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On the Helmholtz operator of variational calculus in fibered-fibered manifolds

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Abstract. A fibered-fibered manifold is a surjective fibered submersion $\pi : Y \to X$ between fibered manifolds. For natural numbers $s \geq r \leq q$ an (r, s, q)th order Lagrangian on a fibered-fibered manifold $\pi : Y \to X$ is a base-preserving morphism $\lambda : J^{r,s,q}Y \to \bigwedge^{\dim X} T^*X$. For $p = \max(q, s)$ there exists a canonical Euler morphism $\mathcal{E}(\lambda) : J^{r+s,2s,r+p}Y \to \mathcal{V}^*Y \otimes \bigwedge^{\dim X} T^*X$ satisfying a decomposition property similar to the one in the fibered manifold case, and the critical fibered sections σ of Y are exactly the solutions of the Euler–Lagrange equation $\mathcal{E}(\lambda) \circ j^{r+s,2s,r+p}\sigma = 0$. In the present paper, similarly to the fibered manifold case, for any morphism $B : J^{r,s,q}Y \to \mathcal{V}^*Y \otimes \bigwedge^m T^*X$ over Y, $s \geq r \leq q$, we define canonically a Helmholtz morphism $\mathcal{H}(B) : J^{s+p,s+p,2p}Y \to \mathcal{V}^*J^{r,s,r}Y \otimes \mathcal{V}^*Y \otimes \bigwedge^{\dim X} T^*X$, and prove that a morphism $B : J^{r+s,2s,r+p}Y \to \mathcal{V}^*Y \otimes \bigwedge T^*M$ over Y is locally variational (i.e. locally of the form $B = \mathcal{E}(\lambda)$ for some (r, s, p)th order Lagrangian λ) if and only if $\mathcal{H}(B) = 0$, where $p = \max(s, q)$. Next, we study naturality of the Helmholtz morphism $\mathcal{H}(B)$ on fibered-fibered manifolds Y of dimension (m_1, m_2, n_1, n_2). We prove that any natural operator of the Helmholtz morphism type is $c\mathcal{H}(B)$, $c \in \mathbb{R}$, if $n_2 \geq 2$.

0. Introduction. The first problem in variational calculus is to characterize critical values. It is known that the critical sections of a fibered manifold $p : X \to X_0$ with respect to an rth order Lagrangian $\lambda : J^rX \to \bigwedge^{\dim X_0}T^*X_0$ can be characterized as the solutions of the so-called Euler–Lagrange equation. There exists a unique Euler map $E(\lambda) : J^{2r}X \to V^*X \otimes \bigwedge^{\dim X_0}T^*X_0$ over X satisfying some decomposition formula. Then the Euler–Lagrange equation is $E(\lambda) \circ j^{2r}\sigma = 0$ with unknown section σ (see [2]).

The second problem is to characterize morphisms $B : J^{2r}X \to V^*X \otimes \bigwedge^{\dim X_0}T^*X_0$ over X which are locally variational (i.e. locally of the form $B = E(\lambda)$ for some rth order Lagrangian λ). In [3], for any natural number r and any morphism $B : J^rY \to V^*X \otimes \bigwedge^{\dim X_0}T^*X_0$ over

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X a canonical Helmholtz morphism $H(B) : J^{2r}X \to V^*J^rX \otimes V^*X \otimes \bigwedge^{\dim X_0} T^*X_0$ over J^rY was described. Next, it was proved that a morphism $B : J^{2r}X \to V^*X \otimes \bigwedge^{\dim X_0} T^*X_0$ over X is locally variational if and only if $H(B) = 0$.

Fibered-fibered manifolds generalize fibered manifolds. They are surjective fibered submersions $\pi : Y \to X$ between fibered manifolds. They appear naturally in differential geometry if we consider transverse natural bundles (in the sense of R. Wolak [7]) over foliated manifolds (see [5]). A simple example of a fibered-fibered manifold is the following. For any four manifolds X_1, X_2, X_3, X_4, the obvious projection $\pi : X_1 \times X_2 \times X_3 \times X_4 \to X_1 \times X_2$ is a fibered-fibered manifold (we consider $X_1 \times X_2 \times X_3 \times X_4$ as the trivial fibered manifold over $X_1 \times X_3$ and $X_1 \times X_2$ as the trivial fibered manifold over X_1). In [5], for fibered-fibered manifolds, using the concept of (r, s, q)-jets on fibered manifolds, [2], we extended the notion of r-jet prolongation bundle to the (r, s, q)-jet prolongation bundle $J^{r,s,q}Y$ for $r, s, q \in \mathbb{N} \setminus \{0\}$, $s \geq r \leq q$. In [6], we solved the first variational problem for fibered-fibered manifolds. We defined (r, s, q)th order Lagrangians as base preserving (over X) morphisms $\lambda : J^{r,s,q}Y \to \bigwedge^{\dim X} T^*X$. Then similarly to the fibered manifold case we defined critical fibered sections of Y. Setting $p = \max(q, s)$ we proved that there exists a canonical Euler morphism $\mathcal{E}(\lambda) : J^{r+s,2s,r+p}Y \to \mathcal{V}^*Y \otimes \bigwedge^{\dim X} T^*X$ of λ over Y satisfying a decomposition property similar to the one in the fibered manifold case, where $\mathcal{V}Y \subset TY$ is the vector subbundle of vectors vertical with respect to two obvious projections from Y (onto X and onto Y_0). Then we deduced that the critical fibered sections σ are exactly the solutions of the Euler–Lagrange equation $\mathcal{E}(\lambda) \circ j^{r+s,2s,r+p}\sigma = 0$. Next, we studied invariance properties of the corresponding Euler operator $\mathcal{E}$. We proved that any natural operator of the Euler morphism type is of the form $c\mathcal{E}$ for some real number c. (A similar result for the Euler operator E from variational calculus on fibered manifolds has been obtained by I. Kolář [1].)

