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Inclusion relationships between classes of functions defined by subordination

by Jacek Dziok (Rzeszów)

Abstract. The purpose of the present paper is to investigate various inclusion relationships between several classes of analytic functions defined by subordination. Many interesting applications involving the well-known classes of functions defined by linear operators are also considered.

1. Introduction. Let $\mathcal{A}$ denote the class of functions which are *analytic* in $\mathcal{U} = \mathcal{U}(1)$, where

$$\mathcal{U}(r) = \{z \in \mathbb{C} : |z| < r\}.$$

We denote by $\mathcal{A}_0$ the class of functions $f \in \mathcal{A}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathcal{U}). \tag{1}$$

We say that a function $f \in \mathcal{A}$ is *subordinate* to a function $F \in \mathcal{A}$, and write $f(z) \prec F(z)$ (or simply $f \prec F$), if there exists a function $\omega \in \mathcal{A}$,

$$|\omega(z)| \le |z| \quad (z \in \mathcal{U}),$$

such that

$$f(z) = F(\omega(z)) \quad (z \in \mathcal{U}).$$

Moreover, we say that f is *subordinate to* F *in* $\mathcal{U}(r)$ if $f(rz) \prec F(rz)$. Then we shall write $f(z) \prec_r F(z)$. In particular, if F is univalent in $\mathcal{U}$, we have the following equivalence:

$$f(z) \prec F(z) \;\Leftrightarrow\; [f(0) = F(0) \wedge f(\mathcal{U}) \subset F(\mathcal{U})]. \tag{2}$$

For functions $f, g \in \mathcal{A}$ of the form

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad g(z) = \sum_{n=0}^{\infty} b_n z^n,$$

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we denote by $f*g$ the *Hadamard product* (or *convolution*) of f and g, defined by

$$(f*g)(z)=\sum_{n=0}^{\infty}a_nb_nz^n \qquad (z\in\mathcal{U}).$$

Let ϕ,φ be given functions from the class $\mathcal{A}_0$ and let $h\in\mathcal{A}$ be a function convex and univalent in $\mathcal{U}$, with $h(0)=1$.

We denote by $\mathcal{W}(\phi(z),\varphi(z);h(z))$ (or simply $\mathcal{W}(\phi,\varphi;h)$) the class of functions $f\in\mathcal{A}_0$ such that

$$\frac{(\phi*f)(z)}{(\varphi*f)(z)}\prec h(z).$$

Moreover, we define

$$\mathcal{W}(\varphi;h):=\mathcal{W}(z\varphi'(z),\varphi;h), \qquad S^*(h):=\mathcal{W}\left(\frac{z}{1-z};h\right).$$

In particular, the classes

$$S^*:=S^*\left(\frac{1+z}{1-z}\right), \qquad S^c:=\mathcal{W}\left(\frac{z}{(1-z)^2};\frac{1+z}{1-z}\right)$$

are the classes of *starlike* functions and *convex* functions, respectively.

It is clear that

$$g\in\mathcal{W}(\varphi;h) \;\Leftrightarrow\; \varphi*g\in S^*(h). \tag{3}$$

We say that a function $f\in\mathcal{A}_0$ belongs to the class $\mathcal{C}(\phi,\varphi;h)$ if there exists a function $g\in\mathcal{W}(\varphi;h)$ such that

$$\frac{(\phi*f)(z)}{(\varphi*g)(z)}\prec h(z).$$

Furthermore, we let

$$\mathcal{C}(\varphi;h):=\mathcal{C}(\varphi,\varphi;h).$$

The equivalence (3) defines the operator

$$J:\mathcal{W}(\varphi;h)\to S^*(h).$$

If

$$\varphi^{(k)}(0)\neq 0 \qquad (k=1,2,\ldots),$$

then the operator J is one-to-one and we have $f\in\mathcal{C}(\phi,\varphi;h)$ if and only if there exists a function $g\in S^*(h)$ such that

$$\frac{(\phi*f)(z)}{g(z)}\prec h(z).$$

In particular, the class

$$\mathrm{CC}:=\bigcup_{\beta\in(-\pi/2,\pi/2)}\mathcal{C}\left(\frac{z}{(1-z)^2},\frac{z}{1-z};\frac{1+e^{-2i\beta}z}{1-z}\right)$$

is the well-known class of *close-to-convex functions.*

It is easy to show the following relationships:

$$\mathcal{W}(\phi, z; h) = \mathcal{C}(\phi, z; h) \subset \mathcal{C}(\phi, \varphi; h) \subset \mathcal{C}\left(\phi, \frac{z}{1-z}; h\right) \quad (\varphi \in \mathcal{A}_0).$$

