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On the dynamics of extendable polynomial endomorphisms of $\mathbb{R}^2$

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Abstract. We extend the results obtained in our previous paper, concerning quasiregular polynomials of algebraic degree two, to the case of polynomial endomorphisms of $\mathbb{R}^2$ whose algebraic degree is equal to their topological degree. We also deal with some other classes of polynomial endomorphisms extendable to $\mathbb{CP}^2$.

1. Introduction. The present paper is a continuation of our two previous papers [Li1] and [Li2]. In [Li1] we proved that each quadratic quasiregular polynomial mapping on $\mathbb{R}^2$ can be complexified and extended to the complex projective space $\mathbb{CP}^2$ in such a way that this extension acts on $\mathbb{CP}^2 \setminus \mathbb{C}^2$ as a Blaschke product. Using this fact we described the dynamics of homogeneous, degree two, quasiregular polynomial mappings on $\mathbb{C}$. Moreover, we proved the existence of Böttcher coordinates near ∞ for a nonhomogeneous quasiregular, degree two, polynomial mapping for which the Blaschke product on $\mathbb{CP}^2 \setminus \mathbb{C}^2$ has a fixed point inside the unit disc.

In [Li2] we found a simple algebraic condition which is equivalent to the extendability of a complexification of a polynomial mapping from $\mathbb{R}^2$ into itself to a polynomial endomorphism of $\mathbb{CP}^2$. The polynomial endomorphisms of $\mathbb{R}^2$ which satisfy this condition will be called extendable polynomial endomorphisms.

The aim of the present paper is to extend the results of [Li1] to a wide class of extendable polynomial endomorphisms of $\mathbb{R}^2$ with an arbitrary algebraic degree.

It turns out that the following condition is crucial: The polynomial endomorphism Q must be extendable and have the topological degree equal to its algebraic degree. We shall always denote the algebraic degree by n.

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We shall prove that in this case the homogeneous leading term Q_n must be equal to

$$a\cdot\prod_{i=1}^{n}(z-p_i\overline{z})$$

where $|p_i|<1$ for all i. (Hence Q_n is quasiregular.)

We shall show that all results of [Li1] remain true in this case. We shall also study the dynamical behaviour of extendable Q for which this condition fails.

Everywhere in this paper $C(0,1)$ will denote the unit circle.

2. Preliminaries. We shall identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$. Each polynomial mapping of $\mathbb{R}^2$ into itself can be written in complex coordinates as

$$Q(z)=\sum_{k=0}^{n}Q_k(z)=\sum_{k=0}^{n}\sum_{i=0}^{k}a_{ki}z^i\overline{z}^{k-i}.$$

We shall complexify $Q(z)$ as in [Li1, Li2] and define

$$f(z,w)=\Big(\sum_{k=0}^{n}\sum_{i=0}^{k}a_{ki}z^iw^{k-i},\sum_{k=0}^{n}\sum_{i=0}^{k}\overline{a}_{ki}w^iz^{k-i}\Big).$$

We shall also consider the homogenization of $f(z,w)$ equal to

$$\widetilde{f}(z,w,t)=\Big(\sum_{k=0}^{n}\Big(\sum_{i=0}^{k}a_{ki}z^iw^{k-i}\Big)t^{n-k},\sum_{k=0}^{n}\Big(\sum_{i=0}^{k}\overline{a}_{ki}w^iz^{k-i}\Big)t^{n-k},t^n\Big).$$

The leading term Q_n can be written as

$$Q_n(z)=a_{ni_0}\cdot\overline{z}^{n-i_0}\cdot\prod_{j=1}^{i_0}(z-p_j\overline{z}).$$

The number i_0 is equal to the greatest i for which $a_{ni}\neq 0$ and $p_1,\ldots,p_{i_0}$ are the roots of the polynomial

$$P(\xi)=\sum_{i=0}^{i_0}a_{ni}\xi^i.$$

In [Li2] we proved the following

THEOREM 2.1. *The complexified mapping f extends to a polynomial endomorphism of $\mathbb{CP}^2$ iff one of the following conditions holds*:

(1) $i_0=n$ *and* $p_i\overline{p}_j\neq 1$ *for each* $i,j=1,\ldots,n$;
(2) $i_0<n$, $p_i\neq 0$ *for* $i=1,\ldots,i_0$ *and* $p_i\overline{p}_j\neq 1$ *for* $i,j=1,\ldots,i_0$.

The restriction of the extended map to $\mathbb{CP}^2\setminus\mathbb{C}^2$ is a quotient of two finite Blaschke products.

Note that each rational map on $\widehat{\mathbb{C}}$ for which the unit circle is invariant is a quotient of two finite Blaschke products.

DEFINITION 2.2. If one of the conditions in Theorem 2.1 is satisfied, then Q will be called an *extendable polynomial mapping.*

REMARK 2.3. An extendable polynomial map Q need not map $\mathbb{C}$ onto $\mathbb{C}$. The mapping $Q(z) = \overline{z}(z - p\overline{z})$, $|p| < 1$, $p \in \mathbb{R}$, maps $\mathbb{C}$ onto a closed sector of $\mathbb{C}$.