The purpose of the present paper is to solve the second problem of variational calculus in fibered-fibered manifolds. Similarly to the fibered manifold case, for any natural numbers $s \geq r \leq q$ and a morphism $B : J^{r,s,q}Y \to \mathcal{V}^*Y \otimes \bigwedge^{\dim X} T^*X$ over Y we define canonically a Helmholtz morphism $\mathcal{H}(B) : J^{s+p,s+p,2p}Y \to \mathcal{V}^*J^{r,s,r}Y \otimes \mathcal{V}^*Y \otimes \bigwedge^{\dim X} T^*X$ over $J^{r,s,r}Y$, where $p = \max(s, q)$. Then we deduce that a morphism $B : J^{r+s,2s,r+p}Y \to \mathcal{V}^*Y \otimes \bigwedge^{\dim X} T^*X$ over Y is locally variational (i.e. locally of the form $B = \mathcal{E}(\lambda)$ for some (r, s, p)th order Lagrangian λ) if and only if $\mathcal{H}(B) = 0$, where $p = \max(s, q)$. Next, we study naturality of the corresponding Helmholtz operator $\mathcal{H}$ on fibered-fibered manifolds Y of (fibered-fibered) dimension (m_1, m_2, n_1, n_2). We prove that any natural operator of the Helmholtz operator type is of the form $c\mathcal{H}$, $c \in \mathbb{R}$, provided $n_2 \geq 2$. (A similar result for the Helmholtz operator H from variational calculus on fibered manifolds has been obtained by I. Kolář and R. Vitolo [3] for $r = 1$ and 2, and by the author [4] for all r.)

A 2-*fibered manifold* is a sequence of two surjective submersions $X \to X_1 \to X_0$. For example, given a fibered manifold $X \to M$ we have the 2-fibered manifolds $TX \to X \to M$, $T^*X \to X \to M$, $J^rX \to X \to M$, etc. Every 2-fibered manifold $X \to X_1 \to X_0$ can be considered as a fibered-fibered manifold $X \to X_1$, where we consider X as a fibered manifold $X \to X_0$ and X_1 as a fibered manifold $X_1 \to X_0$. So, all our results apply to 2-fibered manifolds.

All manifolds and maps are assumed to be of class $\mathcal{C}^\infty$.

1. Background: variational calculus in fibered manifolds

1.1. A *fibered manifold* is a surjective submersion $p : X \to X_0$ between manifolds. If $p' : X' \to X_0'$ is another fibered manifold then a map $f : X \to X'$ is called *fibered* if there exists a (unique) map $f_0 : X_0 \to X_0'$ such that $p' \circ f = f_0 \circ p$.

Denote the set of (local) sections of p by ΓX. The *r-jet prolongation*

$$J^rX = \{j^r_{x_0}\sigma \mid \sigma \in \Gamma X,\, x_0 \in X_0\}$$

of X is a fibered manifold over X_0 with respect to the source projection $p^r : J^rX \to X_0$. If $p' : X' \to X_0'$ is another fibered manifold and $f : X \to X'$ is a fibered map covering a local diffeomorphism $f_0 : X \to X_0'$ then $J^rf : J^rX \to J^rX'$ is given by $J^rf(j^r_x\sigma) = j^r_{f_0(x)}(f \circ \sigma \circ f_0^{-1})$ for $j^r_x\sigma \in J^rX$.

1.2. Let $p : X \to X_0$ be as above. A vector field V on X is *projectable* if there exists a vector field V_0 on X_0 such that V is p-related to V_0. If V is projectable on X, then its flow $\mathrm{Exp}\, tV$ is formed by local fibered diffeomorphisms, and we can define a vector field

$$\mathcal{J}^rV = \frac{\partial}{\partial t}_{|t=0} J^r(\mathrm{Exp}\, tV)$$

on J^rX. If V is p-*vertical* (i.e. $V_0 = 0$), then $\mathcal{J}^rV$ is p^r-vertical.

1.3. An *rth order Lagrangian* on a fibered manifold $p : X \to X_0$ with $\dim X_0 = m$ is a base preserving (over X_0) morphism

$$\lambda : J^rX \to \bigwedge^m T^*X_0.$$

Given a section $\sigma \in \Gamma X$ and a compact subset $K \subset \mathrm{dom}(\sigma)$ contained in a chart domain, the *action* is

$$S(\lambda, \sigma, K) = \int_K (\lambda \circ j^r\sigma).$$

A section $\sigma \in \Gamma X$ is called *critical* if for any compact $K \subset \mathrm{dom}(\sigma)$ contained in a chart domain and any p-vertical vector field η on X with compact support in $p^{-1}(K)$ we have

$$\frac{d}{dt}_{|t=0} S(\lambda, \mathrm{Exp}\, t\eta \circ \sigma, K) = 0.$$

By interchanging differentiation and integration we see that σ is critical iff for any compact K and η as above we have

$$\int_K \langle \delta\lambda, \mathcal{J}^r \eta \rangle \circ j^r \sigma = 0,$$

where $\delta\lambda : VJ^rX \to \bigwedge^m T^*X_0$ is the p^r-vertical part of the differential of λ.

1.4. Given a base preserving morphism $\varphi : J^qX \to \bigwedge^k T^*X_0$, its *formal exterior differential* $D\varphi : J^{q+1}X \to \bigwedge^{k+1} T^*X_0$ is defined by

$$D\varphi(j^{q+1}_{x_0}\sigma) = d(\varphi \circ j^q\sigma)(x_0)$$

for every local section σ of X, where d means the exterior differential at $x_0 \in X_0$ of the local k-form $\varphi \circ j^q\sigma$ on X_0.