If the function h satisfies the condition

$$\Re[h(z)] > 0 \quad (z \in \mathcal{U}), \tag{4}$$

then we denote the classes $\mathcal{W}(\varphi; h)$, $\mathcal{C}(\phi, \varphi; h)$ by $\mathcal{W}_0(\varphi; h)$, $\mathcal{C}_0(\phi, \varphi; h)$, respectively.

The main object of this paper is to investigate various inclusion relationships between the above-defined classes of functions. Many interesting applications for the well-known classes of functions defined by linear operators are also considered.

2. The main inclusion relationships. The following lemmas will be required in our present investigation.

Lemma 1 ([12]). *If $f \in S^c$, $g \in S^*$, $H \in \mathcal{A}$, then*

$$\frac{f * (Hg)}{f * g}(\mathcal{U}) \subseteq \overline{\text{co}}\{H(\mathcal{U})\}, \tag{5}$$

where $\overline{\text{co}}\{H(\mathcal{U})\}$ denotes the closed convex hull of $H(\mathcal{U})$.

We shall also need the lemma due to Eenigenburg, Miller, Mocanu and Reade [4].

Lemma 2. *Let $\beta, \gamma \in C$, $\beta \neq 0$, and suppose that $H \in \mathcal{A}$ is convex univalent in $\mathcal{U}$, with*

$$\Re(\beta H(z) + \gamma) \geq 0 \quad (z \in \mathcal{U}).$$

If $q \in \mathcal{A}$, $q(0) = H(0)$, $0 < r \leq 1$, then the subordination

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} \prec_r H(z)$$

implies that

$$q(z) \prec_r H(z).$$

Making use of Lemma 1, we get the following theorem.

Theorem 1. *Let $\varphi, f \in \mathcal{A}_0$ and suppose that*

$$G = \varphi * f \in S^*.$$

If $f \in \mathcal{W}(\phi, \varphi; h)$ and $\psi \in S^c$, then $f \in \mathcal{W}(\psi * \phi, \psi * \varphi; h)$.

Proof. Let $f \in \mathcal{W}(\phi, \varphi; h)$. Then there exists $\omega \in \mathcal{A}$ with $|\omega(z)| \leq |z|$ $(z \in \mathcal{U})$ such that

$$\frac{(\phi * f)(z)}{(\varphi * f)(z)} = h(\omega(z)) \quad (z \in \mathcal{U}).$$

Thus, applying properties of the convolution we get

$$\frac{[(\psi*\phi)*f](z)}{[(\psi*\varphi)*f](z)}=\frac{\psi(z)*[h(\omega(z))G(z)]}{\psi(z)*G(z)}.$$

Consequently, by using Lemma 1 we conclude that

$$\frac{[(\psi*\phi)*f](z)}{[(\psi*\varphi)*f](z)}\in\overline{\mathrm{co}}\{h(\omega(\mathcal{U}))\}\subset\overline{\mathrm{co}}\{h(\mathcal{U})\}\quad (z\in\mathcal{U}).$$

Because h is convex and univalent in $\mathcal{U}$, by (2) we obtain

$$\frac{[(\psi*\phi)*f](z)}{[(\psi*\varphi)*f](z)}\prec h(z).$$

This gives $f\in\mathcal{W}(\psi*\phi,\psi*\varphi;h)$, and proves the theorem. ∎