3. The homogeneous extendable mappings. Most of the results in this section are analogous to the corresponding ones in [Li1]. The proofs are the same or almost the same. However, we decided to give them here in order to make the paper more readable and self-contained.

In this section we shall always assume that

$$Q(z) = Q_n(z) = a_{ni_0} \cdot \overline{z}^{n-i_0} \cdot \prod_{j=1}^{i_0} (z - p_j\overline{z}), \quad n \geq 2,$$

and that Q is extendable. Then the complexified mapping $f(z, w)$ is homogeneous and we can proceed as in [Li1], using Proposition 7.1 from [H-P] to obtain

PROPOSITION 3.1. *The basin of attraction of zero in $\mathbb{C}$ is a bounded domain in $\mathbb{C}$ which is starlike with respect to zero and is given by*

$$\omega_Q = \{z \in \mathbb{C} : h_Q(z) < 0\}$$

where

$$h_Q(z) = \lim_{m\to\infty} \frac{1}{n^m} \log |Q^{\circ m}(z)|, \quad Q^{\circ m}(z) = Q \circ \overset{m\,\text{times}}{\cdots} \circ Q(z).$$

We have the same situation as in [Li1]: if $h_Q(z) > 0$, then

$$\lim_{m\to\infty} Q^{\circ m}(z) = \infty.$$

We have two superattractors: zero and ∞, and the set

$$\mathcal{J}_Q = \{z \in \mathbb{C} : h_Q(z) = 0\}$$

which separates their basins of attractions.

We also have

THEOREM 3.2. *The set $\mathcal{J}_Q$ is a Jordan curve.*

Proof. The polynomial Q is $\mathbb{R}$-homogeneous and hence

$$h_Q(tz) = \log |t| + h_Q(z).$$

This implies that on each halfline issuing from zero there is exactly one point z_0 for which $h_Q(z_0) = 0$.

Take $e^{i\theta} \in C(0, 1)$ and define $\psi(e^{i\theta}) = z(e^{i\theta})$ to be the unique point on the halfline with origin at zero passing through $e^{i\theta}$ for which $h(z(e^{i\theta})) = 0$. It follows from Proposition 7.1 of [H-P] that h_Q is continuous. Hence ψ is also continuous and univalent. The circle $C(0,1)$ is compact and thus ψ is a homeomorphism from $C(0,1)$ onto $\mathcal{J}_Q$. ∎

We are now going to describe the dynamics of Q on $\mathcal{J}_Q$. Similarly to [Li1], we have the following situation:

Let $\phi(\xi)$ denote the restriction of the complexified map $f(z,w)$ to $\mathbb{CP}^2 \setminus \mathbb{C}^2$. According to Theorem 2.1, $\phi(\xi)$ is a quotient of two finite Blaschke products. As in [Li1] we have

PROPOSITION 3.3. *Assume that n is even. There is a one-to-one correspondence between the fixed points of ϕ on the unit circle and the nonzero fixed points of Q. If the fixed point ξ_0 is repelling, then so is the corresponding point z_0. If the fixed point ξ_0 is attracting or neutral with one Leau leaf, then the corresponding point is a saddle point.*

Proof. We proceed in exactly the same way as in the proof of Proposition 4.4 of [Li1]. If $\xi \in C(0,1)$, then we can take $z_1 \in C(0,1)$ with $z_1/\overline{z}_1 = \xi$. We have

$$\frac{z_1}{\overline{z}_1} = \phi(\xi) = \frac{Q(z_1)}{\overline{Q(z_1)}}.$$

Hence $z_1/Q(z_1)$ is a real number. Let

$$z_0 = z_1\left(\frac{z_1}{Q(z_1)}\right)^{1/(n-1)}.$$

We have

$$Q(z_0) = Q(z_1)\left(\frac{z_1}{Q(z_1)}\right)^{n/(n-1)} = \frac{z_1 \cdot z_1^{1/(n-1)}}{(Q(z_1))^{1/(n-1)}} = z_0.$$

If $Q(z_0) = z_0$, then $\phi(\xi) = \xi$ for $\xi = z_0/\overline{z}_0$. In the neighbourhood of the fixed point z_0, $Q(z)$ is conjugate to ϕ via the map $z \mapsto z/\overline{z} = \xi$ and the inverse branch of this map which maps ξ_0 on z_0. This implies the rest of Proposition 3.3. ∎

If n is *odd*, then the conclusion of Proposition 3.3 is not valid.

If $Q(z_0) = z_0$, then $Q(-z_0) = -z_0$. Both z_0 and $-z_0$ correspond to the same fixed point $\xi_0 = z_0/\overline{z}_0 = -z_0/(-\overline{z}_0)$ of $\phi(\xi)$.