Further, for every morphism $F : J^qX \to \bigotimes^l V^*J^sX \otimes \bigwedge^k T^*X_0$ over J^sX, $s \le q$, and every l-tuple of vertical vector fields $\eta_1, \ldots, \eta_l$ on X, we have the evaluation $F(\mathcal{J}^s\eta_1, \ldots, \mathcal{J}^s\eta_l) : J^qX \to \bigwedge^k T^*X_0$. One verifies easily in coordinates that there exists a unique morphism $DF : J^{q+1}X \to \bigotimes^l V^*J^{s+1}X \otimes \bigwedge^{k+1} T^*X_0$ over $J^{s+1}Y$ satisfying

$$D(F(\mathcal{J}^s\eta_1, \ldots, \mathcal{J}^s\eta_l)) = (DF)(\mathcal{J}^{s+1}\eta_1, \ldots, \mathcal{J}^{s+1}\eta_l)$$

for all $\eta_1, \ldots, \eta_l$. It will also be called the formal exterior differential of F.

1.5. In the following assertion we do not explicitly indicate the pull-back to $J^{2r}X$.

Proposition 1 ([3]). *For every morphism $B : J^rX \to V^*J^rX \otimes \bigwedge^m T^*X_0$ over J^rX, $m = \dim X_0$, there exists a unique pair of morphisms*

$$\mathbf{E}(B) : J^{2r}X \to V^*X \otimes \bigwedge^m T^*X_0, \qquad F(B) : J^{2r}X \to V^*J^rX \otimes \bigwedge^m T^*X_0,$$

over X and J^rX, respectively, such that $B = \mathbf{E}(B) + F(B)$, and $F(B)$ is locally of the form $F(B) = DP$, with $P : J^{2r-1}X \to V^*J^{r-1}X \otimes \bigwedge^{m-1} T^*X_0$ over the identity of $J^{r-1}X$.

Remark 1. If $f : J^qX \to \mathbb{R}$ is a function, we have a coordinate decomposition

$$Df = (D_i f) dx^i,$$

where

$$D_i f = \frac{\partial f}{\partial x^i} + \sum_{|\alpha| \le q} \frac{\partial f}{\partial y^p_\alpha}\, y^p_{\alpha+1_i} : J^{q+1}X \to \mathbb{R}$$

is the so-called formal (or total) derivative of f and (x^i, y^k) are fiber coordinates on X and (x^i, y^k_α) are the induced coordinates on J^qX. The local coordinate form of $\mathbf{E}(B)$ is

$$\mathbf{E}(B) = \sum_{k=1}^{n} \sum_{|\alpha|\leq r} (-1)^{|\alpha|} D_\alpha B^\alpha_k dy^k \otimes d^m x$$

(see [3]), where $d^m x = dx^1 \wedge \cdots \wedge dx^m$, $B = \sum_{k=1}^n \sum_{|\alpha|\leq r} B^\alpha_k dy^k_\alpha \otimes d^m x$ and D_α is the iterated formal derivative corresponding to the multiindex α.

A morphism $\widetilde{B} : J^rX \to V^*X \otimes \bigwedge^m T^*X_0$ over X is called *an Euler morphism*. The morphism $\mathbf{E}(B)$ is called *the formal Euler morphism of* B.

Let $\lambda : J^rX \to \bigwedge^m T^*X_0$ be an rth order Lagrangian. We have $\delta\lambda : J^rX \to V^*J^rX \otimes \bigwedge^m T^*X_0$ over J^rX. The morphism $E(\lambda) := \mathbf{E}(\delta\lambda) : J^{2r}X \to V^*X \otimes \bigwedge^m T^*X_0$ over X is called *the Euler morphism of* λ.

Proposition 1 and the Stokes theorem immediately yield the following well known fact.

Proposition 2 ([2]). *A section $\sigma \in \Gamma X$ is critical iff it satisfies the Euler–Lagrange equation $E(\lambda) \circ j^{2r}\sigma = 0$.*

1.6. Let $B : J^rX \to V^*X \otimes \bigwedge T^*X_0$ be an Euler morphism. We can interpret B as a vertical $\bigwedge^m T^*X_0$-valued 1-form on J^rX by using the canonical projection $VJ^rX \to VX$. Then its vertical differential δB (defined fiberwise) is a vertical $\bigwedge^m T^*X_0$-valued 2-form on J^rX. For every vertical vector field η on X, we have $\langle \delta B, \mathcal{J}^r\eta\rangle : J^rX \to V^*J^rX \otimes \bigwedge^m T^*X_0$. Then we apply the formal Euler operator to obtain $\mathbf{E}(\langle \delta B, \mathcal{J}^r\eta\rangle) : J^{2r}X \to V^*X \otimes \bigwedge^m T^*X_0$ over X.

Proposition 3 ([3]). *There exists a unique morphism*

$$H(B) : J^{2r}X \to V^*J^rX \otimes V^*X \otimes \bigwedge{}^m T^*X_0$$

over J^rX satisfying

$$\mathbf{E}(\langle \delta B, \mathcal{J}^r\eta\rangle) = H(B)(\mathcal{J}^r\eta)$$

for every vertical vector field η on X.

Remark 2. The local coordinate form of $H(B)$ is

$$H(B) = \sum_{k,l=1}^{n} \sum_{|\alpha|\leq r} H^\alpha_{kl} dy^k_\alpha \otimes dy^l \otimes d^m x,$$

where

$$H^\alpha_{kl} = \frac{\partial B_l}{\partial y^k_\alpha} - \sum_{|\beta|\leq r-|\alpha|} (-1)^{|\alpha+\beta|} \frac{(\alpha+\beta)!}{\alpha!\beta!} D_\beta \frac{\partial B_k}{\partial y^l_{\alpha+\beta}}$$

and $B = \sum_{k=1}^n B_k dy^k \otimes d^m x$ (see [3]).

