\infty,f) = 1 - \limsup_{r\to\infty}\frac{\overline{N}(r,\infty;f)}{T(r,f)} \leq \frac{2}{n+m-1},$$

which contradicts the fact that $\Theta(\infty,f) > 2/(n+m-1)$. Thus $f \equiv g$ or $f \equiv -g$. Clearly the case $f \equiv -g$ may arise only when both n and m are even integers.

This proves Corollary 1.5. ∎

Proof of Corollary 1.6. By (3.18) we have

$$[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)} \equiv 1. \tag{3.27}$$

Let $m = 0$. Since $f(z) \neq \infty$ and $g(z) \neq \infty$, by (3.27) and Lemma 2.12 we obtain $f(z) = c_1e^{cz}$, $g(z) = c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$.

Again by Lemma 2.10, (3.27) does not arise for $k = 1$ and $m \geq 1$. By (3.23),

$$f^n(f-1)^m \equiv g^n(g-1)^m. \tag{3.28}$$

Now we consider the following three subcases.

Subcase (i). Let $m = 0$. Then from (3.28) we get $f \equiv tg$ for a constant t such that $t^n = 1$.

Subcase (ii). Let $m = 1$. Then from (3.28) we obtain

$$f^n(f-1) \equiv g^n(g-1). \tag{3.29}$$

Suppose $f \not\equiv g$. Let $h = f/g$ be a constant. Then from (3.29) it follows that $h \neq 1$, $h^n \neq 1$, $h^{n+1} \neq 1$ and $g = (1-h^n)/(1-h^{n+1}) =$ constant, a contradiction. So we suppose that h is not a constant. Since $f \not\equiv g$, we have $h \not\equiv 1$ and so $g = (1-h^n)/(1-h^{n+1})$ and $f = (1-h^n)h/(1-h^{n+1})$. Thus it follows that

$$T(r,f) = nT(r,h) + S(r,f).$$

By the second fundamental theorem of Nevanlinna

$$\overline{N}(r,\infty;f) = \sum_{j=1}^{n} \overline{N}(r,\alpha_j;h) \geq (n-2)T(r,h) + S(r,f),$$

where α_j $(\neq 1)$ $(j = 1,\ldots,n)$ are distinct roots of the equation $h^{n+1}-1 = 0$. So

$$\Theta(\infty,f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,\infty;f}{T(r,f)} \leq 2/n,$$

which contradicts our assumption that $\Theta(\infty,f) > 2/n$. Thus $f \equiv g$.

Subcase (iii). Suppose that $m \geq 2$. Then from (3.28) we get

$$f^n[f^m + \cdots + (-1)^i\, {}^mC_{m-i} f^{m-i} + \cdots + (-1)^m] = g^n[g^m + \cdots + (-1)^i\, {}^mC_{m-i} g^{m-i} + \cdots + (-1)^m]. \tag{3.30}$$

Let $h = f/g$. If h is a constant, then substituting $f = gh$ in (3.30) we obtain

$$g^{n+m}(h^{n+m}-1) + \cdots + (-1)^i\, {}^mC_{m-i} g^{n+m-i}(h^{n+m-i}-1) + \cdots + (-1)^m g^n(h^n-1) = 0,$$

which implies $h^d = 1$, where $d = \gcd\{n+m,\ldots,n+m-i,\ldots,n+1,n\}$. Thus $f \equiv tg$ for a constant t such that $t^d = 1$, $d = \gcd\{n+m,\ldots,n+m-i,\ldots,n+1,n\}$.

If h is not a constant, from (3.28) we see that f and g satisfy the algebraic equation $R(f,g) = 0$, where $R(x,y) = x^n(x-1)^m - y^n(y-1)^m$.

This completes the proof of Corollary 1.6. ∎

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