If $Q(z_0) = -z_0$, then $Q(-z_0) = z_0$. Both z_0 and $-z_0$ correspond to the fixed point $\xi_0 = z_0/z_0 = -z_0/(-z_0)$ of $\phi(\xi)$ but no fixed point of $Q(z)$ corresponds to ξ_0.

Let ξ_0 be a fixed point of $\phi(\xi)$. Assume that n is odd. If $\xi_0 = z_0/\overline{z}_0$ and $z_0/Q(z_0) > 0$, then there exist two fixed points of $Q(z)$ corresponding to ξ_0; if $z_0/Q(z_0) < 0$, then there exist two points corresponding to ξ_0 for which $Q(z) = -z$. However, they are fixed points of $Q^{\circ 2}(z)$.

We shall now consider the branches of

$$(\phi(\xi^2))^{1/2}.$$

We can assume that for $i = 1, \ldots, s$ we have $|p_i| < 1$, and for $i = s+1, \ldots, i_0$ we have $|p_i| > 1$. Hence

$$\phi(\xi) = \frac{B_1(\xi)}{B_2(\xi)}$$

where

$$B_1(\xi) = \frac{a_{ni_0}}{\overline{a}_{ni_0}} \cdot \prod_{j=1}^{s} \frac{\xi - p_j}{1 - \overline{p}_j \xi}, \qquad B_2(\xi) = \xi^{n-i_0} \cdot \prod_{j=s+1}^{i_0} \frac{\overline{p}_j}{p_j} \frac{\xi - 1/\overline{p}_j}{1 - \xi/p_j}.$$

It can be checked that

$$\phi(\xi^2) = \left(\frac{\widetilde{B}_1(\xi)}{\widetilde{B}_2(\xi)} \right)^2$$

where

$$\widetilde{B}_1(\xi) = c \cdot \prod_{j=1}^{s} \left(\xi \frac{|1 - \overline{p}_j \xi^2|}{1 - \overline{p}_j \xi^2} \right), \qquad c^2 = \frac{a_{ni_0}}{\overline{a}_{ni_0}},$$

$$\widetilde{B}_2(\xi) = \xi^{n-i_0} \cdot \prod_{j=s+1}^{i_0} \left(c_j \xi \frac{|1 - \xi^2/p_j|}{1 - \xi^2/p_j} \right), \quad c_j^2 = \frac{\overline{p}_j}{p_j}.$$

This implies that we can find a continuous branch of $(\phi(\xi^2))^{1/2}$ (equal to $\widetilde{B}_1(\xi)/\widetilde{B}_2(\xi)$ or $-\widetilde{B}_1(\xi)/\widetilde{B}_2(\xi)$) for which the following is true:

PROPOSITION 3.4. *On $C(0,1)$ there exists a continuous branch $g(\xi)$ of $(\phi(\xi^2))^{1/2}$ such that $Q(z) = \psi \circ g \circ \psi^{-1}(z)$ on $\mathcal{J}_Q$. The mapping ψ was defined in the proof of Theorem* 3.2.

Proof. We have

$$(\psi^{-1}(z))^2 = z/\overline{z} = \xi.$$

Let $z \in \mathcal{J}_Q$. Then

$$\phi(\xi) = \phi[(\psi^{-1}(z))^2] = \frac{Q(z)}{\overline{Q(z)}} = (\psi^{-1}(Q(z)))^2.$$

We can take $g(\xi)$ for which $\psi^{-1}(Q(z)) = g(\psi^{-1}(z))$. ∎

Since $g(\xi)$ is semiconjugate to $\phi(\xi)$, we can see that the dynamics of $Q(z)$ on $\mathcal{J}_Q$ is closely related to the dynamics of $\phi(\xi)$ on $C(0,1)$. If $\phi(\xi)$ is not equal to a Blaschke product, it can be completely different from those described in [Li1].

Let $Q(z) = \overline{z}(z - p\overline{z})$, $p \in \mathbb{R}$, $-1 < p < 1/3$. In this case we have

$$\phi(\xi) = \frac{1}{\xi} \cdot \frac{\xi - p}{1 - p\xi}.$$

The Julia set $\mathcal{J}_\phi$ is a Cantor set contained in $\mathbb{R}$ and $C(0,1) \cap \mathcal{J}_\phi = \emptyset$. The set $\widehat{\mathbb{C}} \setminus \mathcal{J}_\phi$ is attracted by $\phi^{\circ n}$ to the unique attracting point $\xi_0 = 1$. The point ξ_0 corresponds to $z_0 = 1/(1-p)$. Hence the whole set $\mathcal{J}_Q$ is attracted to z_0 and there is no chaotic dynamics at all.

More details on such mappings will be given in Section 6.

4. The nonhomogeneous case

Definition 4.1. Let $Q(z)$ be an extendable polynomial mapping. We define the *filled-in Julia set* $\mathcal{K}_Q$ as the set of all $z \in \mathbb{C}$ whose forward orbit $\{Q^{\circ m}(z)\}_{m=1,2,\ldots}$ is bounded.

