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A unicity theorem for plurisubharmonic functions

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Abstract. We give sufficient conditions for unicity of plurisubharmonic functions in Cegrell classes.

1. Introduction. Let Ω be an open subset of $\mathbb{C}^n$. An upper semicontinuous function $u : \Omega \to [-\infty, \infty)$ is said to be *plurisubharmonic* if the restriction of u to each complex line is subharmonic (we allow the function identically $-\infty$ to be plurisubharmonic). We write $\mathrm{PSH}(\Omega)$ (resp. $\mathrm{PSH}^-(\Omega)$) for the cone of plurisubharmonic (resp. negative plurisubharmonic) functions on Ω. The domain Ω is said to be *hyperconvex* if there exists a continuous negative plurisubharmonic exhaustion function for Ω.

Let $u, v \in \mathrm{PSH}^-(\Omega)$ be such that $\lim_{z\to\partial\Omega} u(z) = \lim_{z\to\partial\Omega} v(z) = 0$. In this note, we are aiming at sufficient conditions to ensure that $u = v$ near the boundary $\partial\Omega$. Before formulating the main result, it is convenient to recall the following concept. A compact subset K of Ω is said to be *holomorphically convex* if for every $z \in \Omega \setminus K$, there exists a holomorphic function f on Ω such that $\|f\|_K < |f(z)|$.

We will prove the following.

Theorem A. *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$. Let $K \subset \Omega$ be a compact holomorphically convex subset of Ω. Let $u_1, u_2 \in \mathrm{PSH}^-(\Omega)$ be such that the following conditions hold:*

(a) $\lim_{z\to\partial\Omega} u_1(z) = \lim_{z\to\partial\Omega} u_2(z) = 0$.
(b) $(dd^c u_1)^n \le (dd^c u_2)^n$ *on* $\Omega \setminus K$ *and* $\int_\Omega (dd^c u_2)^n < \infty$.
(c) $u_1 \le u_2$ *on* $\Omega \setminus K$.
(d) $\int_K (dd^c u_1)^n \le \int_K (dd^c u_2)^n$.

Then $u_1 = u_2$ *on* $\Omega \setminus K$.

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Here $(dd^c)^n$ is the complex Monge–Ampère operator, which can be defined over the class of locally bounded plurisubharmonic functions (cf. [BT1], [BT2]). Later on, this operator was extensively studied in [Dem], where we can find extensions of $(dd^c)^n$ to certain classes of non-locally bounded plurisubharmonic functions. In particular, this operator can be well defined in the class of negative plurisubharmonic functions which are bounded in a neighborhood of the boundary (e.g., functions like u_1, u_2 in Theorem A). See Lemma 3.3 in [Ce]. We refer the reader to [Ce] also for a comprehensive account on the domain of definition of $(dd^c)^n$. Theorem A is inspired by Theorem 2.4 in [BL], where the authors study a sort of unique continuation of plurisubharmonic functions.

Theorem A can be used to give a sort of *quasi*-unicity property for polynomial maps of n complex variables. Before turning to this result, we introduce some notation. Given a polynomial map $P := (p_1, \ldots, p_n) : \mathbb{C}^n \to \mathbb{C}^n$, we write $\mathcal{Z}(P)$ for the common zero set of $p_1, \ldots, p_n$; we also denote by $\mathcal{P}(P; \lambda)$ the polynomial polyhedron

$$\{z \in \mathbb{C}^n : |p_1(z)| < \lambda, \ldots, |p_n(z)| < \lambda\}.$$

Corollary B. *Let $P := (p_1, \ldots, p_n), Q := (q_1, \ldots, q_n)$ be proper polynomial maps $\mathbb{C}^n \to \mathbb{C}^n$. Assume that the following conditions are satisfied:*

(a) $\prod_{j=1}^n \deg p_j \geq \prod_{j=1}^n \deg q_j$.
(b) *There exists $a > 0$ such that $\mathcal{P}(P; a) = \mathcal{P}(Q; a) =: \Omega$ and Ω is connected.*
(c) *There exists $\varepsilon > 0$ such that $\mathcal{P}(P; b) \subset \mathcal{P}(Q; b)$ for every $b \in (a-\varepsilon, a)$.*

Then for every $1 \leq j \leq n$ there exists $1 \leq k(j) \leq n$ and a constant $|\lambda_j| = 1$ such that $p_j = \lambda_j q_{k(j)}$.

By considering the two polynomial maps $(z_1, \ldots, z_n)$ and $(z_1^2, \ldots, z_n^2)$ we see that assumption (a) is indispensable.