The morphism $H(B) : J^{2r}X \to V^*J^rX \otimes V^*X \otimes \bigwedge^m T^*X_0$ over J^rX is called *the Helmholtz morphism of* B.

We have the following characterization of local variationality.

PROPOSITION 4 ([3]). *An* r*th order Euler morphism* B *is locally variational (i.e. locally of the form* $B = E(\lambda)$ *for some local* r*th order Lagrangian* λ*) if and only if* $H(B) = 0$.

2. Variational calculus in fibered-fibered manifolds

2.1. In [5], we generalized the concept of fibered manifolds as follows. A *fibered-fibered manifold* is a fibered surjective submersion $\pi : Y \to X$ between fibered manifolds $p^Y : Y \to Y_0$ and $p^X : X \to X_0$, i.e. a surjective submersion which sends fibers to fibers such that the restricted maps (between fibers) are submersions. If $\pi' : Y' \to X'$ is another fibered-fibered manifold then a fibered map $f : Y \to Y'$ is called *fibered-fibered* if there exists a (unique) fibered map $f_0 : X \to X'$ such that $\pi' \circ f = f_0 \circ \pi$.

Let $r, s, q \in \mathbb{N} \setminus \{0\}$, $s \geq r \leq q$.

Denote the set of local fibered maps $\sigma : X \to Y$ with $\pi \circ \sigma = \mathrm{id}_{\mathrm{dom}(\sigma)}$ (fibered sections) by $\Gamma_{\mathrm{fib}}Y$. By 12.19 in [2], $\sigma, \varrho \in \Gamma_{\mathrm{fib}}Y$ represent the same (r, s, q)-jet $j^{r,s,q}_x\sigma = j^{r,s,q}_x\varrho$ at a point $x \in X$ iff

$$j^r_x\sigma = j^r_x\varrho, \quad j^s_x(\sigma|X_{x_0}) = j^s_x(\varrho|X_{x_0}), \quad j^q_{x_0}\sigma_0 = j^q_{x_0}\varrho_0,$$

where X_0 and Y_0 are the bases of the fibered manifolds X and Y, $x_0 \in X_0$ is the element under x, X_{x_0} is the fiber of X over x_0, and $\sigma_0, \varrho_0 : X_0 \to Y_0$ are the underlying maps of σ, ϱ. The (r, s, q)*-jet prolongation*

$$J^{r,s,q}Y = \{j^{r,s,q}_x\sigma \mid \sigma \in \Gamma_{\mathrm{fib}}Y,\, x \in X\}$$

of Y is a fibered manifold over X with respect to the source projection $\pi^{r,s,q}_X : J^{r,s,q}Y \to X$ (see [4]). We also have the target projection $\pi^{r,s,q}_Y : J^{r,s,q}Y \to Y$. If $\pi' : Y' \to X'$ is another fibered-fibered manifold and $f : Y \to Y'$ is a fibered-fibered map covering a local fibered diffeomorphism $f_0 : X \to X'$ then $J^{r,s,q}f : J^{r,s,q}Y \to J^{r,s,q}Y'$ is given by $J^{r,s,q}f(j^{r,s,q}_x\sigma) = j^{r,s,q}_{f_0(x)}(f \circ \sigma \circ f_0^{-1})$ for any $j^{r,s,q}_x\sigma \in J^{r,s,q}Y$.

2.2. Let $\pi : Y \to X$ be a fibered-fibered manifold which is a fibered submersion between fibered manifolds $p^Y : Y \to Y_0$ and $p^X : X \to X_0$. A projectable vector field W on the fibered manifold Y is *projectable-projectable* if there exists a π-related (to W) projectable vector field $\underline{W}$ on X. If W is projectable-projectable on Y, then its flow $\mathrm{Exp}\, tW$ is formed by local fibered-fibered diffeomorphisms, and we define a vector field

$$\mathcal{J}^{r,s,q}W = \frac{\partial}{\partial t}_{|t=0} J^{r,s,q}(\mathrm{Exp}\, tW)$$

on $J^{r,s,q}Y$. If additionally W is π-vertical and p^Y-vertical (i.e. W is π-related and p^Y-related to zero vector fields), then $\mathcal{J}^{r,s,q}W$ is $\pi_X^{r,s,q}$-vertical and $p^Y \circ \pi_Y^{r,s,q}$-vertical.

2.3. Let r, s, q be as above.

An (r,s,q)*th order Lagrangian* on a fibered-fibered manifold $\pi : Y \to X$ with $\dim X = m$ is a base preserving (over X) morphism

$$\lambda : J^{r,s,q}Y \to \bigwedge^m T^*X.$$

Given a fibered section $\sigma \in \Gamma_{\mathrm{fib}}Y$ and a compact subset $K \subset \mathrm{dom}(\sigma) \subset X$ contained in a chart domain, the *action* is

$$S(\lambda, \sigma, K) = \int_K (\lambda \circ j^{r,s,q}\sigma).$$

A fibered section $\sigma \in \Gamma_{\mathrm{fib}}Y$ is called *critical* (with respect to λ) if for any compact $K \subset \mathrm{dom}(\sigma)$ contained in a chart domain and any π-vertical and p^Y-vertical vector field η on Y with compact support in $\pi^{-1}(K)$ contained in a chart domain we have

$$\frac{d}{dt}_{|t=0} S(\lambda, \mathrm{Exp}\, t\eta \circ \sigma, K) = 0.$$

Again we see that σ is critical iff for any compact K and η as above we have

$$\int \langle \delta\lambda, \mathcal{J}^{r,s,q}\eta \rangle j^{r,s,q}\sigma = 0,$$

where $\delta\lambda : \mathcal{V}J^{r,s,q}Y \to \bigwedge^m T^*X$ is the restriction of the differential of λ to the vector subbundle $\mathcal{V}J^{r,s,q}Y \subset TJ^{r,sq}Y$ of vectors vertical with respect to the projections from $J^{r,s,q}Y$ onto X and onto Y_0.