We have the following

Proposition 4.2. *If $Q(z)$ is an extendable polynomial mapping of algebraic degree $n \geq 2$, then*

$$\mathcal{K}_Q = \left\{ z : \lim_{m\to\infty} \frac{1}{n^m} \log(1 + |Q^{\circ m}(z)|) = 0 \right\}.$$

Proof. If $z \in \mathcal{K}_Q$, then there exists $M > 0$ such that $|Q^{\circ m}(z)| \leq M$ for all m. Hence

$$0 \leq \lim_{m\to\infty} \frac{1}{n^m} \log(1 + |Q^{\circ m}(z)|) \leq \lim_{m\to\infty} \frac{1}{n^m} \log(1 + M) = 0.$$

Since Q is extendable, we have

$$\inf_{|z|=1} |Q_n(z)| > c > 0.$$

Thus there exist $R > 1$ and $k > 0$ such that $|Q(z)| > k|z|^n > |z|$ for $|z| > R$. If $z \notin \mathcal{K}_Q$, then there exists m_0 for which $|Q^{\circ m_0}(z)| > R$. We have, for $m > m_0$,

$$\frac{1}{n^m} \log(1 + |Q^{\circ m}(z)|) > \frac{1}{n^m} ((m - m_0) \log k + n^{m-m_0} \log R).$$

Hence the left hand side cannot tend to zero as $m \to \infty$. ∎

Consider now the complexification $f(z,w)$ of Q and its homogenization $\widetilde{f}(z,w,t)$. Define

$$\mathcal{K}_f = \{(z,w) \in \mathbb{C}^2 : \{f^{\circ m}(z,w)\}_{m=1,2,\ldots} \text{ is bounded}\}$$

and

$$\mathcal{K}_{\widetilde{f}} = \{(z,w,t) \in \mathbb{C}^3 : \{\widetilde{f}^{\circ m}(z,w,t)\}_{m=1,2,\ldots} \text{ is bounded}\}.$$

Since Q is extendable, $\widetilde{f}(z,w,t) = 0$ iff $(z,w,t) = (0,0,0)$. We can again use Proposition 7.1 from [H-P] to obtain

Proposition 4.3. *Let*

$$h_{\widetilde{f}}(z,w,t) = \lim_{m\to\infty} \frac{1}{n^m} \log \|\widetilde{f}^{\circ m}(z,w,t)\|.$$

The set

$$A = \{(z, w, t) : h_{\widetilde{f}}(z, w, t) < 0\}$$

is the basin of attraction of zero, the set

$$B = \{(z, w, t) : h_{\widetilde{f}}(z, w, t) > 0\}$$

is equal to the set of all (z, w, t) *for which*

$$\lim_{m\to\infty} \|\widetilde{f}^{\circ m}(z, w, t)\| = \infty,$$

and the set

$$\mathcal{J}_{\widetilde{f}} = \{(z, w, t) : h_{\widetilde{f}}(z, w, t) = 0\} = \partial\mathcal{K}_{\widetilde{f}}$$

is a topological surface in $\mathbb{C}^3$ *homeomorphic to the unit sphere. We have*

$$\mathcal{K}_f \times \{1\} = \mathcal{J}_{\widetilde{f}} \cap \{(z, w, 1) : (z, w) \in \mathbb{C}^2\}$$

and $\mathcal{K}_Q$ *can be identified with the set* $\mathcal{J}_{\widetilde{f}} \cap \{(z, \overline{z}, 1)\}_{z\in\mathbb{C}}$.

Proof. The only thing which needes to be proved is that $\mathcal{J}_{\widetilde{f}}$ is homeomorphic to the sphere S. The proof is the same as the proof of Theorem 3.2. We take $\xi_0 \in S$ and define $\psi(\xi_0) = p(\xi_0)$, the unique point on the halfline joining 0 and ξ_0 for which $h_{\widetilde{f}}(p) = 0$. The uniqueness of p follows from the formula $h_{\widetilde{f}}(t\xi) = \log|t| + h_{\widetilde{f}}(\xi)$. Since $h_{\widetilde{f}}$ is continuous on $\mathbb{C}^3$ by Proposition 7.1 of [H-P], so also is ψ. The mapping ψ is one-to-one and S is compact, thus ψ is a homeomorphism. ∎

5. Properly extendable mappings. At the end of Section 3 we have seen that: if $f(z, w)$ acts on $\mathbb{CP}^2 \setminus \mathbb{C}^2$ as a quotient $\phi(\xi) = B_1(\xi)/B_2(\xi)$ of two Blaschke products, then it can happen that $\mathcal{J}_\phi \cap C(0, 1) = \emptyset$. To avoid such a situation and deal only with Blaschke products we shall need some additional assumptions. We have:

PROPOSITION 5.1. *Let* Q *be an extendable polynomial mapping. The following conditions are equivalent*:

