On the Helmholtz operator of variational calculus in fibered-fibered manifolds

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Abstract. A fibered-fibered manifold is a surjective fibered submersion $\pi : Y \to X$ between fibered manifolds. For natural numbers $s \geq r \leq q$ an $(r,s,q)$th order Lagrangian on a fibered-fibered manifold $\pi : Y \to X$ is a base-preserving morphism $\lambda : J^{r,s,q}Y \to \bigwedge^{\dim X} T^*X$. For $p = \max(q,s)$ there exists a canonical Euler morphism $E(\lambda) : J^{r+s,2s,r+p}Y \to V^*Y \otimes \bigwedge^{\dim X} T^*X$ satisfying a decomposition property similar to the one in the fibered manifold case, and the critical fibered sections $\sigma$ of $Y$ are exactly the solutions of the Euler–Lagrange equation $E(\lambda) \circ j^{r+s,2s,r+p} \sigma = 0$. In the present paper, similarly to the fibered manifold case, for any morphism $B : J^{r,s,q}Y \to V^*Y \otimes \bigwedge^{\dim X} T^*X$ over $Y$, $s \geq r \leq q$, we define canonically a Helmholtz morphism $\mathcal{H}(B) : J^{s+p,s+p,2p}Y \to V^*J^{r,s,r}Y \otimes V^*Y \otimes \bigwedge^{\dim X} T^*X$, and prove that a morphism $B : J^{r+s,2s,r+p}Y \to V^*Y \otimes \bigwedge^{\dim X} T^*X$ over $Y$ is locally variational (i.e. locally of the form $B = E(\lambda)$ for some $(r,s,p)$th order Lagrangian $\lambda$) if and only if $\mathcal{H}(B) = 0$, where $p = \max(s,q)$. Next, we study naturality of the Helmholtz morphism $\mathcal{H}(B)$ on fibered-fibered manifolds $Y$ of dimension $(m_1, m_2, n_1, n_2)$. We prove that any natural operator of the Helmholtz morphism type is $c\mathcal{H}(B)$, $c \in \mathbb{R}$, if $n_2 \geq 2$.

0. Introduction. The first problem in variational calculus is to characterize critical values. It is known that the critical sections of a fibered manifold $p : X \to X_0$ with respect to an $r$th order Lagrangian $\lambda : J^rX \to \bigwedge^{\dim X_0} T^*X_0$ can be characterized as the solutions of the so-called Euler–Lagrange equation. There exists a unique Euler map $E(\lambda) : J^{2r}X \to V^*X \otimes \bigwedge^{\dim X_0} T^*X_0$ over $X$ satisfying some decomposition formula. Then the Euler–Lagrange equation is $E(\lambda) \circ j^{2r} \sigma = 0$ with unknown section $\sigma$ (see [2]).

The second problem is to characterize morphisms $B : J^{2r}X \to V^*X \otimes \bigwedge^{\dim X_0} T^*X_0$ over $X$ which are locally variational (i.e. locally of the form $B = E(\lambda)$ for some $r$th order Lagrangian $\lambda$). In [3], for any natural number $r$ and any morphism $B : J^rY \to V^*X \otimes \bigwedge^{\dim X_0} T^*X_0$ over

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X a canonical Helmholtz morphism $H(B) : J^{2r}X \to V^*J^rX \otimes V^*X \otimes \bigwedge^\dim X_0 T^*X_0$ over $J^rY$ was described. Next, it was proved that a morphism $B : J^{2r}X \to V^*X \otimes \bigwedge^\dim X_0 T^*X_0$ over $X$ is locally variational if and only if $H(B) = 0$.

Fibered-fibered manifolds generalize fibered manifolds. They are surjective fibered submersions $\pi : Y \to X$ between fibered manifolds. They appear naturally in differential geometry if we consider transverse natural bundles (in the sense of R. Wolak [7]) over foliated manifolds (see [5]). A simple example of a fibered-fibered manifold is the following. For any four manifolds $X_1, X_2, X_3, X_4$, the obvious projection $\pi : X_1 \times X_2 \times X_3 \times X_4 \to X_1 \times X_2$ is a fibered-fibered manifold (we consider $X_1 \times X_2 \times X_3 \times X_4$ as the trivial fibered manifold over $X_1 \times X_3$ and $X_1 \times X_2$ as the trivial fibered manifold over $X_1$).

In [5], for fibered-fibered manifolds, using the concept of $(r,s,q)$-jets on fibered manifolds, [2], we extended the notion of $r$-jet prolongation bundle to the $(r,s,q)$-jet prolongation bundle $J^{r,s,q}Y$ for $r, s, q \in \mathbb{N}\setminus\{0\}$, $s \geq r \leq q$. In [6], we solved the first variational problem for fibered-fibered manifolds. We defined $(r,s,q)$th order Lagrangians as base preserving (over $X$) morphisms $\lambda : J^{r,s,q}Y \to \bigwedge^\dim Y T^*X$. Then similarly to the fibered manifold case we defined critical fibered sections of $Y$. Setting $p = \max(q,s)$ we proved that there exists a canonical Euler morphism $\mathcal{E}(\lambda) : J^{r+s,2s,r+p}Y \to \mathcal{V}^*Y \otimes \bigwedge^\dim X T^*X$ of $\lambda$ over $Y$ satisfying a decomposition property similar to the one in the fibered manifold case, where $\mathcal{V}Y \subset TY$ is the vector subbundle of vectors vertical with respect to two obvious projections from $Y$ (onto $X$ and onto $Y_0$). Then we deduced that the critical fibered sections $\sigma$ are exactly the solutions of the Euler–Lagrange equation $\mathcal{E}(\lambda) \circ j^{r+s,2s,r+p} \sigma = 0$. Next, we studied invariance properties of the corresponding Euler operator $\mathcal{E}$. We proved that any natural operator of the Euler morphism type is of the form $c\mathcal{E}$ for some real number $c$. (A similar result for the Euler operator $E$ from variational calculus on fibered manifolds has been obtained by I. Kolár [1].)

The purpose of the present paper is to solve the second problem of variational calculus in fibered-fibered manifolds. Similarly to the fibered manifold case, for any natural numbers $s \geq r \leq q$ and a morphism $B : J^{r,s,q}Y \to \mathcal{V}^*Y \otimes \bigwedge^\dim Y T^*X$ over $Y$ we define canonically a Helmholtz morphism $\mathcal{H}(B) : J^{s+p,s+p,2p}Y \to \mathcal{V}^*J^{r,s,r}Y \otimes \mathcal{V}^*Y \otimes \bigwedge^\dim X T^*X$ over $J^{r,s,r}Y$, where $p = \max(s,q)$. Then we deduce that a morphism $B : J^{r+s,2s,r+p}Y \to \mathcal{V}^*Y \otimes \bigwedge^\dim X T^*X$ over $Y$ is locally variational (i.e. locally of the form $B = \mathcal{E}(\lambda)$ for some $(r,s,p)$th order Lagrangian $\lambda$) if and only if $\mathcal{H}(B) = 0$, where $p = \max(s,q)$. Next, we study naturality of the corresponding Helmholtz operator $\mathcal{H}$ on fibered-fibered manifolds $Y$ of (fibered-fibered) dimension $(m_1, m_2, n_1, n_2)$. We prove that any natural operator of the Helmholtz
operator type is of the form $cH$, $c \in \mathbb{R}$, provided $n_2 \geq 2$. (A similar result for the Helmholtz operator $H$ from variational calculus on fibered manifolds has been obtained by I. Kolář and R. Vitolo [3] for $r = 1$ and 2, and by the author [4] for all $r$.)

A 2-fibered manifold is a sequence of two surjective submersions $X \rightarrow X_1 \rightarrow X_0$. For example, given a fibered manifold $X \rightarrow M$ we have the 2-fibered manifolds $TX \rightarrow X \rightarrow M$, $T^*X \rightarrow X \rightarrow M$, etc. Every 2-fibered manifold $X \rightarrow X_1 \rightarrow X_0$ can be considered as a fibered-fibered manifold $X \rightarrow X_1$, where we consider $X$ as a fibered manifold $X \rightarrow X_0$ and $X_1$ as a fibered manifold $X_1 \rightarrow X_0$. So, all our results apply to 2-fibered manifolds.

All manifolds and maps are assumed to be of class $C^\infty$.