2. Proofs. Throughout this note, we will write Ω for a bounded hyperconvex domain in $\mathbb{C}^n$. Recall that $\mathcal{E}_0(\Omega)$ is the Cegrell class of bounded functions $u \in \mathrm{PSH}^-(\Omega)$ such that

$$\lim_{z \to \partial\Omega} u(z) = 0 \quad \text{and} \quad \int_{\Omega} (dd^c u)^n < \infty.$$

First, we need the following useful fact.

Lemma C. *Let K be a compact holomorphically convex subset of Ω. Let $z_0 \in \Omega \setminus K$. Then there exist a neighborhood U of z_0 and $\psi \in \mathcal{E}_0(\Omega) \cap \mathcal{C}(\Omega)$ such that $\psi \geq -1$ on Ω, $\psi \equiv -1$ on a neighborhood of K and ψ is strictly plurisubharmonic on U.*

Proof. First, we claim that there exists a continuous bounded plurisubharmonic function u on Ω such that

$$\sup_{z\in K} u < u(z_0).$$

For this, we use the following argument due to Poletsky (see Lemma 4.1 in [Po]). Let ρ be a bounded negative continuous plurisubharmonic exhaustion function for Ω. Choose $0 < \varepsilon < \varepsilon'$ such that

$$K \cup \{z_0\} \subset \{\rho < -\varepsilon'\} \subset \{\rho < -\varepsilon\}.$$

Since K is holomorphically convex in Ω, we can find a bounded function $v \in \mathrm{PSH}(\Omega) \cap \mathcal{C}(\Omega)$ and a constant $\alpha < -1$ such that

$$-\alpha < v|_K < -1, \qquad v(z_0) > -1.$$

Let β be the maximum of v on $\{z : \rho(z) = -\varepsilon\}$. Then $\beta > -1$. Consequently, the function

$$\rho' := 2\frac{\alpha+\beta}{\varepsilon'-\varepsilon}\max\{\rho+\varepsilon', 0\} - \alpha$$

is smaller than v on $\{\rho < -\varepsilon'\}$ and larger than v on $\{\rho = -\varepsilon\}$. It follows that the function u equal to $\max\{\rho', v\}$ on $\{\rho < -\varepsilon\}$ and ρ' on $\{\rho \ge -\varepsilon\}$ is plurisubharmonic on Ω. The claim now follows since u is bounded on Ω and $u = v$ on $\{\rho < -\varepsilon'\}$.

Set

$$-\infty < a := \sup_{z\in K} u(z) < u(z_0) < b := \sup_{z\in\Omega} u(z) < \infty.$$

Choose an increasing convex function $\chi : (-\infty, b) \to \mathbb{R}$ such that $\chi(a) < -1 < \chi(b) < 0$. Then for small $\varepsilon > 0$ we can find a small neighborhood U of z_0 such that the function $\hat{u}(z) := \chi \circ u(z) + \varepsilon|z|^2$ is continuous strictly plurisubharmonic on U and

$$\sup_K \hat{u} < -1 < \inf_U \hat{u} < \sup_\Omega \hat{u} < 0.$$

Set $\tilde{u} := \max\{\hat{u}, -1\}$. Then $\tilde{u} \in \mathrm{PSH}^-(\Omega)$, $\tilde{u} \equiv -1$ on a neighborhood of K, and $\tilde{u} = u$ is strictly plurisubharmonic on U. Let B be an open ball contained in Ω. It is well known that the relative extremal function

$$\rho''(z) := \sup\{\theta(z) : \theta \in \mathrm{PSH}^-(\Omega),\ \theta|_B \le -1\}$$

belongs to $\mathcal{C}(\Omega) \cap \mathcal{E}_0(\Omega)$. It follows that for $A > 0$ sufficiently large, $\psi := \max(A\rho'', \tilde{u}) \in \mathrm{PSH}^-(\Omega)$, $\psi = \tilde{u}$ on U, $\psi \equiv -1$ on a neighborhood of K and $\psi = A\rho''$ on a small neighborhood of $\partial\Omega$. By Stokes' theorem we also have

$$\int_\Omega (dd^c\psi)^n = A^n \int_\Omega (dd^c\rho'')^n < \infty.$$

Thus $\psi \in \mathcal{E}_0(\Omega) \cap C(\Omega)$. The proof is complete.

Now we are able to give

Proof of Theorem A. We proceed in two steps.