(1) *the algebraic degree of* Q *is equal to its topological degree*;
(2) *the leading term* Q_n *of* Q *is equal to* $c \cdot \prod_{j=1}^{n}(z - p_j\overline{z})$ *with* $|p_j| < 1$ *for* $j = 1, \ldots, n$;
(3) *the complexification* $f(z, w)$ *of* Q *acts on* $\mathbb{CP}^2 \setminus \mathbb{C}^2$ *as a finite Blaschke product.*

Proof. The implication (2)⇒(3) is obvious. We shall prove (3)⇒(2). Let (3) hold. We have

$$Q_n(z) = a_{ni_0} \cdot \overline{z}^{n-i_0} \cdot \prod_{i=1}^{i_0}(z - p_i\overline{z}).$$

The complexification $f(z,w)$ acts on $\mathbb{CP}^2 \setminus \mathbb{C}^2$ as

$$\phi(\xi) = \frac{a_{ni_0}}{\overline{a}_{ni_0}} \cdot \frac{1}{\xi^{n-i_0}} \cdot \prod_{i=1}^{i_0} \frac{1-p_i\xi}{1-\overline{p}_i\xi}.$$

The mapping $\phi(\xi)$ is a finite Blaschke product iff $n = i_0$ and $|p_i| < 1$ for each i. Thus (2) holds.

The equivalence (1)$\Leftrightarrow$(2) can be proved in the same way as Proposition 2.1 in [Li1]. The topological degree of Q is equal to the topological degree of its leading term Q_n. The topological degree of Q_n is equal to the growth of the argument of $Q(z)$ (when going once around zero counterclockwise) divided by 2π. This is equal to $s-(n-i_0)-(n-s)$ where s is the number of p_i with $|p_i|<1$.

Hence the algebraic degree of Q is equal to its topological degree iff $i_0 = s = n$. ∎

Definition 5.2. We shall say that Q is *properly extendable* if Q is an extendable polynomial mapping and the algebraic degree of Q is equal to its topological degree.

In [Li1] we proved that if $n=2$ and Q is quasiregular, i.e.

$$\left|\frac{\partial Q/\partial \overline{z}}{\partial Q/\partial z}\right| < k < 1$$

a.e. on $\mathbb{C}$, then Q is properly extendable; if $n>2$, this fact is not true.

Lemma 2.5 of [Li2] says that the homogeneous polynomial

$$Q(z) = \prod_{i=1}^{n}(z - p_i\overline{z})$$

is quasiregular iff

$$|\xi| = 1 \Rightarrow \sum_{i=1}^{n} \frac{1-|p_i|^2}{|\xi - p_i|^2} > 0.$$

This implies

Example 5.3. Let

$$Q(z) = (z-p\overline{z})^s(z-q\overline{z})^r,$$

$s>r$, $|p|\neq 1$, $|q|\neq 1$, $p\overline{q}\neq 1$. If

$$\frac{s-r}{s+1} > |p|$$

and

$$|q| > \frac{(s+r)-(s-r)|p|}{(s-r)-(s+r)|p|} > 1,$$

then $Q(z)$ is quasiregular. The algebraic degree of Q is equal to $n = s + r$ and the topological degree of Q is equal to $d = s - r$. Hence an extendable quasiregular polynomial mapping may not be properly extendable if its algebraic degree is greater than two.

Assume now that $Q(z)$ is a properly extendable *homogeneous* mapping. We can now use Theorem 3.2, Propositions 3.3 and 3.4 and the extensive knowledge of the dynamics of Blaschke products (see [Sh-Su, H, C-G]) to describe the dynamics of Q on $\mathcal{J}_Q$. The situation is the same as in Section 4 of [Li1]. We shall get the following

PROPOSITION 5.4. *Let Q be a homogeneous properly extendable mapping. Let ϕ denote the restriction of f (the complexification of Q) to $\mathbb{CP}^2 \setminus \mathbb{C}^2$. Then ϕ is a finite Blaschke product. The dynamics of $Q|_{\mathcal{J}_Q}$ is the same as the dynamics of a continuous branch of $(\phi(\xi^2))^{1/2}$ on $C(0,1)$. Hence:*

1. *If the Julia set of ϕ is equal to $C(0,1)$, then the dynamics of Q is chaotic on the whole $\mathcal{J}_Q$. If, in addition, ϕ has a fixed point in the unit disc, then the dynamics of $Q|_{\mathcal{J}_Q}$ is chaotic, expanding, ergodic and mixing.*
2. *If the Julia set of ϕ is a Cantor subset $\mathcal{J}_\phi$ of $C(0,1)$, then the chaotic dynamics of Q is supported by a Cantor set*
$$\mathcal{C}_Q = \left\{ z \in \mathcal{J}_Q : \psi^{-1}(z)/\overline{\psi^{-1}(z)} \in \mathcal{J}_\phi \right\}$$
(ψ is the same as in Proposition 3.4). There exists $\xi_0 \in C(0,1)$ which is either an attracting point for ϕ or a rationally neutral point with one Leau leaf equal to $\widehat{\mathbb{C}} \setminus \mathcal{J}_\phi$. In this case, if the degree of Q is even, then there exists $z_0 \in \mathcal{J}_Q$ such that $\mathcal{J}_Q \setminus \mathcal{C}_Q$ is attracted to z_0; if the degree of Q is odd, there are two possibilities:
   - (a) *There exist two attracting (on $\mathcal{J}_Q$!) points $z_1, z_2 \in \mathcal{J}_Q$, and $\mathcal{J}_Q \setminus \mathcal{C}_Q$ is the union of their basins of attraction in $\mathcal{J}_Q$.*
   - (b) *There exists one period two attracting cycle in $\mathcal{J}_Q$ and $\mathcal{J}_Q \setminus \mathcal{C}_Q$ is its basin of attraction.*