2.4. Given a base preserving morphism $\varphi : J^{\widetilde{p},\widetilde{p},\widetilde{p}}Y \to \bigwedge^k T^*X$, its *formal exterior differential* $\mathcal{D}\varphi : J^{\widetilde{p}+1,\widetilde{p}+1,\widetilde{p}+1}Y \to \bigwedge^{k+1} T^*X$ over X is defined by

$$\mathcal{D}\varphi(j_x^{\widetilde{p}+1,\widetilde{p}+1,\widetilde{p}+1}\sigma) = d(\varphi \circ j^{\widetilde{p},\widetilde{p},\widetilde{p}}\sigma)(x)$$

for every local fibered section σ of Y, where d means the exterior differential at $x \in X$ of the local k-form $\varphi \circ j^{\widetilde{p},\widetilde{p},\widetilde{p}}\sigma$ on X.

For every morphism $F : J^{\widetilde{p},\widetilde{p},\widetilde{p}}Y \to \bigotimes^l \mathcal{V}^* J^{\overline{p},\overline{p},\overline{p}}Y \otimes \bigwedge^k T^*X$, $\overline{p} \le \widetilde{p}$, over $J^{\overline{p},\overline{p},\overline{p}}Y$, and every l-tuple of π-vertical and p^Y-vertical vector fields $\eta_1, \dots, \eta_l$ on Y, we have the evaluation $F(\mathcal{J}^{\overline{p},\overline{p},\overline{p}}\eta_1, \dots, \mathcal{J}^{\overline{p},\overline{p},\overline{p}}\eta_l) : J^{\widetilde{p},\widetilde{p},\widetilde{p}}Y \to \bigwedge^k T^*X$. One verifies easily in coordinates that there exists a unique morphism $\mathcal{D}F : J^{\widetilde{p}+1,\widetilde{p}+1,\widetilde{p}+1}Y \to \bigotimes^l \mathcal{V}^* J^{\overline{p}+1,\overline{p}+1,\overline{p}+1}Y \otimes \bigwedge^{k+1} T^*X$ over $J^{\overline{p}+1,\overline{p}+1,\overline{p}+1}Y$ satisfying

$$\mathcal{D}(F(\mathcal{J}^{\overline{p},\overline{p},\overline{p}}\eta_1, \dots, \mathcal{J}^{\overline{p},\overline{p},\overline{p}}\eta_l)) = (\mathcal{D}F)(\mathcal{J}^{\overline{p}+1,\overline{p}+1,\overline{p}+1}\eta_1, \dots, \mathcal{J}^{\overline{p}+1,\overline{p}+1,\overline{p}+1}\eta_l)$$

for all $\eta_1, \dots, \eta_l$. Here and throughout, $\mathcal{V}J^{\overline{p},\overline{p},\overline{p}}Y$ is the vector subbundle of $TJ^{\overline{p},\overline{p},\overline{p}}Y$ of vectors vertical with respect to the obvious projections from

$J^{\overline{p},\overline{p},\overline{p}}Y$ onto X and onto Y_0. Also in this case $\mathcal{D}F$ will be called the formal exterior differential of F.

2.5. In the following assertion we do not explicitly indicate the pullbacks to $J^{2p,2p,2p}Y$ and $J^{p,p,p}Y$.

PROPOSITION 5. *Let r, s, q be natural numbers with $s \ge r \le q$, and set $p = \max(q, s)$. For every morphism $B : J^{r,s,q}Y \to \mathcal{V}^* J^{r,s,q}Y \otimes \bigwedge^m T^*X$ over $J^{r,s,q}Y$, there is a unique pair of morphisms*

$$\widetilde{\mathbf{E}}(B) : J^{2p,2p,2p}Y \to \mathcal{V}^*Y \otimes \bigwedge^m T^*X$$

and

$$\mathcal{F}(B) : J^{2p,2p,2p}Y \to \mathcal{V}^* J^{p,p,p}Y \otimes \bigwedge^m T^*X,$$

over Y and $J^{p,p,p}Y$, respectively, such that $B = \widetilde{\mathbf{E}}(B) + \mathcal{F}(B)$, and $\mathcal{F}(B)$ is locally of the form $\mathcal{F}(B) = \mathcal{D}P$, $P : J^{2p-1,2p-1,2p-1}Y \to \mathcal{V}^* J^{p-1,p-1,p-1}Y \otimes \bigwedge^{m-1} T^*X$. Here $\mathcal{V}Y$, $\mathcal{V}J^{p-1,p-1,p-1}Y$ and $\mathcal{V}J^{p,p,p}Y$ are as in Sections 2.3 *and* 2.4.

Proof. Let $\pi^{p,p,p}_{r,s,q} : J^{p,p,p}Y \to J^{r,s,q}Y$ be the jet projection and let $i_p : J^{p,p,p}Y \to J^pY$ be the canonical inclusion, where in J^pY we consider Y as a fibered manifold over X. Using a suitable partition of unity on X and local fibered-fibered coordinate arguments we produce a morphism $\widetilde{B} : J^pY \to V^*J^pY \otimes \bigwedge^m T^*X$ over J^pY such that $(i_p)^*\widetilde{B} = (\pi^{p,p,p}_{r,s,q})^*B$. Then by the decomposition formula (Proposition 1) there exists a pair of morphisms

$$\mathbf{E}(\widetilde{B}) : J^{2p}Y \to V^*Y \otimes \bigwedge^m T^*X, \quad F(\widetilde{B}) : J^{2p}X \to V^*J^pY \otimes \bigwedge^m T^*X$$