1. Background: variational calculus in fibered manifolds

1.1. A fibered manifold is a surjective submersion $p : X \rightarrow X_0$ between manifolds. If $p' : X' \rightarrow X'_0$ is another fibered manifold then a map $f : X \rightarrow X'$ is called fibered if there exists a (unique) map $f_0 : X_0 \rightarrow X'_0$ such that $p' \circ f = f_0 \circ p$.

Denote the set of (local) sections of $p$ by $\Gamma X$. The $r$-jet prolongation $J^rX = \{ j^r_{x_0} \sigma \mid \sigma \in \Gamma X, x_0 \in X_0 \}$ of $X$ is a fibered manifold over $X_0$ with respect to the source projection $p^r : J^rX \rightarrow X_0$. If $p' : X' \rightarrow X'_0$ is another fibered manifold and $f : X \rightarrow X'$ is a fibered map covering a local diffeomorphism $f_0 : X_0 \rightarrow X'_0$ then $J^rf : J^rX \rightarrow J^rX'$ is given by $J^rf(j^r_x \sigma) = j^r_{f_0(x)}(f \circ \sigma \circ f_0^{-1})$ for $j^r_x \sigma \in J^rX$.

1.2. Let $p : X \rightarrow X_0$ be as above. A vector field $V$ on $X$ is projectable if there exists a vector field $V_0$ on $X_0$ such that $V$ is $p$-related to $V_0$. If $V$ is projectable on $X$, then its flow $\exp_tV$ is formed by local fibered diffeomorphisms, and we can define a vector field

$$J^rV = \left. \frac{\partial}{\partial t} \right|_{t=0} J^r(\exp tV)$$

on $J^rX$. If $V$ is $p$-vertical (i.e. $V_0 = 0$), then $J^rV$ is $p^r$-vertical.

1.3. An $r$th order Lagrangian on a fibered manifold $p : X \rightarrow X_0$ with $\dim X_0 = m$ is a base preserving (over $X_0$) morphism $\lambda : J^rX \rightarrow \bigwedge^m T^*X_0$.

Given a section $\sigma \in \Gamma X$ and a compact subset $K \subset \text{dom}(\sigma)$ contained in a chart domain, the action is

$$S(\lambda, \sigma, K) = \int_K (\lambda \circ j^r \sigma).$$
A section $\sigma \in \Gamma X$ is called critical if for any compact $K \subset \text{dom}(\sigma)$ contained in a chart domain and any $p$-vertical vector field $\eta$ on $X$ with compact support in $p^{-1}(K)$ we have

$$\frac{d}{dt}_{|t=0} S(\lambda, \text{Exp} t\eta \circ \sigma, K) = 0.$$  

By interchanging differentiation and integration we see that $\sigma$ is critical iff for any compact $K$ and $\eta$ as above we have

$$\int_K \langle \delta \lambda, J^r \eta \rangle \circ j^r \sigma = 0,$$

where $\delta \lambda : VJ^r X \to \bigwedge^m T^* X_0$ is the $p^r$-vertical part of the differential of $\lambda$.

1.4. Given a base preserving morphism $\varphi : J^q X \to \bigwedge^k T^* X_0$, its formal exterior differential $D\varphi : J^{q+1} X \to \bigwedge^{k+1} T^* X_0$ is defined by

$$D\varphi(j^{q+1}_0) = d(\varphi \circ j^q_0)(x_0)$$

for every local section $\sigma$ of $X$, where $d$ means the exterior differential at $x_0 \in X_0$ of the local $k$-form $\varphi \circ j^q_0$ on $X_0$.

Further, for every morphism $F : J^q X \to \bigotimes^l V^* J^s X \otimes \bigwedge^k T^* X_0$ over $J^s X$, $s \leq q$, and every $l$-tuple of vertical vector fields $\eta_1, \ldots, \eta_l$ on $X$, we have the evaluation $F(J^s \eta_1, \ldots, J^s \eta_l) : J^q X \to \bigwedge^{k+1} T^* X_0$. One verifies easily in coordinates that there exists a unique morphism $DF : J^{q+1} X \to \bigotimes^l V^* J^{s+1} X \otimes \bigwedge^{k+1} T^* X_0$ over $J^{s+1} Y$ satisfying

$$D(F(J^s \eta_1, \ldots, J^s \eta_l)) = (DF)(J^{s+1} \eta_1, \ldots, J^{s+1} \eta_l)$$

for all $\eta_1, \ldots, \eta_l$. It will also be called the formal exterior differential of $F$.

1.5. In the following assertion we do not explicitly indicate the pull-back to $J^{2r} X$.

**PROPOSITION 1** ([3]). For every morphism $B : J^r X \to V^* J^r X \otimes \bigwedge^m T^* X_0$ over $J^r X$, $m = \dim X_0$, there exists a unique pair of morphisms

$$E(B) : J^{2r} X \to V^* X \otimes \bigwedge^m T^* X_0, \quad F(B) : J^{2r} X \to V^* J^r X \otimes \bigwedge^m T^* X_0,$$

over $X$ and $J^r X$, respectively, such that $B = E(B) + F(B)$, and $F(B)$ is locally of the form $F(B) = DP$, with $P : J^{2r-1} X \to V^* J^{r-1} X \otimes \bigwedge^{m-1} T^* X_0$ over the identity of $J^{r-1} X$.

**REMARK 1.** If $f : J^q X \to \mathbb{R}$ is a function, we have a coordinate decomposition

$$D f = (D_i f) dx^i,$$

where

$$D_i f = \frac{\partial f}{\partial x^i} + \sum_{|\alpha| \leq q} \frac{\partial f}{\partial y_{\alpha}^p} y_{\alpha + 1}^p : J^{q+1} X \to \mathbb{R}$$
is the so-called formal (or total) derivative of \( f \) and \((x^i, y^k_\alpha)\) are fiber coordinates on \( X \) and \((x^i, y^k_\alpha)\) are the induced coordinates on \( J^qX \). The local coordinate form of \( E(B) \) is

\[
E(B) = \sum_{k=1}^{n} \sum_{|\alpha| \leq r} (-1)^{|\alpha|} D_\alpha B^k dy^k \otimes d^m x
\]

(see [3]), where \( d^m x = dx^1 \wedge \cdots \wedge dx^n \), \( B = \sum_{k=1}^{n} \sum_{|\alpha| \leq r} B^k dy^k_\alpha \otimes d^m x \) and \( D_\alpha \) is the iterated formal derivative corresponding to the multiindex \( \alpha \).

A morphism \( \tilde{B} : J^r X \to V^* X \otimes \bigwedge^m T^* X_0 \) over \( X \) is called an Euler morphism. The morphism \( E(B) \) is called the formal Euler morphism of \( B \).

Let \( \lambda : J^r X \to \bigwedge^m T^* X_0 \) be an \( r \)th order Lagrangian. We have \( \delta \lambda : J^r X \to V^* J^r X \otimes \bigwedge^m T^* X_0 \) over \( J^r X \). The morphism \( E(\lambda) := E(\delta \lambda) : J^{2r} X \to V^* X \otimes \bigwedge^m T^* X_0 \) over \( X \) is called the Euler morphism of \( \lambda \).

Proposition 1 and the Stokes theorem immediately yield the following well known fact.

**Proposition 2 ([2]).** A section \( \sigma \in \Gamma X \) is critical if it satisfies the Euler–Lagrange equation \( E(\lambda) \circ j^{2r} \sigma = 0 \).

**1.6.** Let \( B : J^r X \to V^* X \otimes \bigwedge^m T^* X_0 \) be an Euler morphism. We can interpret \( B \) as a vertical \( \bigwedge^m T^* X_0 \)-valued 1-form on \( J^r X \) by using the canonical projection \( V J^r X \to V X \). Then its vertical differential \( \delta B \) (defined fiberwise) is a vertical \( \bigwedge^m T^* X_0 \)-valued 2-form on \( J^r X \). For every vertical vector field \( \eta \) on \( X \), we have \( \langle \delta B, J^r \eta \rangle : J^r X \to V^* J^r X \otimes \bigwedge^m T^* X_0 \). Then we apply the formal Euler operator to obtain \( E(\langle \delta B, J^r \eta \rangle) : J^{2r} X \to V^* X \otimes \bigwedge^m T^* X_0 \) over \( X \).

**Proposition 3 ([3]).** There exists a unique morphism

\[
H(B) : J^{2r} X \to V^* J^r X \otimes V^* X \otimes \bigwedge^m T^* X_0
\]

over \( J^r X \) satisfying

\[
E(\langle \delta B, J^r \eta \rangle) = H(B)(J^r \eta)
\]

for every vertical vector field \( \eta \) on \( X \).