Proof. The conjugacy between $(\phi(\xi^2))^{1/2}$ and $Q|_{\mathcal{J}_Q}$ was proved in Proposition 3.4. The semiconjugacy between $Q|_{\mathcal{J}_Q}$ and $\phi(\xi)$ implies that the dynamics of $Q|_{\mathcal{J}_Q}$ has the same properties as the dynamics of $\phi|_{C(0,1)}$ if only $\mathcal{J}_\phi = C(0,1)$. The dynamics of $\mathcal{J}_\phi$ in this case was described in [Sh-Su, H] and also in [C-G]. This proves (1).

Suppose now that $\mathcal{J}_\phi$ is a Cantor set. In this case ϕ cannot have a fixed point inside the disc. The Denjoy–Wolff theorem implies that there exists a fixed point $\xi_0 \in C(0,1)$ such that the unit disc is attracted to ξ_0. The set $\widehat{\mathbb{C}} \setminus \overline{B(0,1)}$ must also be attracted to it. The classification of periodic components of the Fatou set of ϕ implies that ξ_0 is either an attracting point for ϕ or the neutral rational point with one Leau leaf (see [C-G]).

Now, if the degree of Q is even then by Proposition 3.3 there exists exactly one fixed point z_0 of Q, $z_0 \in \mathcal{J}_Q$, for which $z_0/\overline{z}_0 = \xi_0$. We have

$$\xi_0 = (\psi^{-1}(z_0))^2.$$

There exists an arc γ in $C(0,1)$ which is attracted to ξ_0 by $\phi^{\circ n}$. If ξ_0 is an attracting point for ϕ then ξ_0 lies in the interior of γ and if ξ_0 is rational neutral then ξ_0 is one of the ends of this arc. We shall now take the branch of the square root for which $\xi_0^{1/2} = \psi^{-1}(z_0)$.

The arc

$$\Gamma = \psi((\gamma)^{1/2}) \subset \mathcal{J}_Q$$

is attracted to z_0, and the set

$$\bigcup_{n=1}^{\infty} (Q^{\circ n})^{-1}(\Gamma)$$

is equal to $\mathcal{J}_Q \setminus \mathcal{C}_Q$.

If the degree of Q is odd then we consider $Q^{\circ 2}$. The point ξ_0 is an attracting (or rationally neutral) fixed point for $Q^{\circ 2}$. Hence we can apply the above considerations to $Q^{\circ 2}$ instead of Q and use the remarks after Proposition 3.3. This will prove (a) and (b). ■

Suppose now that $Q(z)$ is a nonhomogeneous properly extendable mapping and that $\phi(\xi)$ has a fixed point inside the unit disc. Considering conjugation we can assume as in [Li1] that this fixed point is zero. Then $\mathcal{J}_\phi = C(0,1)$ and ϕ is uniformly expanding on $C(0,1)$ by the result of Tischler [T]. Hence Theorem 4.3 from Bedford–Jonsson's paper [B-J] and its proof can be applied in this case. In fact we can repeat word for word the outline of the proof of Theorem 5.1 from [Li1] and obtain

THEOREM 5.5. *There exists a neighbourhood V of ∞ in the Riemann sphere $\widehat{\mathbb{C}}$ and a homeomorphism ψ which maps V onto some neighbourhood of ∞ conjugating Q to Q_n, the leading term of Q.*

It would be highly desirable to find a more straightforward approach to Theorem 5.5. The mathematical machinery from [B-J] is complicated and does not work if $\phi(\xi)$ is not uniformly expanding on $\mathcal{J}_\phi$.

We end this section with the following

PROPOSITION 5.6. *Let $Q(z)$ be an extendable polynomial mapping (maybe nonhomogeneous) for which the function $\phi(\xi)$ on $\mathbb{CP}^2 \setminus \mathbb{C}^2$ is equal to $1/\mathcal{B}(\xi)$ where $\mathcal{B}$ is a Blaschke product. Then $Q^{\circ 2}(z)$ is properly extendable.*

Proof. Let

$$\mathcal{B}(\xi) = e^{i\theta} \cdot \prod_{i=1}^{n} \frac{z - p_i}{1 - \overline{p}_i z}.$$

We have $\phi \cdot \phi(\xi) = \mathcal{B}_1 \circ \mathcal{B}(\xi)$, where

$$\mathcal{B}_1(z) = e^{-i\theta} \cdot \prod_{i=1}^{n} \frac{z - \overline{p}_i}{1 - p_i z}.$$

Since the superposition of Blaschke products is a Blaschke product, Proposition 5.1 implies that $Q^{\circ 2}$ is properly extendable. ∎

6. Non-properly extendable polynomial mappings which have algebraic degree two. Let $Q(z)$ denote a homogeneous extendable mapping which has algebraic degree two. There are three possibilities:

(1) The topological degree of Q is 2. In this case Q is quasiregular. The properties of such maps were described in [Li1].