satisfying $\widetilde{B} = \mathbf{E}(\widetilde{B}) + F(\widetilde{B})$, and $F(\widetilde{B})$ is locally of the form $F(\widetilde{B}) = D\widetilde{P}$, with $\widetilde{P} : J^{2p-1}Y \to V^*J^{p-1}Y \otimes \bigwedge^{m-1} T^*X$. Taking the pullback $(i_{2p})^*$ of both sides of the last formula and using the obvious equality $\mathcal{D}((\pi^{p,p,p}_{r,s,q})^*B) =$ the restriction of $(i_{2p})^*D(\widetilde{B})$ to $\mathcal{V}J^{2p,2p,2p}Y$, we have the desired decomposition, provided we put $\widetilde{\mathbf{E}}(B) =$ the restriction of $(i_{2p})^*\mathbf{E}(\widetilde{B})$ to $\mathcal{V}Y$ and $\mathcal{F}(B) =$ the restriction of $(i_{2p})^*F(\widetilde{B})$ to $\mathcal{V}J^{p,p,p}Y$. Since locally $F(\widetilde{B}) = D\widetilde{P}$, $\mathcal{F}(B) = \mathcal{D}P$ for $P =$ the restriction of $(i_{2p-1})^*\widetilde{P}$ to $\mathcal{V}J^{p-1,p-1,p-1}Y$. Using Remark 1 it is easy to see (see Remark 3) that the definition of $\widetilde{\mathbf{E}}(B)$ does not depend on the choice of $\widetilde{B}$. ∎

REMARK 3. Let (x^i, X^I, y^k, Y^K) for $i = 1, \ldots, m_1$, $I = 1, \ldots, m_2$, $k = 1, \ldots, n_1$ and $K = 1, \ldots, n_2$ be a fibered-fibered local coordinate system on a fibered-fibered manifold Y. For any $f : J^{\widetilde{p},\widetilde{p},\widetilde{p}}Y \to \mathbb{R}$ we have the decomposition

$$\mathcal{D}(f) = \mathcal{D}_i(f)dx^i + \mathcal{D}_I(f)dX^I,$$

where $\mathcal{D}_i(f) : J^{\widetilde{p}+1,\widetilde{p}+1,\widetilde{p}+1}Y \to \mathbb{R}$ and $\mathcal{D}_I(f) : J^{\widetilde{p}+1,\widetilde{p}+1,\widetilde{p}+1}Y \to \mathbb{R}$ are the "total" derivatives of f. Let $F : J^{\widetilde{p}}Y \to \mathbb{R}$ be such that $F \circ i_{\widetilde{p}} = f$. From the clear equality $D(F)\circ i_{\widetilde{p}+1} = \mathcal{D}(f)$ we easily deduce that $\mathcal{D}_i(f) = D_i(F)\circ i_{\widetilde{p}+1}$ and $\mathcal{D}_I(f) = D_I(F)\circ i_{\widetilde{p}+1}$. In particular, since D_i, D_I and $D_{i'}, D_{I'}$ commute, so do $\mathcal{D}_i, \mathcal{D}_I$ and $\mathcal{D}_{i'}, \mathcal{D}_{I'}$. From the formulas for D_i and D_I (see Remark 1) and from the above formulas for $\mathcal{D}_i$ and $\mathcal{D}_I$ we easily see that in local coordinates

$$\mathcal{D}_i(f) = \frac{\partial f}{\partial x^i} + \sum_{k=1}^{n_1} \sum_{|\widetilde{\alpha}|\le\widetilde{p}} \frac{\partial f}{\partial y^k_{\widetilde{\alpha}}}\, y^k_{\widetilde{\alpha}+1_i} + \sum_{K=1}^{n_2} \sum_{|\widetilde{\beta}|+|\widetilde{\gamma}|\le\widetilde{p}} \frac{\partial f}{\partial Y^K_{(\widetilde{\beta},\widetilde{\gamma})}}\, Y^K_{(\widetilde{\beta}+1_i,\widetilde{\gamma})}$$

and

$$\mathcal{D}_I(f) = \frac{\partial f}{\partial X^I} + \sum_{K=1}^{n_2} \sum_{|\widetilde{\beta}|+|\widetilde{\gamma}|\le\widetilde{p}} \frac{\partial f}{\partial Y^K_{(\widetilde{\beta},\widetilde{\gamma})}}\, Y^K_{(\widetilde{\beta},\widetilde{\gamma}+1_I)},$$

where $(x^i, X^I, y^k_{\widetilde{\alpha}}, Y^K_{(\widetilde{\beta},\widetilde{\gamma})})$ is the induced coordinate system on $J^{\widetilde{p},\widetilde{p},\widetilde{p}}Y$, $\widetilde{\alpha} = (\widetilde{\alpha}^1, \ldots, \widetilde{\alpha}^{m_1})$, $\widetilde{\beta} = (\widetilde{\beta}^{m_1}, \ldots, \widetilde{\beta}^{m_1})$ and $\widetilde{\gamma} = (\widetilde{\gamma}^1, \ldots, \widetilde{\gamma}^{m_2})$.