**Remark 2.** The local coordinate form of \( H(B) \) is

\[
H(B) = \sum_{k,l=1}^{n} \sum_{|\alpha| \leq r} H^\alpha_{kl} dy^k_\alpha \otimes dy^l \otimes d^m x,
\]

where

\[
H^\alpha_{kl} = \frac{\partial B_l}{\partial y^k_\alpha} - \sum_{|\beta| \leq r - |\alpha|} (-1)^{|\alpha + \beta|} \frac{(\alpha + \beta)!}{\alpha! \beta!} D_\beta \frac{\partial B_k}{\partial y^{\alpha+\beta}}
\]

and \( B = \sum_{k=1}^{n} B_k dy^k \otimes d^m x \) (see [3]).
The morphism \( H(B) : J^{2r}X \rightarrow V^*J^rX \otimes V^*X \otimes \bigwedge^m T^*X_0 \) over \( J^rX \) is called the Helmholz morphism of \( B \).

We have the following characterization of local variationality.

**Proposition 4 ([3]).** An \( r \)th order Euler morphism \( B \) is locally variational (i.e. locally of the form \( B = E(\lambda) \) for some local \( r \)th order Lagrangian \( \lambda \)) if and only if \( H(B) = 0 \).

### 2. Variational calculus in fibered-fibered manifolds

#### 2.1. In [5], we generalized the concept of fibered manifolds as follows. A fibered-fibered manifold is a fibered surjective submersion \( \pi : Y \rightarrow X \) between fibered manifolds \( p^Y : Y \rightarrow Y_0 \) and \( p^X : X \rightarrow X_0 \), i.e. a surjective submersion which sends fibers to fibers such that the restricted maps (between fibers) are submersions. If \( \pi' : Y' \rightarrow X' \) is another fibered-fibered manifold then a fibered map \( f : Y \rightarrow Y' \) is called fibered-fibered if there exists a (unique) fibered map \( f_0 : X \rightarrow X' \) such that \( \pi' \circ f = f_0 \circ \pi \).

Let \( r, s, q \in \mathbb{N} \setminus \{0\} \), \( s \geq r \leq q \).

Denote the set of local fibered maps \( \sigma : X \rightarrow Y \) with \( \pi \circ \sigma = \text{id}_{\text{dom}(\sigma)} \) (fibered sections) by \( \mathring{f}_{\text{fib}}Y \). By 12.19 in [2], \( \sigma, \varrho \in \mathring{f}_{\text{fib}}Y \) represent the same \((r, s, q)\)-jet \( j^r_{x,s,q}\sigma = j^r_{x,s,q}\varrho \) at a point \( x \in X \) iff

\[
j^r_x\sigma = j^r_x\varrho, \quad j^s_x(\sigma|_{X_0}) = j^s_x(\varrho|_{X_0}), \quad j^q_{x_0}\sigma_0 = j^q_{x_0}\varrho_0,
\]

where \( X_0 \) and \( Y_0 \) are the bases of the fibered manifolds \( X \) and \( Y \), \( x_0 \in X_0 \) is the element under \( x \), \( X_{x_0} \) is the fiber of \( X \) over \( x_0 \), and \( \sigma_0, \varrho_0 : X_0 \rightarrow Y_0 \) are the underlying maps of \( \sigma, \varrho \). The \((r, s, q)\)-jet prolongation

\[
J^{r,s,q}Y = \{ j^r_{x,s,q}\sigma | \sigma \in \mathring{f}_{\text{fib}}Y, x \in X \}
\]

of \( Y \) is a fibered manifold over \( X \) with respect to the source projection \( \pi^{r,s,q}_X : J^{r,s,q}Y \rightarrow X \) (see [4]). We also have the target projection \( \pi^{r,s,q}_Y : J^{r,s,q}Y \rightarrow Y \). If \( \pi' : Y' \rightarrow X' \) is another fibered-fibered manifold and \( f : Y \rightarrow Y' \) is a fibered-fibered map covering a local fibered diffeomorphism \( f_0 : X \rightarrow X' \) then \( J^{r,s,q}f : J^{r,s,q}Y \rightarrow J^{r,s,q}Y' \) is given by \( J^{r,s,q}f(j^r_{x,s,q}\sigma) = j^r_{f_0(x)}(f \circ \sigma \circ f_0^{-1}) \) for any \( j^r_{x,s,q}\sigma \in J^{r,s,q}Y \).

#### 2.2. Let \( \pi : Y \rightarrow X \) be a fibered-fibered manifold which is a fibered submersion between fibered manifolds \( p^Y : Y \rightarrow Y_0 \) and \( p^X : X \rightarrow X_0 \). A projectable vector field \( W \) on the fibered manifold \( Y \) is projectable-projectable if there exists a \( \pi \)-related (to \( W \)) projectable vector field \( W_0 \) on \( X \). If \( W \) is projectable-projectable on \( Y \), then its flow \( \text{Exp}_tW \) is formed by local fibered-fibered diffeomorphisms, and we define a vector field

\[
J^{r,s,q}W = \frac{\partial}{\partial t}|_{t=0} J^{r,s,q}(\text{Exp}_tW)
\]
on $J^{r,s,q}Y$. If additionally $W$ is $\pi$-vertical and $p^Y$-vertical (i.e. $W$ is $\pi$-related and $p^Y$-related to zero vector fields), then $J^{r,s,q}W$ is $\pi^X_{r,s,q}$-vertical and $p^Y \circ \pi^X_{r,s,q}$-vertical.

2.3. Let $r, s, q$ be as above.

An $(r,s,q)$th order Lagrangian on a fibered-fibered manifold $\pi : Y \to X$ with $\dim X = m$ is a base preserving (over $X$) morphism

$$\lambda : J^{r,s,q}Y \to \bigwedge^m T^* X.$$ 

Given a fibered section $\sigma \in \Gamma_{\text{fib}}Y$ and a compact subset $K \subset \text{dom}(\sigma) \subset X$ contained in a chart domain, the action is

$$S(\lambda, \sigma, K) = \int_K (\lambda \circ j^{r,s,q} \sigma).$$

A fibered section $\sigma \in \Gamma_{\text{fib}}Y$ is called critical (with respect to $\lambda$) if for any compact $K \subset \text{dom}(\sigma)$ contained in a chart domain and any $\pi$-vertical and $p^Y$-vertical vector field $\eta$ on $Y$ with compact support in $\pi^{-1}(K)$ contained in a chart domain we have

$$\frac{d}{dt} |_{t=0} S(\lambda, \text{Exp} \eta \circ \sigma, K) = 0.$$ 

Again we see that $\sigma$ is critical iff for any compact $K$ and $\eta$ as above we have

$$\int \langle \delta \lambda, j^{r,s,q} \eta \rangle j^{r,s,q} \sigma = 0,$$

where $\delta \lambda : \mathcal{V}J^{r,s,q}Y \to \bigwedge^m T^* X$ is the restriction of the differential of $\lambda$ to the vector subbundle $\mathcal{V}J^{r,s,q}Y \subset T J^{r,s,q}Y$ of vectors vertical with respect to the projections from $J^{r,s,q}Y$ onto $X$ and onto $Y_0$.

2.4. Given a base preserving morphism $\varphi : J^{\tilde{p},\tilde{p},\tilde{p}}Y \to \bigwedge^k T^* X$, its formal exterior differential $D \varphi : J^{\tilde{p}+1,\tilde{p}+1,\tilde{p}+1}Y \to \bigwedge^{k+1} T^* X$ over $X$ is defined by

$$D \varphi(j_x^{\tilde{p}+1,\tilde{p}+1,\tilde{p}+1} \sigma) = d(\varphi \circ j^{\tilde{p},\tilde{p},\tilde{p}} \sigma)(x)$$

for every local fibered section $\sigma$ of $Y$, where $d$ means the exterior differential at $x \in X$ of the local $k$-form $\varphi \circ j^{\tilde{p},\tilde{p},\tilde{p}} \sigma$ on $X$.

For every morphism $F : J^{\tilde{p},\tilde{p},\tilde{p}}Y \to \bigotimes^1 \mathcal{V}^* J^{p,p,p}Y \otimes \bigwedge^k T^* X$, $\tilde{p} \leq \tilde{p}$, over $J^{\tilde{p},\tilde{p},\tilde{p}}Y$, and every $l$-tuple of $\pi$-vertical and $p^Y$-vertical vector fields $\eta_1, \ldots, \eta_l$ on $Y$, we have the evaluation $F(j^{\tilde{p},\tilde{p},\tilde{p}} \eta_1, \ldots, J^{\tilde{p},\tilde{p},\tilde{p}} \eta_l) : J^{\tilde{p},\tilde{p},\tilde{p}}Y \to \bigwedge^k T^* X$.