STEP 1. We show that $(u_2 - u_1)T = 0$ on $\Omega \setminus K$, where

$$T := \sum_{l=0}^{n-1} (dd^c u_1)^l \wedge (dd^c u_2)^{n-l-1}.$$

After linear changes of coordinates, it is enough to prove

$$(u_2 - u_1) dd^c |z|^2 \wedge T = 0 \quad \text{on } \Omega \setminus K.$$

Fix $z_0 \in \Omega \setminus K$. By Lemma C, there exists a small neighborhood $U \subset \Omega \setminus K$ of z_0 and $\psi \in \mathcal{E}_0(\Omega)$ such that $\psi \equiv -1$ on a neighborhood of K and ψ is strictly plurisubharmonic on U. Notice also that, by assumptions (a) and (b),

$$\max\{u_1, -j\} \downarrow u_1, \quad \int_\Omega (dd^c \max\{u_1, -j\})^n = \int_\Omega (dd^c u_1)^n < \infty \quad \forall j \geq 1.$$

Here the latter equality follows from Stokes' theorem. This implies that $u_1 \in \mathcal{F}(\Omega)$. See Section 4 in [Ce] for details on the Cegrell class $\mathcal{F}(\Omega)$. Similarly, we also have $u_2 \in \mathcal{F}(\Omega)$. By Corollary 5.6 in [Ce] we have

$$-\infty < \int_\Omega \psi dd^c u_1 \wedge T, \quad -\infty < \int_\Omega \psi dd^c u_2 \wedge T.$$

These facts allow us to apply Cegrell's integration by part formula (Corollary 3.4 in [Ce]) to get

$$\begin{aligned} \int_\Omega u_1 dd^c \psi \wedge T &= \int_\Omega \psi dd^c u_1 \wedge T, \\ \int_\Omega u_2 dd^c \psi \wedge T &= \int_\Omega \psi dd^c u_2 \wedge T. \end{aligned} \tag{1}$$

Since $\psi = -1$ on K, we have $u_1 \leq u_2$ on $\Omega \setminus K$, and since $dd^c\psi = 0$ on a neighborhood of K, we may apply (1) to obtain

$$\begin{aligned} 0 &\leq \int_U (u_2 - u_1) dd^c \psi \wedge T \leq \int_\Omega (u_2 - u_1) dd^c \psi \wedge T \\ &= \int_\Omega \psi dd^c (u_2 - u_1) \wedge T = \int_\Omega \psi [(dd^c u_2)^n - (dd^c u_1)^n] \\ &= \int_K [(dd^c u_1)^n - (dd^c u_2)^n] + \int_{\Omega \setminus K} \psi [(dd^c u_2)^n - (dd^c u_1)^n] \leq 0. \end{aligned}$$

Here the last equality follows from (b), (d) and the fact that $\psi < 0$ in Ω. Since ψ is strictly plurisubharmonic on U we get $(u_2 - u_1) dd^c |z|^2 \wedge T = 0$ on U. The desired conclusion follows.

STEP 2. We show $u_1 = u_2$ on $\Omega \setminus K$. Assume towards a contradiction that there exists $a \in \Omega \setminus K$ such that $u_1(a) < u_2(a)$. Since K is holomorphically convex in Ω, we can find a small ball $B \subset \Omega \setminus K$ around the point a and a (non-constant) holomorphic function f on Ω such that $f(B) \cap f(K) = \emptyset$. Let Z be the set of points $x \in \Omega$ such that the complex hypersurface $\{z \in \Omega : f(z) = f(x)\}$ is smooth. By Sard's theorem, the set $\Omega \setminus Z$ has Lebesgue measure zero. Thus we may choose a point $\xi \in Z \cap B$ such that $u_1(\xi) < u_2(\xi)$.

Denote by S_ξ the connected component of $\{z \in \Omega : f(z) = f(\xi)\}$ that contains ξ. Then $S_\xi \cap K = \emptyset$.

It follows from Step 1 that

$$(u_2 - u_1)(dd^c u_1)^{n-1} = 0 \quad \text{ on } \Omega \setminus K.$$

Denote by u_1', u_2' the restrictions of u_1, u_2 to S_ξ. Now we apply the slicing theory of Bedford–Taylor (see Section 4 in [BT2], in particular the remark following Corollary 4.3) to obtain

$$(u_2' - u_1')(dd^c u_1')^{n-1} = 0 \quad \text{ on } S_\xi.$$

Since u_1, u_2 are not locally bounded on Ω, we cannot directly apply Bedford–Taylor's results. Instead, we follow their method: first we notice that the formula is obvious when u_1, u_2 are smooth, and then, by using Proposition 5.1 in [Ce] on the continuity of the complex Monge–Ampère operator in $\mathcal{F}(\Omega)$, we get the desired equality (see [BT2, p. 149] for a similar argument).