Let $(x^i, X^I, y^k_\alpha, Y^K_{(\beta,\gamma)})$ be the induced coordinates on $J^{p,p,p}Y$, where $p = \max(s,q)$. Then using the formula in Remark 1 it is easy to see that the local coordinate form of $\widetilde{\mathbf{E}}(B)$ is

$$\widetilde{\mathbf{E}}(B) = \sum_{K=1}^{n_2} \sum_{|\beta|+|\gamma|\le p} (-1)^{|\beta|+|\gamma|} \mathcal{D}_{(\beta,\gamma)} B^{(\beta,\gamma)}_K dY^K \otimes (d^{m_1}x \wedge d^{m_2}X),$$

where $d^{m_1}x = dx^1 \wedge \cdots \wedge dx^{m_1}$, $d^{m_2}X = dX^1 \wedge \cdots \wedge dX^{m_2}$, $(\pi^{p,p,p}_{r,s,q})^* B = \sum_{K=1}^{n_2} \sum_{|\beta|+|\gamma|\le p} B^{(\beta,\gamma)}_K d^K_{(\beta,\gamma)} \otimes (d^{m_1} \wedge d^{m_2}X)$ and $\mathcal{D}_{(\beta,\gamma)}$ denotes the iterated "total" derivative with $\beta = (\beta^1, \ldots, \beta^{m_1})$, $\gamma = (\gamma^1, \ldots, \gamma^{m_2})$.

From the above local formula it follows that $\widetilde{\mathbf{E}}(B)$ can be factorized through $J^{r+s,2s,r+p}Y$, $p = \max(s,q)$.

A morphism $\widetilde{B} : J^{r,s,q}Y \to \mathcal{V}^*Y \otimes \bigwedge^m T^*X$ over Y is called *an Euler morphism*. The morphism $\widetilde{\mathbf{E}}(B) : J^{2p,2p,2p}Y \to \mathcal{V}^*Y \otimes \bigwedge^m T^*X$ over Y is called *the formal Euler morphism of* B.

Let λ be an (r,s,q)th order Lagrangian on Y, and $p = \max(s,q)$. We have $\delta\lambda : J^{r,s,q}Y \to \mathcal{V}^* J^{r,s,q}Y \otimes \bigwedge^m T^*X$. The morphism $\mathcal{E}(\lambda) = \widetilde{\mathbf{E}}(\delta\lambda) : J^{2p,2p,2p}Y \to \mathcal{V}^*Y \otimes \bigwedge^m T^*X$ over Y is called *the Euler morphism of* λ.

By the above-mentioned property of $\widetilde{\mathbf{E}}(B)$ it follows that $\mathcal{E}(\lambda)$ can also be factorized through $J^{r+s,2s,r+p}Y$.

Proposition 5 and the Stokes theorem yield the following fact.

PROPOSITION 6 ([6]). *A fibered section $\sigma \in \Gamma_{\mathrm{fib}}Y$ is critical iff it satisfies the Euler–Lagrange equation $\mathcal{E}(\lambda) \circ j^{2p,2p,2p}\sigma = 0$. By the above-mentioned property of $\mathcal{E}(\lambda)$ this equation is $\mathcal{E}(\lambda) \circ j^{r+s,2s,r+p}\sigma = 0$.*

2.6. Let $B : J^{r,s,q}Y \to \mathcal{V}^*Y \otimes \bigwedge^m T^*X$ be an Euler morphism, and $p = \max(s,q)$. Using the canonical projections $\mathcal{V}^*J^{r,s,q}Y \to \mathcal{V}Y$, we can interpret B as a vertical $\bigwedge^m T^*X$-valued 1-form on $J^{r,s,q}Y$. Then the vertical differential δB (defined fiberwise) is a vertical $\bigwedge^m T^*X$-valued 2-form on $J^{r,s,q}Y$. For every π-vertical and p^Y-vertical vector field η on Y, we have $\langle \delta B, \mathcal{J}^{r,s,q}\eta\rangle : J^{r,s,q}Y \to \mathcal{V}^*J^{r,s,q}Y \otimes \bigwedge^m T^*X$ over $J^{r,s,q}Y$. Then we can apply the formal Euler operator to obtain $\widetilde{\mathbf{E}}(\langle \delta B, \mathcal{J}^{r,s,q}\eta\rangle) : J^{2p,2p,2p}Y \to \mathcal{V}^*Y \otimes \bigwedge^m T^*X$ over Y.

Proposition 7. *There exists a unique morphism*

$$\mathcal{H}(B) : J^{2p,2p,2p}Y \to \mathcal{V}^*J^{p,p,p}Y \otimes \mathcal{V}^*Y \otimes \bigwedge^m T^*X$$

over $J^{p,p,p}Y$ *satisfying*

$$\widetilde{\mathbf{E}}(\langle \delta B, \mathcal{J}^{r,s,q}\eta\rangle) = \mathcal{H}(B)(\mathcal{J}^{p,p,p}\eta)$$

for every π*-vertical and* p^Y*-vertical vector field* η *on* Y.

Proof. That $\mathcal{H}(B)$ is unique is clear. We prove the existence.

As in the proof of Proposition 5, we have a morphism $\widetilde{B} : J^pY \to V^*Y \otimes \bigwedge^m T^*X$ over Y such that $(\pi^{p,p,p}_{r,s,q})^*B =$ the restriction of $(i_p)^*\widetilde{B}$ to $\mathcal{V}^*Y$. Then by Proposition 3, $\mathbf{E}(\langle \delta\widetilde{B}, \mathcal{J}^p\eta\rangle) = H(\widetilde{B})(\mathcal{J}^p\eta)$, where $H(\widetilde{B})$ is the Helmholtz morphism of $\widetilde{B}$. Applying the pull-back $(i_{2p})^*$ to both sides of the last equality and using the definition of $\widetilde{\mathbf{E}}(\langle \delta B, \mathcal{J}^{r,s,q}\eta\rangle)$ (see the proof of Proposition 5) we obtain the desired equality for $\mathcal{H}(B) =$ the restriction of $(i_{2p})^*H(\widetilde{B})$ to $\mathcal{V}J^{p,p,p}Y \times_Y \mathcal{V}Y$. One can show (see Remark 4 below) that the definition of $\mathcal{H}(B)$ is independent of the choice of $\widetilde{B}$. ■