Corollary 1. *If $\psi\in S^c$, then*

$$\mathcal{W}_0(\varphi;h)\subset\mathcal{W}_0(\psi*\varphi;h).$$

Proof. Let $f\in\mathcal{W}_0(\varphi;h)$. Then, by (4), the function $\varphi*f$ belongs to the class S^*. Moreover, putting

$$\phi(z):=z\varphi'(z)\quad (z\in\mathcal{U}),$$

we have

$$(\psi*\phi)(z)=z(\psi*\phi)'(z)\quad (z\in\mathcal{U}).$$

Thus, Corollary 1 follows directly from Theorem 1. ∎

Theorem 2. *If $\psi\in S^c$, then*

$$\mathcal{C}_0(\phi,\varphi;h)\subset\mathcal{C}_0(\psi*\phi,\psi*\varphi;h).$$

Proof. Let $f\in\mathcal{C}_0(\phi,\varphi;h)$. Then there exists a function $g\in\mathcal{W}_0(\varphi;h)$ and $\omega\in\mathcal{A}$ with $|\omega(z)|\le|z|$ $(z\in\mathcal{U})$ such that

$$\frac{(\phi*f)(z)}{(\varphi*g)(z)}=h(\omega(z))\quad (z\in\mathcal{U}). \tag{6}$$

Since $g\in\mathcal{W}_0(\varphi;h)$ and the function h satisfies (4), the function $G=\varphi*g$ belongs to the class S^*. Thus, applying (6) and properties of the convolution we get

$$\frac{[(\psi*\phi)*f](z)}{[(\psi*\varphi)*g](z)}=\frac{\psi(z)*[h(\omega(z))G(z)]}{\psi(z)*G(z)}.$$

Consequently, by using Lemma 1 we conclude that

$$\frac{[(\psi*\phi)*f](z)}{[(\psi*\varphi)*g](z)}\in\overline{\mathrm{co}}\{h(\omega(\mathcal{U}))\}\subset\overline{\mathrm{co}}\{h(\mathcal{U})\}\quad (z\in\mathcal{U}).$$

Because h is convex and univalent in $\mathcal{U}$, by (2) we obtain

$$\frac{[(\psi*\phi)*f](z)}{[(\psi*\varphi)*g](z)}\prec h(z).$$

Moreover, by Corollary 1 we have $g\in\mathcal{W}_0(\psi*\varphi;h)$. This gives $f\in\mathcal{C}_0(\psi*\phi,\psi*\varphi;h)$, and proves the theorem. ∎

It is clear that the function

$$\psi(z) := \log\left(\frac{1}{1-z}\right) \quad (z \in \mathcal{U})$$

belongs to the class S^c and

$$\varphi(z) = \psi(z) * z\varphi'(z), \quad \phi(z) = \psi(z) * z\phi'(z) \quad (z \in \mathcal{U}).$$

Thus, by using Theorems 1 and 2, and Corollary 1 we obtain the following two corollaries:

COROLLARY 2. *If* $f \in \mathcal{W}(z\phi'(z), z\varphi'(z); h)$ *and* $\varphi * f \in S^*$, *then* $f \in \mathcal{W}(\phi, \varphi; h)$.

COROLLARY 3.

$$\mathcal{W}_0(z\varphi'(z); h) \subset \mathcal{W}_0(\varphi; h), \quad \mathcal{C}_0(z\phi'(z), z\varphi'(z); h) \subset \mathcal{C}_0(\phi, \varphi; h).$$

THEOREM 3. *Let* $\varphi \in \mathcal{A}_0$ *and*

$$\phi_a(z) := \frac{1}{a} z\varphi'(z) + \left(1 - \frac{1}{a}\right)\varphi(z) \quad (z \in \mathcal{U},\ 0 < a). \tag{7}$$

If the function h *satisfies the condition*

$$\Re[h(z)] > \max\{1-a, 0\} \quad (z \in \mathcal{U}), \tag{8}$$

then

$$\mathcal{W}(\phi_a; h) \subset \mathcal{W}(\varphi; h).$$

Proof. Let $f \in \mathcal{W}(\phi_a; h)$. Then from the definition of the class $\mathcal{W}(\phi_a; h)$ we have

$$\frac{(z\phi_a'(z)) * f(z)}{(\phi_a * f)(z)} \prec h(z). \tag{9}$$

If we put

$$R = \sup\{r : (\varphi * f)(z) \neq 0,\ z \in \mathcal{U}(r)\},$$

then the function

$$q(z) := \frac{(z\varphi'(z)) * f(z)}{(\varphi * f)(z)} \tag{10}$$

is analytic in $\mathcal{U}(R)$ and $q(0) = 1$. Thus, by (7) we have

$$a\frac{(\phi_a * f)(z)}{(\varphi * f)(z)} = q(z) + a - 1 \quad (z \in \mathcal{U}(R)).$$