Since $u_1' \le u_2'$ on S_ξ we infer

$$\int_{\{u_1' < u_2'\}} (dd^c u_1')^{n-1} = 0.$$

Notice that Ω contains no *compact* complex variety of positive dimension, so $\partial(S_\xi \cap \Omega) \subset \partial\Omega$. Therefore

$$\lim_{z \to \partial(S_\xi \cap \Omega),\, z \in S_\xi} (u_1'(z) - u_2'(z)) = 0.$$

An application of the comparison principle to the smooth complex hypersurface S_ξ (see Corollary 3.7.5 in [Kl]) yields $u_1' = u_2'$ on S_ξ. In particular,

$$u_1(\xi) = u_1'(\xi) = u_2'(\xi) = u_2(\xi).$$

This contradicts the choice of ξ. The proof is complete.

Remark. If we suppose that $u_1 \le u_2$ entirely on Ω then Theorem A is a direct consequence of Lemma C and Lemma 3.5 in [ACCH]. Indeed, let z_0 be an arbitrary point in $\Omega \setminus K$. Choose $\psi \in \mathcal{E}_0(\Omega)$ such that $\psi \ge -1$, $\psi|_K = -1$ and ψ is strictly plurisubharmonic near z_0. By Lemma 3.5 in [ACCH] we have

$$\frac{1}{n!} \int_\Omega (u_2 - u_1)^n (dd^c \psi)^n + \int_\Omega (-\psi)(dd^c u_2)^n \le \int_\Omega (-\psi)(dd^c u_1)^n.$$

By assumptions on u_1, u_2 and the choice of ψ we get $\int_\Omega (u_2-u_1)^n(dd^c\psi)^n = 0$. In particular, this implies $u_1(z_0) = u_2(z_0)$.

Proof of Corollary B. Since the polynomial maps $P, Q : \mathbb{C}^n \to \mathbb{C}^n$ are proper we deduce that Ω is bounded. Hence Ω is a hyperconvex domain. Set

$$u = \max\{\log|p_1|, \dots, \log|p_n|\}, \qquad v = \max\{\log|q_1|, \dots, \log|q_n|\}.$$

According to Proposition 4.12 in [Dem] we have

$$(dd^cu)^n = \sum_{\lambda\in\mathcal{Z}(P)} m_\lambda \delta_\lambda.$$

Here δ_λ is the Dirac mass at λ and m_λ is the multiplicity of $\mathcal{Z}(P)$ at λ. Set $K := \overline{\mathcal{P}(P; a-\varepsilon)}$. It follows that

$$\int_{\mathbb{C}^n} (dd^cu)^n = \int_K (dd^cu)^n = \sum_{\lambda\in\mathcal{Z}(P)} m_\lambda = \prod_{j=1}^n \deg p_j.$$

Here the last equality follows from Bézout's theorem. In the same way we obtain

$$\int_{\mathbb{C}^n} (dd^cv)^n = \int_K (dd^cv)^n = \prod_{j=1}^n \deg q_j.$$

Combining the above equality with (a) we get $\int_K (dd^cu)^n \ge \int_K (dd^cv)^n$. On the other hand, note that $(dd^cu)^n = (dd^cv)^n = 0$ on $\Omega \setminus K$. It also follows from (b) and (c) that $u = v$ on $\partial\Omega$ and $u \ge v$ on $\Omega\setminus K$. Therefore, we may apply Theorem A to $u - \log a, v - \log a$ and obtain $u = v$ on $\Omega \setminus K$.

Next, we set

$$U = \{z \in \Omega\setminus K : |p_i(z)| \neq |p_j(z)|,\ |q_i(z)| \neq |q_j(z)| \text{ for all } i \neq j\}.$$

Since $P, Q : \mathbb{C}^n \to \mathbb{C}^n$ are proper, we deduce that U is open and dense in $\Omega\setminus K$. Fix $1 \le j \le n$, and choose $z_0 \in U$ such that $|p_j(z_0)| = u(z_0)$. Then there exists $k(j)$ such that $|q_{k(j)}(z_0)| = v(z_0)$. By continuity, we can find a small neighborhood V of z in U such that

$$|p_j(z)| = u(z) = v(z) = |q_{k(j)}(z)| \quad \forall z \in V.$$

Observe that the map $\varphi := p_j/q_{k(j)} : \mathbb{C}^n \setminus \{q_{k(j)} = 0\} \to \mathbb{C}$ is either open or constant. Since φ maps an open subset of V onto the unit circle, we infer that φ must be constant. Thus $p_j \equiv \lambda_j q_{k(j)}$ for some constant $|\lambda_j| = 1$. The proof is complete.

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