One verifies easily in coordinates that there exists a unique morphism $DF : J^{\tilde{p}+1,\tilde{p}+1,\tilde{p}+1}Y \to \bigotimes^1 \mathcal{V}^* J^{p+1,p+1,p+1}Y \otimes \bigwedge^{k+1} T^* X$ over $J^{p+1,p+1,p+1}Y$ satisfying

$$D(F(j^{\tilde{p},\tilde{p},\tilde{p}} \eta_1, \ldots, J^{\tilde{p},\tilde{p},\tilde{p}} \eta_l)) = (DF)(J^{p+1,p+1,p+1} \eta_1, \ldots, J^{p+1,p+1,p+1} \eta_l)$$

for all $\eta_1, \ldots, \eta_l$. Here and throughout, $\mathcal{V}J^{p,p,p}Y$ is the vector subbundle of $T J^{p,p,p}Y$ of vectors vertical with respect to the obvious projections from
\(J^{p,p,p}Y\) onto \(X\) and onto \(Y_0\). Also in this case \(DF\) will be called the formal exterior differential of \(F\).

2.5. In the following assertion we do not explicitly indicate the pullbacks to \(J^{2p,2p,2p}Y\) and \(J^{p,p,p}Y\).

**Proposition 5.** Let \(r, s, q\) be natural numbers with \(s \geq r \leq q\), and set \(p = \max(q,s)\). For every morphism \(B : J^{r,s,q}Y \to \mathcal{V}^*J^{r,s,q}Y \otimes \wedge^mT^*X\) over \(J^{r,s,q}Y\), there exists a unique pair of morphisms

\[
\mathcal{E}(B) : J^{2p,2p,2p}Y \to \mathcal{V}^*Y \otimes \wedge^mT^*X
\]

and

\[
\mathcal{F}(B) : J^{2p,2p,2p}Y \to \mathcal{V}^*J^{p,p,p}Y \otimes \wedge^mT^*X,
\]

over \(Y\) and \(J^{p,p,p}Y\), respectively, such that \(B = \mathcal{E}(B) + \mathcal{F}(B)\), and \(\mathcal{F}(B)\) is locally of the form \(\mathcal{F}(B) = DP, P : J^{2p-1,2p-1,2p-1}Y \to \mathcal{V}^*J^{p-1,p-1,p-1}Y \otimes \wedge^{m-1}T^*X\). Here \(\mathcal{V}Y, \mathcal{V}J^{p-1,p-1,p-1}Y\) and \(\mathcal{V}J^{p,p,p}Y\) are as in Sections 2.3 and 2.4.

**Proof.** Let \(\pi^{r,s,q}_{r,s,q} : J^{p,p,p}Y \to J^{r,s,q}Y\) be the jet projection and let \(i_p : J^{p,p,p}Y \to J^pY\) be the canonical inclusion, where in \(J^pY\) we consider \(Y\) as a fibered-fibered manifold over \(X\). Using a suitable partition of unity on \(X\) and local fibered-fibered coordinate arguments we produce a morphism \(\tilde{B} : J^pY \to V^*J^pY \otimes \wedge^mT^*X\) over \(J^pY\) such that \((i_p)^*\tilde{B} = (\pi^{r,s,q}_{r,s,q})^*B\). Then by the decomposition formula (Proposition 1) there exists a pair of morphisms

\[
\mathcal{E}(\tilde{B}) : J^{2p}Y \to V^*Y \otimes \wedge^mT^*X, \quad \mathcal{F}(\tilde{B}) : J^{2p}X \to V^*J^pY \otimes \wedge^mT^*X
\]

satisfying \(\tilde{B} = \mathcal{E}(\tilde{B}) + \mathcal{F}(\tilde{B})\), and \(\mathcal{F}(\tilde{B})\) is locally of the form \(\mathcal{F}(\tilde{B}) = DP, P : J^{2p-1}Y \to V^*J^{p-1}Y \otimes \wedge^{m-1}T^*X\). Taking the pullback \((i_{2p})^*\) of both sides of the last formula and using the obvious equality \(\mathcal{D}((\pi^{r,s,q}_{r,s,q})^*B) = \) the restriction of \((i_{2p})^*\mathcal{D}(\tilde{B})\) to \(\mathcal{V}J^{2p,2p,2p}Y\), we have the desired decomposition, provided we put \(\mathcal{E}(B) = \) the restriction of \((i_{2p})^*\mathcal{E}(\tilde{B})\) to \(\mathcal{V}Y\) and \(\mathcal{F}(B) = \) the restriction of \((i_{2p})^*\mathcal{F}(\tilde{B})\) to \(\mathcal{V}J^{p,p,p}Y\). Since locally \(\mathcal{F}(\tilde{B}) = DP, \mathcal{F}(B) = DP\) for \(P = \) the restriction of \((i_{2p-1})^*\tilde{P}\) to \(\mathcal{V}J^{p-1,p-1,p-1}Y\). Using Remark 1 it is easy to see (see Remark 3) that the definition of \(\mathcal{E}(B)\) does not depend on the choice of \(\tilde{B}\).

**Remark 3.** Let \((x^i, X^I, y^k, Y^K)\) for \(i = 1, \ldots, m_1, I = 1, \ldots, m_2, k = 1, \ldots, n_1\) and \(K = 1, \ldots, n_2\) be a fibered-fibered local coordinate system on a fibered-fibered manifold \(Y\). For any \(f : J^{p\tilde{p},\tilde{p},\tilde{p}}Y \to \mathbb{R}\) we have the decomposition

\[
\mathcal{D}(f) = \mathcal{D}_i(f)dx^i + \mathcal{D}_I(f)dX^I,
\]

where \(\mathcal{D}_i(f) : J^{p+1,\tilde{p}+1,\tilde{p}+1}Y \to \mathbb{R}\) and \(\mathcal{D}_I(f) : J^{\tilde{p}+1,\tilde{p}+1,\tilde{p}+1}Y \to \mathbb{R}\) are the “total” derivatives of \(f\). Let \(F : J^pY \to \mathbb{R}\) be such that \(F \circ i_p = f\). From the
clear equality $D_i(F) \circ i_{\tilde{p}+1} = D(f)$ we easily deduce that $D_i(f) = D_i(F) \circ i_{\tilde{p}+1}$ and $D_I(f) = D_I(F) \circ i_{\tilde{p}+1}$. In particular, since $D_i, D_I$ and $D_{i'}, D_{I'}$ commute, so do $D_i, D_I$ and $D_{i'}, D_{I'}$. From the formulas for $D_i$ and $D_I$ (see Remark 1) and from the above formulas for $D_i$ and $D_I$ we easily see that in local coordinates

$$D_i(f) = \frac{\partial f}{\partial x^i} + \sum_{k=1}^{n_i} \sum_{|\alpha| \leq p} \frac{\partial f}{\partial y^k_{\alpha}} y^k_{\alpha+1} + \sum_{K=1}^{n_2} \sum_{|\beta| + |\gamma| \leq \tilde{p}} \frac{\partial f}{\partial Y^K_{(\beta, \gamma)}} Y^K_{(\beta+1, \gamma)}$$

and

$$D_I(f) = \frac{\partial f}{\partial X^i} + \sum_{K=1}^{n_2} \sum_{|\beta| + |\gamma| \leq \tilde{p}} \frac{\partial f}{\partial Y^K_{(\beta, \gamma)}} Y^K_{(\beta, \gamma+1)},$$

where $(x^i, X^I, y^k_{\alpha}, Y^K_{(\beta, \gamma)}$) is the induced coordinate system on $J^p\tilde{p}Y$, $\tilde{\alpha} = (\tilde{\alpha}^1, \ldots, \tilde{\alpha}^{m_1})$, $\tilde{\beta} = (\tilde{\beta}^{m_1}, \ldots, \tilde{\beta}^{m_1})$ and $\tilde{\gamma} = (\tilde{\gamma}^1, \ldots, \tilde{\gamma}^{m_2})$.