Taking the logarithmic derivative and using (10) we get

$$\frac{(z\phi_a'(z)) * f(z)}{\phi_a(z) * f(z)} \prec_R q(z) + \frac{zq'(z)}{q(z) + a - 1}.$$

Hence, by (9) we conclude that

$$q(z) + \frac{zq'(z)}{q(z) + a - 1} \prec_R h(z).$$

Now an application of Lemma 2 leads to

$$q(z) \prec_R h(z). \tag{11}$$

By (10) and from the definition of the class $\mathcal{W}(\varphi; h)$ it is sufficient to verify that $R = 1$. From (11), (10) and (8) we conclude that the function $\varphi * f$ is starlike in $\mathcal{U}(R)$. Thus, we see that $(\varphi * f)(z)$ cannot vanish on $|z| = R$ if $R < 1$. Hence, $R = 1$ and this proves Theorem 3. ∎

THEOREM 4. *Let ϕ_a be defined by* (7). *If the function h satisfies* (8), *then*

$$\mathcal{C}(\phi_a; h) \subset \mathcal{C}(\varphi; h).$$

Proof. Let $f \in \mathcal{C}(\phi_a; h)$. Thus, there exists a function $g \in \mathcal{W}(\phi_a; h)$ such that

$$\frac{(\phi_a * f)(z)}{(\phi_a * g)(z)} \prec h(z). \tag{12}$$

Since h satisfies (8), by (3) the function $\varphi * g$ belongs to the class $S^*(h) \subset S^*$. Therefore, the function

$$q(z) := \frac{(\varphi * f)(z)}{(\varphi * g)(z)}$$

is analytic in $\mathcal{U}$. Upon differentiating both sides of the equality

$$(\varphi * f)(z) = q(z)(\varphi * g)(z)$$

with respect to z, and then simplifying, we obtain

$$\frac{z(\varphi * f)'(z)}{(\varphi * g)(z)} = zq'(z) + P(z)q(z) \quad (z \in \mathcal{U}),$$

where

$$P(z) = \frac{z(\varphi * g)'(z)}{(\varphi * g)(z)} \quad (z \in \mathcal{U}).$$

Thus, by (7) we get

$$\begin{aligned} \frac{(\phi_a * f)(z)}{(\phi_a * g)(z)} &= \frac{z(\varphi * f)'(z) + (a-1)(\varphi * f)(z)}{z(\varphi * g)'(z) + (a-1)(\varphi * g)(z)} \\ &= \frac{zq'(z) + P(z)q(z) + (a-1)q(z)}{P(z) + (a-1)} = q(z) + \frac{zq'(z)}{P(z) + (a-1)}. \end{aligned}$$

Hence, by (12) we conclude that

$$q(z) + \frac{zq'(z)}{P(z) + a - 1} \prec h(z).$$

Since $P(z) \prec h(z)$, by (8) Lemma 2 yields $q(z) \prec h(z)$, and this proves Theorem 4. ∎

3. Applications. The classes $\mathcal{W}(\phi, \varphi; h)$ and $\mathcal{C}(\phi, \varphi; h)$ generalize well-known important classes, which were investigated in earlier works. Most of these classes were defined by using linear operators and special functions.

For complex parameters

$$a_1, \ldots, a_r, b_1, \ldots, b_s \quad (b_j \neq 0, -1, -2, \ldots; \; j = 1, \ldots, s),$$

the *generalized hypergeometric function* ${}_rF_s(a_1, \ldots, a_r; b_1, \ldots, b_s; z)$ is defined by

$${}_rF_s(a_1, \ldots, a_r; b_1, \ldots, b_s; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_r)_n}{(b_1)_n \cdots (b_s)_n} \frac{z^n}{n!}$$
$$(r \leq s+1; \; r, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; \; z \in \mathcal{U}),$$

where

$$(\lambda)_n = \begin{cases} 1 & (n = 0), \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n \in \mathbb{N}) \end{cases}$$

is the Pochhammer symbol.