(2) The topological degree of Q is -2. In this case $\phi(\xi)$ is the inverse of a Blaschke product and $Q^{\circ 2}$ is properly extendable. Moreover, the mapping $\overline{Q}$ is quasiregular. We have $\mathcal{J}_\phi = \mathcal{J}_{\phi\circ\phi} \subset C(0,1)$ and we can use the results of Section 5 to study the behaviour of $Q^{\circ 2}$ and Q on $\mathcal{J}_Q$.

(3) The topological degree of Q is zero. In this case we have either

   (a) $Q(z) = a\overline{z}(z - p\overline{z})$, $0 < |p| < 1$, or
   (b) $Q(z) = a(z - p\overline{z})(z - q\overline{z})$, $|p| < 1$, $|q| > 1$, $p \neq 1/\overline{q}$.

PROPOSITION 6.1. *In both cases* (a) *and* (b) *we have* $Q(\mathbb{C}) \neq \mathbb{C}$. *The set* $Q(\mathbb{C})$ *is a closed sector with vertex at zero.*

Proof. Assume first that $Q(z) = \overline{z}(z - p\overline{z})$, $p \in \mathbb{R}$, $0 < |p| < 1$. Elementary calculations show that the equation $Q(z) = w$ can have a solution iff

$$|\Im w| \le \Re w \cdot \frac{|p|}{\sqrt{1 - p^2}}.$$

If $Q(z) = a\overline{z}(z - p\overline{z})$, $0 < |p| < 1$, then

$$Q(p^{1/2} z) = a|p|\overline{z}(z - |p|\overline{z}).$$

Hence

$$Q(\mathbb{C}) = a|p| \cdot Q_1(\mathbb{C})$$

where $Q_1(z) = \overline{z}(z - |p|\overline{z})$.

If $Q(z) = a(z - p\overline{z})(z - q\overline{z})$, $|p| < 1$, $|q| > 1$, $p \neq 1/\overline{q}$, then

$$Q\left(\frac{\overline{z} + qz}{1 - |q|^2}\right) = a\,\frac{q - p}{1 - |q^2|}\,\overline{z}\left(z - \frac{pq - 1}{q - p}\overline{z}\right).$$

Again

$$Q(\mathbb{C}) = a\,\frac{q-p}{1-|q|^2}\cdot Q_1(\mathbb{C}),$$

$$Q_1(z) = \overline{z}\biggl(z - \frac{pq-1}{q-p}\,\overline{z}\biggr). \ \blacksquare$$

Let us now say something about the dynamics of Q. The simplest case is again

$$Q(z) = \overline{z}(z - p\overline{z}), \quad p \in \mathbb{R},\, 0 < |p| < 1.$$

In this case the complexification of Q acts on $\mathbb{CP}^2 \setminus \mathbb{C}^2$ as

$$\phi(\xi) = \frac{\xi - p}{\xi(1-p\xi)}.$$

What is important, the real axis is *completely* invariant with respect to ϕ. If a rational map of $\widehat{\mathbb{C}}$ has a completely invariant circle C in $\widehat{\mathbb{C}}$, then the Julia set of this map is contained in C and this map is conjugate to a Blaschke product via a homography which maps C onto the unit circle $C(0,1)$. Hence the Julia set of ϕ is contained in $\mathbb{R}$.

If $1 > p > 1/3$, then $\xi = 1$ is a repelling fixed point for ϕ, and ϕ has two attracting points equal to

$$\xi_1 = \frac{(1-p) - i\sqrt{(p+1)(3p-1)}}{2p},$$

$$\xi_2 = \frac{(1-p) + i\sqrt{(p+1)(3p-1)}}{2p},$$

both in $C(0,1)$. We have $\phi(-1) = 1$ and $\mathcal{J}_\phi = \mathbb{R}$. Hence the dynamics of ϕ on $C(0,1)$ is the following: $\phi(-1) = 1$, which is the repelling point, the upper arc of $C(0,1) \setminus \{-1,1\}$ is attracted to ξ_2 and the lower one to ξ_1.