Let $(x^i, X^I, y^k_{\alpha}, Y^K_{(\beta, \gamma)}$) be the induced coordinates on $J^{p,p}Y$, where $p = \max(s, q)$. Then using the formula in Remark 1 it is easy to see that the local coordinate form of $E(B)$ is

$$\tilde{E}(B) = \sum_{K=1}^{n_2} \sum_{|\beta| + |\gamma| \leq \tilde{p}} (-1)^{|\beta| + |\gamma|} D_{(\beta, \gamma)} B^{(K)} \otimes (d^{m_1}X \wedge d^{m_2}X),$$

where $d^{m_1}X = dx^1 \wedge \cdots \wedge dx^{m_1}$, $d^{m_2}X = dX^1 \wedge \cdots \wedge dX^{m_2}$, $(\pi_{r,s,q}^{p,p,p} B = \sum_{K=1}^{n_2} \sum_{|\beta| + |\gamma| \leq \tilde{p}} B^{(K)}_{(\beta, \gamma)} \otimes (d^{m_1}X \wedge d^{m_2}X)$ and $D_{(\beta, \gamma)}$ denotes the iterated “total” derivative with $\beta = (\beta^1, \ldots, \beta^{m_1})$, $\gamma = (\gamma^1, \ldots, \gamma^{m_2})$.

From the above local formula it follows that $\tilde{E}(B)$ can be factorized through $J^{r+s,2s,r+p}Y$, $p = \max(s, q)$.

A morphism $\tilde{B} : J^{r,s,q}Y \rightarrow \mathcal{V}^*Y \otimes \bigwedge^m T^*X$ over $Y$ is called an Euler morphism. The morphism $\tilde{E}(B) : J^{2p,2p,2p}Y \rightarrow \mathcal{V}^*Y \otimes \bigwedge^m T^*X$ over $Y$ is called the formal Euler morphism of $B$.

Let $\lambda$ be an $(r, s, q)th$ order Lagrangian on $Y$, and $p = \max(s, q)$. We have $\delta \lambda : J^{r,s,q}Y \rightarrow \mathcal{V}^*J^{r,s,q}Y \otimes \bigwedge^m T^*X$. The morphism $E(\lambda) = \tilde{E}(\delta \lambda) : J^{2p,2p,2p}Y \rightarrow \mathcal{V}^*Y \otimes \bigwedge^m T^*X$ over $Y$ is called the Euler morphism of $\lambda$.

By the above-mentioned property of $\tilde{E}(B)$ it follows that $E(\lambda)$ can also be factorized through $J^{r+s,2s,r+p}Y$.

Proposition 5 and the Stokes theorem yield the following fact.

**Proposition 6 ([6]).** A fibered section $\sigma \in \Gamma_{\tilde{F}Y}$ is critical iff it satisfies the Euler–Lagrange equation $E(\lambda) \circ j^{2p,2p,2p} \sigma = 0$. By the above-mentioned property of $E(\lambda)$ this equation is $E(\lambda) \circ j^{r+s,2s,r+p} \sigma = 0$. 
2.6. Let $B : J^{r,s,q}Y \to \mathcal{V}^*Y \otimes \wedge^mT^*X$ be an Euler morphism, and $p = \max(s,q)$. Using the canonical projections $\mathcal{V}^*J^{r,s,q}Y \to \mathcal{V}Y$, we can interpret $B$ as a vertical $\wedge^mT^*X$-valued $1$-form on $J^{r,s,q}Y$. Then the vertical differential $\delta B$ (defined fiberwise) is a vertical $\wedge^mT^*X$-valued $2$-form on $J^{r,s,q}Y$. For every $\pi$-vertical and $p^Y$-vertical vector field $\eta$ on $Y$, we have $\langle \delta B, \mathcal{J}^{r,s,q}\eta \rangle : J^{r,s,q}Y \to \mathcal{V}^*J^{r,s,q}Y \otimes \wedge^mT^*X$ over $J^{r,s,q}Y$. Then we can apply the formal Euler operator to obtain $\tilde{E}(\langle \delta B, \mathcal{J}^{r,s,q}\eta \rangle) : J^{2p,2p,2p}Y \to \mathcal{V}^*Y \otimes \wedge^mT^*X$ over $Y$.

**Proposition 7.** There exists a unique morphism

$$\mathcal{H}(B) : J^{2p,2p,2p}Y \to \mathcal{V}^*J^{p,p,p}Y \otimes \mathcal{V}^*Y \otimes \wedge^mT^*X$$

over $J^{p,p,p}Y$ satisfying

$$\tilde{E}(\langle \delta B, \mathcal{J}^{r,s,q}\eta \rangle) = \mathcal{H}(B)(\mathcal{J}^{p,p,p}\eta)$$

for every $\pi$-vertical and $p^Y$-vertical vector field $\eta$ on $Y$.

**Proof.** That $\mathcal{H}(B)$ is unique is clear. We prove the existence.

As in the proof of Proposition 5, we have a morphism $\tilde{B} : J^pY \to \mathcal{V}^*Y \otimes \wedge^mT^*X$ over $Y$ such that $(\pi^{r,s,q}_p)^*B = \text{the restriction of} (i_p)^*\tilde{B}$ to $\mathcal{V}^*Y$. Then by Proposition 3, $E(\langle \delta \tilde{B}, \mathcal{J}^p\eta \rangle) = H(\tilde{B})(\mathcal{J}^p\eta)$, where $H(\tilde{B})$ is the Helmholtz morphism of $\tilde{B}$. Applying the pull-back $(i_{2p})^*$ to both sides of the last equality and using the definition of $\tilde{E}(\langle \delta B, \mathcal{J}^{r,s,q}\eta \rangle)$ (see the proof of Proposition 5) we obtain the desired equality for $\mathcal{H}(B) = \text{the restriction of} (i_{2p})^*H(\tilde{B})$ to $\mathcal{V}J^{p,p,p}Y \times_Y \mathcal{V}Y$. One can show (see Remark 4 below) that the definition of $\mathcal{H}(B)$ is independent of the choice of $\tilde{B}$.

**Remark 4.** It follows from the formula in Remark 2 and from the definition of $\mathcal{H}(B)$ in the proof of Proposition 7 that the local coordinate form of $\mathcal{H}(B)$ is

$$\mathcal{H}(B) = \sum_{K,L=1}^{n_2} \sum_{|\beta|+|\gamma| \leq p} \mathcal{H}_{KL}^{(\beta,\gamma)} dY^K_{(\beta,\gamma)} \otimes dY^L \otimes d^{m_1}x \otimes d^{m_2}X,$$

where

$$\mathcal{H}_{KL}^{(\beta,\gamma)} = \frac{\partial B_L}{\partial Y^K_{(\beta,\gamma)}} - \sum_{|\bar{\beta}|+|\bar{\gamma}| \leq p-|\beta|-|\gamma|} (-1)^{|\bar{\beta}|+|\bar{\gamma}|} \begin{pmatrix} (\beta + \bar{\beta})(\gamma + \bar{\gamma})! \\ \beta!\beta!\gamma!\gamma! \end{pmatrix} D_{(\bar{\beta},\bar{\gamma})} \frac{\partial B_K}{\partial Y^L_{(\beta+\bar{\beta},\gamma+\bar{\gamma})}}$$

and $B = \sum_{K=1}^{n_1} B_K dY^K \otimes d^{m_1}x \wedge d^{m_2}X$.

From this local formula it follows easily that $\mathcal{H}(B)$ can be factorized through $(J^{s+p,s+p}2pY \times_{J^{r,s,q}Y} \mathcal{V}J^{r,s,q}Y) \times_Y \mathcal{V}Y$. 


We have the following characterization of local variationality.

**Proposition 8.** Let \( s \geq r \leq q \) be natural numbers and \( p = \max(s, q) \). A \((2p, 2p, 2p)\)-th order Euler morphism \( B \) is locally variational (i.e. locally of the form \( E(\lambda) \)) for some \((p, p, p)\)-th order Lagrangian \( \lambda \) if and only if \( \mathcal{H}(B) = 0 \).

Moreover, if a \((2p, 2p, 2p)\)-th order Euler morphism \( B \) is locally variational and factorizes through \( J^{r+s,2s,r+p}Y \), then locally \( B = E(\lambda) \) for some \((r, s, p)\)-th order Lagrangian.