Next we consider the function $\varphi(a_1, \ldots, a_r; b_1, \ldots, b_s; z)$ given in the following way:

$$\varphi(a_1, \ldots, a_r; b_1, \ldots, b_s; z) := z\,{}_rF_s(a_1, \ldots, a_r; b_1, \ldots, b_s; z) \quad (z \in \mathcal{U}).$$

In particular, the function

$$\Phi(a, c; z) := \varphi(a; c; z) = \sum_{n=1}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} z^n \quad (z \in \mathcal{U}) \tag{13}$$

is called the *incomplete Beta function*. If, for convenience, we put

$$\varphi_a(z) = \varphi(a_1, \ldots, a_r; b_1, \ldots, b_s; z) \quad (a := a_1), \tag{14}$$

then we obtain the following relationships:

$$a\varphi_{a+1}(z) = z(\varphi_a)'(z) + (a-1)\varphi_a(z) \quad (z \in \mathcal{U}), \tag{15}$$

$$\varphi_a(z) = \Phi(a, c; z) * \varphi_c(z) \quad (z \in \mathcal{U}). \tag{16}$$

Corresponding to a function φ_a defined by (14) we consider the following classes:

$$\mathcal{V}_a(r, s; h) := \mathcal{W}(\varphi_a; h), \qquad \mathcal{M}_a(r, s; h) := \mathcal{C}(\varphi_a; h).$$

By using the linear operator

$$H(a_1; r, s)f(z) = \varphi(a_1, \ldots, a_r; b_1, \ldots, b_s; z) * f(z), \tag{17}$$

introduced by Dziok and Srivastava [3], we can define the class $\mathcal{V}_a(r, s; h)$ alternatively in the following way:

$$f \in \mathcal{V}_a(r, s; h) \;\Leftrightarrow\; a\frac{H(a+1; r, s)f(z)}{H(a; r, s)f(z)} + 1 - a \prec h(z). \tag{18}$$

Corollary 4. *If the function h satisfies* (8), *and $m \in \mathbb{N}$, $a > 0$, then*

(19) $$\mathcal{V}_{a+m}(r,s;h) \subset \mathcal{V}_a(r,s;h), \quad \mathcal{M}_{a+m}(r,s;h) \subset \mathcal{M}_a(r,s;h).$$

Proof. It is clear that it is sufficient to prove the corollary for $m = 1$. Let φ_a be defined by (14). If we put $\varphi = \varphi_a$ in (7) and apply (15), then we obtain $\phi_a = \varphi_{a+1}$. Hence, by using Theorems 3 and 4, we conclude that

$$\mathcal{W}(\varphi_{a+1};h) \subset \mathcal{W}(\varphi_a;h), \quad \mathcal{C}(\varphi_{a+1};h) \subset \mathcal{C}(\varphi_a;h).$$

This clearly forces the inclusions (19) for $m = 1$. ∎

It is natural to ask about the inclusion relations (19) when m is positive real. Using Corollary 1 and Theorem 2 we shall give a partial answer to this question.

Corollary 5. *If the incomplete Beta function $\Phi(a,c;z)$ defined by* (13) *belongs to the class S^c, then*

(20) $$\mathcal{V}_c(r,s;h) \subset \mathcal{V}_a(r,s;h), \quad \mathcal{M}_c(r,s;h) \subset \mathcal{M}_a(r,s;h).$$

Proof. Let us put

$$\psi(z) = \Phi(a,c;z), \quad \varphi(z) = \varphi_a(z) \quad (z \in \mathcal{U}),$$

where φ_a is defined by (14). Then using Corollary 1, Theorem 2 and relationship (16) we obtain the inclusion relations (20). ∎

A sufficient condition for convexity of the incomplete Beta function $\Phi(a,c;z)$ is given by the following lemma.