We have the same dynamics for $Q|_{\mathcal{J}_Q}$. If $p = 1/3$, then $\xi = 1$ is a triple neutral fixed point of ϕ with two Leau leafs. Again $\mathcal{J}_\phi = \mathbb{R}$, but this time the whole $C(0,1)$ is attracted to $\xi = 1$. If $p < 1/3$, then $\mathcal{J}_\phi$ is a Cantor set in $\mathbb{R}$, $\mathcal{J}_\phi \cap C(0,1) = \emptyset$ and the circle $C(0,1)$ is attracted to the fixed point $\xi = 1$. Hence, if $p \le 1/3$, then the whole set $\mathcal{J}_Q$ is attracted by Q to a single point.

We cannot expect such a simple dynamics for a general $Q(z)$ for which the topological degree of Q is 0.

Let us take a quadratic polynomial $w_c(z) = z^2 + c$, $c \in \mathbb{R}$. The real axis is invariant for $w_c(z)$. Hence $w_c(z)$ can be conjugate (via a homography mapping $\mathbb{R}$ onto $C(0,1)$) to a rational map $g_c(z)$ for which the unit circle is invariant. That means that $g_c(z)$ is a quotient of finite Blaschke products. If the Julia set $\mathcal{J}_{w_c}$ is not contained in $\mathbb{R}$, then the Julia set of g_c cannot be contained in $C(0,1)$. Since g_c has degree two and is a quotient of Blaschke products, there exists $Q(z)$ which is homogeneous, extendable and of algebraic degree two such that g_c is equal to the action of the complexification of Q on $\mathbb{CP}^2 \setminus \mathbb{C}^2$. (In fact all $c \cdot Q$, $c \in \mathbb{R}$, have this property but they are conjugate via $z \mapsto c^{-1}z$). When $\mathcal{J}_{g_c} \not\subset C(0,1)$, Q must be of topological degree zero. If $c = -2$, then $\mathcal{J}_{g_c}$ is an arc in $C(0,1)$. If $-2 < c < -3/4$, then $\mathcal{J}_{g_c} \cap C(0,1)$ consists of countably many points. Thus the dynamics of Q on $\mathcal{J}_Q$ must be far more complicated than the dynamics of $Q_p(z) = \overline{z}(z - p\overline{z})$, $p \in \mathbb{R}$, $0 < |p| < 1$.

REMARK 6.2. We can take any polynomial $w(z)$ with *real* coefficients and use the above procedure to obtain an extendable homogeneous polynomial mapping $Q_w(z)$. We can also use the same procedure for rational functions

$$R(z) = w(z)/p(z)$$

where w, p are polynomials with real coefficients, and obtain an extendable homogeneous polynomial mapping Q_R whose action on $\mathcal{J}_{Q_R}$ is semiconjugate to the action of R on $\mathbb{R} \cup \{\infty\}$.

Especially interesting is the case when

$$R(z) = 1 - 2/z^2$$

since in this case $\mathcal{J}_R = \widehat{\mathbb{C}}$. The corresponding map Q_R is an extendable map with algebraic degree two and topological degree zero. If $\phi(\xi)$ is the restriction of the complexification of $Q_R(z)$ to $\mathbb{CP}^2 \setminus \mathbb{C}^2$, then $\mathcal{J}_\phi = \widehat{\mathbb{C}}$.

There is also another method of attaching to every polynomial $w(z)$ with real coefficients an extendable homogeneous polynomial map Q^1_w. For

$$w(x) = \sum_{i=0}^{n} c_i x^i, \quad c_i \in \mathbb{R},\, c_n \neq 0,$$

we can take the homogenization

$$Q^1_w(x,y) = \Big(\sum_{i=0}^{n} c_i x^i y^{n-i}, y^n\Big)$$

which is an extendable homogeneous polynomial mapping. In the case of rational functions this method can lead to nonextendable maps.

Consider now the homogeneous polynomial mappings of algebraic degree three which have the form

$$Q(z) = e^{\pi i\theta} z^2 (z - a\overline{z}), \quad a, \theta \in \mathbb{R},\, a > 3.$$

The corresponding mapping $\phi(\xi)$ on $\mathbb{CP}^2 \setminus \mathbb{C}^2$ is equal to

$$\phi(\xi) = e^{2\pi i\theta} \cdot \xi^2 \cdot \frac{\xi - a}{1 - a\xi}.$$

For each $a > 3$ and diophantine number α there exists $\theta \in \mathbb{R}$ such that $\phi(\xi)$ is conjugate to the rotation

$$\xi \mapsto e^{2\pi\alpha i} \cdot \xi$$

via a homeomorphism G of the circle. (In fact, this is the famous example of the Herman ring, see [C-G, Ch. VI].) We have

$$G(t) = e^{2\pi i F(t)}$$

where $F(t)$ is a homeomorphism of $\mathbb{R}$ for which $F(t+1) = F(t) + 1$. Let

$$G_0(t) = e^{\pi i F(2t)}.$$

The homeomorphism $G_0 \circ \psi^{-1}$ conjugates $Q|_{\mathcal{J}_Q}$ to the rotation

$$\xi \mapsto e^{\pi \alpha i} \cdot \xi.$$

Such an example is impossible for polynomials of degree two since they cannot have topological degree one.

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