**Proof.** Suppose locally \( B = E(\lambda) \). Choose a local \( p \)-th order Lagrangian \( \Lambda : J^pY \to \bigwedge^mT^*X \) such that \( \lambda \circ \pi_{r,s,q}^p = (i_p)^*\Lambda \). We see that \( \delta\lambda \) is the restriction of \((i_p)^*\delta\Lambda\) to \( \mathcal{V} \). Hence \( \mathcal{H}(B) = \mathcal{H}(E(\lambda)) \) is the restriction of \((i_{4p})^*H(E(\Lambda))\) to \( \mathcal{V}J^{2p}Y \times \mathcal{V} \). Since \( H(E(\Lambda)) = 0 \) (see Proposition 4), also \( \mathcal{H}(B) = 0 \).

To prove the converse we choose local fibered-fibered coordinates \((x^i, X^I, y^k, Y^K)\) on \( U \subset Y \). In this coordinate system we have the obvious projection \( \Pi : J^p\mathcal{U} = \mathbb{R}^M \to J^{p,\tilde{p}}\mathcal{U} = \mathbb{R}^N \) for any \( \tilde{p} \). Let \( \mathcal{H}(B) = 0 \). Then (using the local formula) we have \( H(\Pi^*B) = 0 \). Proposition 4 yields \( \Pi^*B = E(\Lambda) \) for some \( p \)-th order Lagrangian \( \Lambda \) on \( U \). Thus \( B = E(\lambda) \) for \( \lambda = (i_p)^*\Lambda \).

The “moreover” part can be deduced in the following way. By the assumption, there is \( \tilde{\lambda} \) of order \((p, p, p)\) such that \( B = E(\lambda) \) over \( U \), where \((U, x^i, X^I, y^k, Y^K)\) are fibered-fibered coordinates. Using these coordinates we can consider the obvious inclusion \( J : J^{r,s,p}U = \mathbb{R}^M \to J^{p,p,p}U = \mathbb{R}^N \), \( J(v) = (v, 0) \). Then (using the local expression of \( \tilde{E}(\delta\lambda) \)) we see that \( B = E(J^*\tilde{\lambda}) \).

**3. On naturality of the Helmholtz operator.** We say that a fibered manifold \( p : X \to X_0 \) is of dimension \((m, n)\) if \( \dim X_0 = m \) and \( \dim X = m + n \). All \((m, n)\)-dimensional fibered manifolds and their local fibered diffeomorphisms form a category which we denote by \( \mathcal{FM}_{m,n} \) and which is local and admissible in the sense of [2].

Similarly, a fibered-fibered manifold \( \pi : Y \to X \) is of dimension \((m_1, m_2, n_1, n_2)\) if the fibered manifold \( X \) is of dimension \((m_1, n_1)\) and the fibered manifold \( Y \) is of dimension \((m_1 + n_1, m_2 + n_2)\). All \((m_1, m_2, n_1, n_2)\)-dimensional fibered-fibered manifolds and their fibered-fibered local diffeomorphisms form a category which we denote by \( \mathcal{FM}_{m_1,m_2,n_1,n_2} \) and which is local and admissible in the sense of [2]. The standard \((m_1, m_2, n_1, n_2)\)-dimensional trivial fibered-fibered manifold \( \pi : \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \) will be denoted by \( \mathbb{R}^{m_1,m_2,n_1,n_2} \). Any \((m_1, m_2, n_1, n_2)\)-dimensional fibered-fibered manifold is locally \( \mathcal{FM}_{m_1,m_2,n_1,n_2} \)-object \( \mathbb{R}^{m_1,m_2,n_1,n_2} \)-isomorphic to the standard \( \mathcal{FM}_{m_1,m_2,n_1,n_2} \)-object \( \mathbb{R}^{m_1,m_2,n_1,n_2} \).
Given two fibered manifolds $Z_1 \rightarrow M$ and $Z_2 \rightarrow M$ over the same base $M$, we denote the space of all base preserving fibered manifold morphisms of $Z_1$ into $Z_2$ by $C_\infty^\infty_\text{FM}(Z_1, Z_2)$. In [3], [4], the authors studied the $r$th order Helmholtz morphism $H(B)$ of variational calculus on an $(m, n)$-dimensional fibered manifold $p : X \rightarrow X_0$ as the Helmholtz operator

$$H : C_\infty^\infty(X, J^r X, V^* X \otimes \bigwedge^m T^* X_0) \rightarrow C_\infty^\infty(J^2r X, V^* J^r X \otimes V^* X \otimes \bigwedge^m T^* X_0).$$

They deduced the following classification theorem:

**Theorem 1 ([3], [4]).** Any $\mathcal{F}_\text{FM}_{m, n}$-natural operator (in the sense of [2]) of the type of the Helmholtz operator is of the form $cH$, $c \in \mathbb{R}$, provided $n \geq 2$.

The purpose of the present section is to obtain a similar result in the fibered-fibered manifold case. Namely, we study the Helmholtz morphism $\mathcal{H}(B)$ of variational calculus on an $(m_1, m_2, n_1, n_2)$-dimensional fibered-fibered manifold $\pi : Y \rightarrow X$ as the Helmholtz operator

$$\mathcal{H} : C_\infty^\infty(Y, J^r, s, q Y, V^* Y \otimes \bigwedge^m T^* X) \rightarrow C_\infty^\infty(J^p, p, p Y, V^* J^r, s, r Y \otimes V^* Y \otimes \bigwedge^m T^* X),$$

where $s \geq r \leq q$ are natural numbers, $p = \max(s, q)$ and $m = m_1 + m_2 = \dim X$. We prove the following classification theorem.

**Theorem 2.** Any $\mathcal{F}_\text{FM}_{m_1, m_2, n_1, n_2}$-natural operator (in the sense of [2]) of the type of the Helmholtz operator is of the form $c\mathcal{H}$, $c \in \mathbb{R}$, provided $n_2 \geq 2$.

**Remark 5.** In view of Remark 3 the assertion of Theorem 2 also holds for natural operators

$$D : C_\infty^\infty(Y, J^r, s, q Y, V^* Y \otimes \bigwedge^m T^* X) \rightarrow C_\infty^\infty(J^{s+p, p, 2p} Y, V^* J^{r, s, r} Y \otimes V^* Y \otimes \bigwedge^m T^* X).$$

**Remark 6.** The assumption of the last theorem means that for any $\mathcal{F}_\text{FM}_{m_1, m_2, n_1, n_2}$-morphism $f : Y \rightarrow Y'$ and any morphisms

$$B \in C_\infty^\infty(Y, J^r, s, q Y, V^* Y \otimes \bigwedge^m T^* X)$$

and

$$B' \in C_\infty^\infty(Y', J^r, s, q Y', V^* Y' \otimes \bigwedge^m T^* X'),$$

if $B$ and $B'$ are $f$-related then so are $D(B)$ and $D(B')$. Moreover $D$ is regular and local. The regularity means that $D$ transforms a smoothly parametrized family of appropriate type morphisms into a smoothly parametrized family of appropriate type morphisms. The locality means that $D(B)_u$ depends on the germ of $B$ at $\pi_{r, s, q}^p (u)$. 
Proof of Theorem 2. Let $D$ be an operator in question.
Let $(x^i, X^I, y^k, Y^K)$ be the usual fibered-fibered coordinate system on $\mathbb{R}^{m_1,m_2,n_1,n_2}$, $i = 1, \ldots, m_1$, $I = 1, \ldots, m_2$, $k = 1, \ldots, n_1$, $K = 1, \ldots, n_2$.