Lemma 3 ([11]). *If either*

$$0 < a \le c \quad \text{and} \quad c \ge 2,$$

or

$$\Re(3-c) \le \Re(a) \le \Re(c) \quad \text{and} \quad \operatorname{Im} a = \operatorname{Im} c,$$

then the incomplete Beta function $\Phi(a,c;z)$ defined by (13) *belongs to the class S^c.*

Combining Corollary 5 with Lemma 3 we obtain the following result.

Corollary 6. *If either*

$$0 < a \le c \quad \text{and} \quad c \ge 2,$$

or

$$\Re(3-c) \le \Re(a) \le \Re(c) \quad \text{and} \quad \operatorname{Im} a = \operatorname{Im} c,$$

then the inclusions (20) *hold.*

By setting $c = a + 1$ in Corollary 6, we obtain the following consequence.

Corollary 7. *If $\Re(a) \ge 1$, then the inclusions* (19) *hold.*

The linear operator $H(a_1; r, s)$ defined by (17) includes other linear operators, which were considered in earlier works, such as (for example) the linear operators introduced by Carlson–Shaffer, Ruscheweyh, Bernardi–Libera–Livingston, Owa, and Srivastava–Owa for details (see [15] and [16]). These operators generate well-known special cases of the class $\mathcal{V}_a(r, s; h)$.

We can also obtain various new or well-known classes by choosing the function h in the classes defined above.

First, we define

$$\Omega_{k,\alpha} := \{u + iv : u - \alpha > k\sqrt{(u-1)^2 + v^2}\} \quad (k \geq 0,\, 0 \leq \alpha < 1).$$

Note that $\Omega_{k,\alpha}$ is the convex domain contained in the right half plane, with $1 \in \Omega_{k,\alpha}$. More precisely, it is an elliptic domain for $k > 1$, a hyperbolic domain for $0 < k < 1$, a parabolic domain for $k = 1$ and finally $\Omega_{k,\alpha}$ is the halfplane $\{w : \Re(w) > \alpha\}$ for $k = 0$.

Let us denote by $h_{k,\alpha}$ the univalent function which maps the unit disc $\mathcal{U}$ onto the conic domain $\Omega_{k,\alpha}$, with $h_{k,\alpha}(0) = 1$. Obviously, $h_{k,\alpha}$ is convex in $\mathcal{U}$ and

$$\Re[h_{k,\alpha}(z)] > 0 \quad (z \in \mathcal{U}).$$

It is easy to check that the function f belongs to the class $\mathcal{W}(\phi, \varphi; h_{\ k,\alpha})$ if and only if

$$\Re\left(\frac{(\phi * f)(z)}{(\varphi * g)(z)} - \alpha\right) > k\left|\frac{(\phi * f)(z)}{(\varphi * g)(z)} - 1\right| \quad (z \in \mathcal{U}).$$

In particular, the class $\mathcal{W}(\varphi; h_{k,\alpha})$ was investigated by Raina and Bansal [9] and the class $\mathcal{V}_a(r, s; h_{k,\alpha})$ was studied by Srivastava et al. [10]. If we put $a_1 = 2$, $a_2 = b_1 = 1$, then

$$\text{UCV} := \mathcal{V}_2(2, 1; h_{1,0}) \quad \text{and} \quad k\text{-UCV} := \mathcal{V}_2(2, 1; h_{k,0})$$

are the classes of *uniformly convex* functions and *k-uniformly convex* functions introduced by Goodman [5] and Kanas and Wiśniowska [7], respectively.

The class $\mathcal{V}_{\alpha_1}(r, s; (1 + Az)/(1 + Bz))$ was introduced by Dziok and Srivastava [3].

The classes $\mathcal{W}(\phi, \varphi; h)$ and $\mathcal{C}(\phi, \varphi; h)$ generalize also many classes defined by linear operators, which are not special cases of the operator (17). We can mention here the Sălăgean operator [13], the Noor operator [8], the Choi–Saigo–Srivastava operator [1], the Jung–Kim–Srivastava operator [6], and others.

REMARK 1. If we apply the results presented in this paper to the classes discussed above, we can obtain a lot of partial results. Some of these results were obtained in earlier works (see for example [1, 2, 14]).

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Jacek Dziok
Institute of Mathematics
University of Rzeszów
35-310 Rzeszów, Poland
E-mail: jdziok@univ.rzeszow.pl

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