Since an $\mathcal{FM}_{m_1,m_2,n_1,n_2}$-map
\[(x^i, X^I, y^k - \sigma^k(x^i, X^I), Y^K - \Sigma^K(x^i, X^I))\]
sends $j^{2p,2p,2p}_{(0,0)}(x^i, X^I, \sigma^k, \Sigma^K)$ to
\[\Theta = j^{2p,2p,2p}_{(0,0)}(x^i, X^I, 0, 0) \in (j^{2p,2p,2p}(\mathbb{R}^{m_1,m_2,n_1,n_2}))(0,0,0,0),\]
\[j^{2p,2p,2p}(\mathbb{R}^{m_1,m_2,n_1,n_2})\] is the $\mathcal{FM}_{m_1,m_2,n_1,n_2}$-orbit of $\Theta$. Then $D$ is uniquely determined by the evaluations
\[\langle D(B)\Theta, w \otimes v \rangle \in \wedge^m T^*_0 \mathbb{R}^m\]
for all
\[B \in C_{\mathbb{R}^{m_1,m_2,n_1,n_2}}(J^{r,s,q}(\mathbb{R}^{m_1,m_2,n_1,n_2}), \mathcal{V}^s\mathbb{R}^{m_1,m_2,n_1,n_2} \otimes \wedge^m T^*\mathbb{R}^m),\]
\[w \in \mathcal{V}_{\pi_{p,p,p}}^p(\mathbb{R}^{m_1,m_2,n_1,n_2}), \quad v \in T_0\mathbb{R}^{n_2} = \mathcal{V}(0,0,0,0)\mathbb{R}^{m_1,m_2,n_1,n_2}.\]

Using the invariance of $D$ with respect to $\mathcal{FM}_{m_1,m_2,n_1,n_2}$-maps of the form $id_{\mathbb{R}^m} \times \psi$ for appropriate linear $\psi$ (since $n_2 \geq 2$) we find that $D$ is uniquely determined by the evaluations
\[\left\langle D(B)\Theta, \frac{d}{dt} \left( t^{p,p,p}_{(0,0)}(x^i, X^I, 0, \ldots, 0, f(x^i, X^I), 0, \ldots, 0) \right) \otimes \frac{\partial}{\partial Y^2} \right\rangle\]
for all
\[B \in C_{\mathbb{R}^{m_1,m_2,n_1,n_2}}(J^{r,s,q}(\mathbb{R}^{m_1,m_2,n_1,n_2}), \mathcal{V}^s\mathbb{R}^{m_1,m_2,n_1,n_2} \otimes \wedge^m T^*\mathbb{R}^m)\]
and all $f : \mathbb{R}^m \to \mathbb{R}$, where $f(x^i, X^I)$ is at position $Y^1$.

Using the invariance of $D$ with respect to the $\mathcal{FM}_{m_1,m_2,n_1,n_2}$-map
\[(x^1, \ldots, x^{m_1}, X^1, \ldots, X^{m_2}, y^1, \ldots, y^{n_1}, Y^1 + f(x^i, X^I)Y^1, Y^2, \ldots, Y^{n_2})\]
preserving $\Theta$ we can assume $f = 1$, i.e. $D$ is uniquely determined by the evaluations
\[\left\langle D(B)\Theta, \frac{d}{dt} \left( t^{p,p,p}_{(0,0)}(x^i, X^I, 0, \ldots, 0, 1, 0, \ldots, 0) \right) \otimes \frac{\partial}{\partial Y^2} \right\rangle \in \wedge^m T^*_0 \mathbb{R}^m\]
for all
\[B \in C_{\mathbb{R}^{m_1,m_2,n_1,n_2}}(J^{r,s,q}(\mathbb{R}^{m_1,m_2,n_1,n_2}), \mathcal{V}^s\mathbb{R}^{m_1,m_2,n_1,n_2} \otimes \wedge^m T^*\mathbb{R}^m),\]
where 1 is at position $Y^1$.

Consider a morphism
\[B \in C_{\mathbb{R}^{m_1,m_2,n_1,n_2}}(J^{r,s,q}(\mathbb{R}^{m_1,m_2,n_1,n_2}), \mathcal{V}^s\mathbb{R}^{m_1,m_2,n_1,n_2} \otimes \wedge^m T^*\mathbb{R}^m).\]
Using the invariance of $D$ with respect to the $\mathcal{F} \mathcal{M}_{m_1, m_2, n_1, n_2}$-maps

$$\psi_{\tau, T} = \left( x^i, X^I, \frac{1}{\tau^k} y^k, \frac{1}{T^K} Y^K \right)$$

for $\tau^k \neq 0$ and $T^K \neq 0$ we get the homogeneity condition

$$\left\langle D((\psi_{\tau, T})_*)B, \frac{d}{dt_0} (t_j^{p, p, p}(x^i, X^I, 0, \ldots, 0, 1, 0, \ldots, 0)) \otimes \frac{\partial}{\partial Y^2_0} \right\rangle$$

$$= T^1 T^2 \left\langle D(B), \frac{d}{dt_0} (t_j^{p, p, p}(x^i, X^I, 0, \ldots, 0, 1, 0, \ldots, 0)) \otimes \frac{\partial}{\partial Y^2_0} \right\rangle$$

for $\tau = (\tau^k)$ and $T = (T^K)$. By Corollary 19.8 in [1] of the non-linear Peetre theorem we can assume that $B$ is a polynomial (of arbitrary degree). The regularity of $D$ implies that

$$\left\langle D(B), \frac{d}{dt_0} (t_j^{p, p, p}(x^i, X^I, 0, \ldots, 0, 1, 0, \ldots, 0)) \otimes \frac{\partial}{\partial Y^2_0} \right\rangle$$

is smooth with respect to the coordinates of $B$. Then by the homogeneous function theorem (and the above type of homogeneity) we deduce that

$$\left\langle D(B), \frac{d}{dt_0} (t_j^{p, p, p}(x^i, X^I, 0, \ldots, 0, 1, 0, \ldots, 0)) \otimes \frac{\partial}{\partial Y^2_0} \right\rangle$$

depends linearly on the coordinates of $B$ on all $x^0 X^a Y^1_{(\beta, \gamma)} dY^2 \otimes d^{m_1} x \land d^{m_2} X$ and $x^0 X^a Y^2_{(\beta, \gamma)} dY^1 \otimes d^{m_1} x \land d^{m_2} X$, it depends bilinearly on the coordinates of $B$ on all $x^0 X^a dY^1 \otimes d^{m_1} x \land d^{m_2} X$ and $x^0 X^a dY^2 \otimes d^{m_1} x \land d^{m_2} X$, and it is independent of the other coordinates of $B$, where (of course) $(x^i, X^I, y^k, Y^K)$ is the induced coordinate system on the prolongation $J^{r, s, q}_0(\mathbb{R}^{m_1, m_2, n_1, n_2})$ and $d^{m_1} x = dx^1 \land \cdots \land dx^{m_1}$ and $d^{m_2} X = dX^1 \land \cdots \land dX^{m_2}$. (Here and in what follows, $\alpha, \beta$ are arbitrary $m_1$-tuples and $\gamma$ is an arbitrary $m_2$-tuple with $|\alpha| \leq q$, $|\beta| + |\gamma| \leq r$ or $|\gamma| \leq s$ if $\beta = (0)$).

In other words (and more precisely),

$$\left\langle D(B), \frac{d}{dt_0} (t_j^{r, s, q}(x^i, X^I, 0, \ldots, 0, 1, 0, \ldots, 0)) \otimes \frac{\partial}{\partial Y^2_0} \right\rangle$$

is determined by the values

$$\left\langle D(x^0 X^a Y^1_{(\beta, \gamma)}) dY^1 \otimes d^{m_1} x \land d^{m_2} X, \frac{d}{dt_0} (t_j^{r, s, q}(x^i, X^I, 0, \ldots, 0, 1, 0, \ldots, 0)) \otimes \frac{\partial}{\partial Y^2_0} \right\rangle,$$

$$\left\langle D(x^0 X^a Y^2_{(\beta, \gamma)}) dY^2 \otimes d^{m_1} x \land d^{m_2} X, \frac{d}{dt_0} (t_j^{r, s, q}(x^i, X^I, 0, \ldots, 0, 1, 0, \ldots, 0)) \otimes \frac{\partial}{\partial Y^2_0} \right\rangle,$$
\[ \left< D(x^\rho X^\sigma dY^1 \otimes d^{m_1}x \wedge d^{m_2}X + x^\rho X^\sigma dY^2 \otimes d^{m_1}x \wedge d^{m_2}X)\rangle, \right. \\
\frac{d}{dt_0} \left( t_{j(0,0)}^{r,s,q} (x^i, X^I, 0, \ldots, 0, 1, 0, \ldots, 0) \right) \otimes \frac{\partial}{\partial Y^2_0} \right>.
\]
Furthermore, \( \langle D(B)\rangle, \frac{d}{dt_0} \left( t_{j(0,0)}^{r,s,q} (x^i, X^I, 0, \ldots, 0, 1, 0, \ldots, 0) \right) \otimes \frac{\partial}{\partial Y^2_0} \) is linear in \( B \) for \( B \) from the \( \mathbb{R} \)-vector subspace spanned by all elements \( x^\rho X^\sigma Y^1_{(\beta,\gamma)} dY^2 \otimes d^{m_1}x \wedge d^{m_2}X \) and \( x^\rho X^\sigma Y^2_{(\beta,\gamma)} dY^1 \otimes d^{m_1}x \wedge d^{m_2}X \); moreover,

\[ \left< D(dY^1 \otimes d^{m_1}x \wedge d^{m_2}X + B)\rangle, \right. \\
\frac{d}{dt_0} \left( t_{j(0,0)}^{r,s,q} (x^i, X^I, 0, \ldots, 0, 1, 0, \ldots, 0) \right) \otimes \frac{\partial}{\partial Y^2_0} \right>
\]
for \( B \) from the vector subspace (over \( \mathbb{R} \)) spanned by all \( x^\rho X^\sigma Y^1_{(\beta,\gamma)} dY^2 \otimes d^{m_1}x \wedge d^{m_2}X \) and \( x^\rho X^\sigma Y^2_{(\beta,\gamma)} dY^1 \otimes d^{m_1}x \wedge d^{m_2}X \); and

\[ (1) \left< D(ax^\rho X^\sigma dY^1 \otimes d^{m_1}x \wedge d^{m_2}X + bx^\rho X^\sigma dY^2 \otimes d^{m_1}x \wedge d^{m_2}X)\rangle, \right. \\
\frac{d}{dt_0} \left( t_{j(0,0)}^{r,s,q} (x^i, X^I, 0, \ldots, 0, 1, 0, \ldots, 0) \right) \otimes \frac{\partial}{\partial Y^2_0} \right>
\]
for all real numbers \( a \) and \( b \).

Then by the invariance of \( D \) with respect to \( (\tau^i x^i, \tau^I X^I, y^k, Y^K) \) for \( \tau^i \neq 0 \) and \( \tau^I \neq 0 \) we get

\[ \left< D(x^\rho X^\sigma Y^2_{(\beta,\gamma)} dY^1 \otimes d^{m_1}x \wedge d^{m_2}X)\rangle, \right. \\
\frac{d}{dt_0} \left( t_{j(0,0)}^{r,s,q} (x^i, X^I, 0, \ldots, 0, 1, 0, \ldots, 0) \right) \otimes \frac{\partial}{\partial Y^2_0} \right>
\]

\[ = \left< D(x^\rho X^\sigma Y^1_{(\beta,\gamma)} dY^2 \otimes d^{m_1}x \wedge d^{m_2}X)\rangle, \right. \\
\frac{d}{dt_0} \left( t_{j(0,0)}^{r,s,q} (x^i, X^I, 0, \ldots, 0, 1, 0, \ldots, 0) \right) \otimes \frac{\partial}{\partial Y^2_0} \right> = 0 \]
for \((\beta, \gamma) \neq (\varphi, \sigma)\), and
\[
\left\langle D(x^\beta X^\sigma dY^1 \otimes d^{m_1} x \wedge d^{m_2} X + x^\beta X^{\sigma^2} dY^2 \otimes d^{m_1} x \wedge d^{m_2} X) \Theta, \frac{d}{dt_0} (t_j^{r,s,q}(x^i, X^I, 0, \ldots, 0, 1, 0, \ldots, 0)) \otimes \frac{\partial}{\partial Y^2_0} \right\rangle = 0
\]
for all \(\varphi, \tilde{\varphi}, \sigma\) and \(\tilde{\sigma}\).

Hence \(D\) is determined by the evaluations
\[
\left\langle D(x^\beta X^\gamma Y^1_{(\beta, \gamma)} dY^2 \otimes d^{m_1} x \wedge d^{m_2} X) \Theta, \frac{d}{dt_0} (t_j^{r,s,q}(x^i, X^I, 0, \ldots, 0, 1, 0, \ldots, 0)) \otimes \frac{\partial}{\partial Y^2_0} \right\rangle,
\]
\[
\left\langle D(x^\beta X^\gamma Y^2_{(\beta, \gamma)} dY^1 \otimes d^{m_1} x \wedge d^{m_2} X) \Theta, \frac{d}{dt_0} (t_j^{r,s,q}(x^i, X^I, 0, \ldots, 0, 1, 0, \ldots, 0)) \otimes \frac{\partial}{\partial Y^2_0} \right\rangle.
\]

Suppose \(\beta^{i_0} \neq 0\) for some \(i_0 = 1, \ldots, m_1\). We use the invariance of \(D\) with respect to the locally defined \(\mathcal{F} \mathcal{M}_{m_1, m_2, n_1, n_2}\)-map
\[
\psi^{i_0} = (x^i, X^I, y^k, Y^1, Y^2 + x^{i_0} Y^2, Y^3, \ldots, Y^{n_2})^{-1}
\]
preserving \(x^i, X^I, \Theta, Y^1, I_j^{r,s,q}(x^i, X^I, 0, \ldots, 0, 1, 0, \ldots, 0), \frac{\partial}{\partial Y^2_0}\) and sending \(Y^2_{(\beta, \gamma)}\) to \(Y^2_{(\beta, \gamma)} + x^{i_0} Y^2_{(\beta, \gamma)}\). Applying this invariance to
\[
\left\langle D(x^{\beta-1i_0} X^\gamma Y^2_{(\beta, \gamma)} dY^1 \otimes d^{m_1} x \wedge d^{m_2} X) \Theta, \frac{d}{dt_0} (t_j^{r,s,q}(x^i, X^I, 0, \ldots, 0, 1, 0, \ldots, 0)) \otimes \frac{\partial}{\partial Y^2_0} \right\rangle
\]
it follows that the value (3) is zero if it is zero for \(\beta - 1i_0\) instead of \(\beta\). Continuing this process and a similar one for the \(\mathcal{F} \mathcal{M}_{m_1, m_2, n_1, n_2}\)-morphism
\[
\Psi^{i_0} = (x^i, X^I, y^k, Y^1, Y^2 + X^{i_0} Y^2, Y^3, \ldots, Y^{n_0})^{-1}
\]
instead of \(\psi^{i_0}\) we see that (3) is zero if it is zero for \((\beta, \gamma) = ((0), (0))\).

By similar arguments (since \(\psi^{i_0}\) sends \(dY^2\) to \(dY^2 + x^{i_0} dY^2\) and \(\Psi^{i_0}\) sends \(dY^2\) to \(dY^2 + X^{i_0} dY^2\)), from the equality
\[
\left\langle D(x^{\beta-1i_0} X^\gamma Y^1_{(\beta, \gamma)} dY^2 \otimes d^{m_1} x \wedge d^{m_2} X) \Theta, \frac{d}{dt} (t_j^{r,s,q}(x^i, X^I, 0, \ldots, 0, 1, 0, \ldots, 0)) \otimes \frac{\partial}{\partial Y^2_0} \right\rangle = 0
\]
for $\beta_{i0} \neq 0$ (or a similar equality for $\gamma_{I0} \neq 0$) we find that (2) is zero if $(\beta, \gamma) \neq ((0), (0))$.

In other words, $D$ is uniquely determined by the values (2) and (3) for $(\beta, \gamma) = ((0), (0))$.

Using the invariance of $D$ with respect to the (local) $\mathcal{F}M_{m_1,m_2,n_1,n_2}$-map

$$(x^i, X^I, y^k, Y^1 + Y^1 Y^2, Y^2, \ldots, Y^{n_1})^{-1}$$

preserving $\Theta$, $j^{r,s,q}_{(0,0)}(x^i, X^I, 0, \ldots, 0, 1, 0, \ldots, 0)$ and $\frac{\partial}{\partial Y^2_0}$, from the equality

$$\left\langle D(dY^1 \otimes d^{m_1}x \wedge d^{m_2}X)\Theta, \right.$$

$$\left. \frac{d}{dt_0} \left(tj^{r,s,q}_{(0,0)}(x^i, X^I, 0, \ldots, 0, 1, 0, \ldots, 0)\right) \otimes \frac{\partial}{\partial Y^2_0} \right\rangle = 0$$

(see (1)) we deduce that

$$\left\langle D(Y^2_{(0),(0)})dY^1 \otimes d^{m_1}x \wedge d^{m_2}X)\Theta, \right.$$

$$\left. \frac{d}{dt_0} \left(tj^{r,s,q}_{(0,0)}(x^i, X^I, 0, \ldots, 0, 1, 0, \ldots, 0)\right) \otimes \frac{\partial}{\partial Y^2_0} \right\rangle = 0$$

$$\left\langle D(Y^1_{(0),(0)})dY^2 \otimes d^{m_1}x \wedge d^{m_2}X)\Theta, \right.$$

$$\left. \frac{d}{dt_0} \left(tj^{r,s,q}_{(0,0)}(x^i, X^I, 0, \ldots, 0, 1, 0, \ldots, 0)\right) \otimes \frac{\partial}{\partial Y^2_0} \right\rangle.$$


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