Remark 4. It follows from the formula in Remark 2 and from the definition of $\mathcal{H}(B)$ in the proof of Proposition 7 that the local coordinate form of $\mathcal{H}(B)$ is

$$\mathcal{H}(B) = \sum_{K,L=1}^{n_2} \sum_{|\beta|+|\gamma|\le p} \mathcal{H}^{(\beta,\gamma)}_{KL}\, dY^K_{(\beta,\gamma)} \otimes dY^L \otimes d^{m_1}x \otimes d^{m_2}X,$$

where

$$\mathcal{H}^{(\beta,\gamma)}_{KL} = \frac{\partial B_L}{\partial Y^K_{(\beta,\gamma)}} - \sum_{|\widetilde{\beta}|+|\widetilde{\gamma}|\le p-|\beta|-|\gamma|} (-1)^{|\widetilde{\beta}|+|\widetilde{\gamma}|} \frac{(\beta+\widetilde{\beta})!(\gamma+\widetilde{\gamma})!}{\beta!\widetilde{\beta}!\gamma!\widetilde{\gamma}!} \mathcal{D}_{(\widetilde{\beta},\widetilde{\gamma})} \frac{\partial B_K}{\partial Y^L_{(\beta+\widetilde{\beta},\gamma+\widetilde{\gamma})}}$$

and $B = \sum_{K=1}^{n_1} B_K dY^K \otimes d^{m_1}x \wedge d^{m_2}X$.

From this local formula it follows easily that $\mathcal{H}(B)$ can be factorized through $(J^{s+p,s+p,2p}Y \times_{J^{r,s,r}Y} \mathcal{V}J^{r,s,r}Y) \times_Y \mathcal{V}Y$.

We have the following characterization of local variationality.

PROPOSITION 8. *Let* $s \geq r \leq q$ *be natural numbers and* $p = \max(s, q)$*. A* $(2p, 2p, 2p)$*th order Euler morphism* B *is locally variational* (*i.e. locally of the form* $\mathcal{E}(\lambda)$ *for some* (p, p, p)*th order Lagrangian* λ) *if and only if* $\mathcal{H}(B) = 0$.

Moreover, *if a* $(2p, 2p, 2p)$*th order Euler morphism* B *is locally variational and factorizes through* $J^{r+s,2s,r+p}Y$, *then locally* $B = \mathcal{E}(\lambda)$ *for some* (r, s, p)*th order Lagrangian.*

Proof. Suppose locally $B = \mathcal{E}(\lambda)$. Choose a local pth order Lagrangian $\Lambda : J^pY \to \bigwedge^m T^*X$ such that $\lambda \circ \pi^{p,p,p}_{r,s,q} = (i_p)^*\Lambda$. We see that $\delta\lambda$ is the restriction of $(i_p)^*\delta\Lambda$ to $\mathcal{V}Y$. Hence $\mathcal{H}(B) = \mathcal{H}(\mathcal{E}(\lambda))$ is the restriction of $(i_{4p})^*H(E(\Lambda))$ to $\mathcal{V}J^{2p}Y \times_Y \mathcal{V}Y$. Since $H(E(\Lambda)) = 0$ (see Proposition 4), also $\mathcal{H}(B) = 0$.

To prove the converse we choose local fibered-fibered coordinates (x^i, X^I, y^k, Y^K) on $U \subset Y$. In this coordinate system we have the obvious projection $\Pi : J^{\widetilde{p}}U = \mathbb{R}^M \to J^{\widetilde{p},\widetilde{p},\widetilde{p}}U = \mathbb{R}^N$ for any $\widetilde{p}$. Let $\mathcal{H}(B) = 0$. Then (using the local formula) we have $H(\Pi^*B) = 0$. Proposition 4 yields $\Pi^*B = E(\Lambda)$ for some pth order Lagrangian Λ on U. Thus $B = \mathcal{E}(\lambda)$ for $\lambda = (i_p)^*\Lambda$.

The "moreover" part can be deduced in the following way. By the assumption, there is $\widetilde{\lambda}$ of order (p, p, p) such that $B = \mathcal{E}(\widetilde{\lambda})$ over U, where (U, x^i, X^I, y^k, Y^K) are fibered-fibered coordinates. Using these coordinates we can consider the obvious inclusion $J : J^{r,s,p}U = \mathbb{R}^M \to J^{p,p,p}U = \mathbb{R}^N$, $J(v) = (v, 0)$. Then (using the local expression of $\widetilde{\mathbf{E}}(\delta\lambda)$) we see that $B = \mathcal{E}(J^*\widetilde{\lambda})$. ∎

3. On naturality of the Helmholtz operator. We say that a fibered manifold $p : X \to X_0$ is of *dimension* (m, n) if $\dim X_0 = m$ and $\dim X = m + n$. All (m, n)-dimensional fibered manifolds and their local fibered diffeomorphisms form a category which we denote by $\mathcal{FM}_{m,n}$ and which is local and admissible in the sense of [2].

Similarly, a fibered-fibered manifold $\pi : Y \to X$ is of *dimension* (m_1, m_2, n_1, n_2) if the fibered manifold X is of dimension (m_1, n_1) and the fibered manifold Y is of dimension $(m_1 + n_1, m_2 + n_2)$. All (m_1, m_2, n_1, n_2)-dimensional fibered-fibered manifolds and their fibered-fibered local diffeomorphisms form a category which we denote by $\mathcal{FM}_{m_1,m_2,n_1,n_2}$ and which is local and admissible in the sense of [2]. The standard (m_1, m_2, n_1, n_2)-dimensional trivial fibered-fibered manifold $\pi : \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$ will be denoted by $\mathbb{R}^{m_1,m_2,n_1,n_2}$. Any (m_1, m_2, n_1, n_2)-dimensional fibered-fibered manifold is locally $\mathcal{FM}_{m_1,m_2,n_1,n_2}$-isomorphic to the standard $\mathcal{FM}_{m_1,m_2,n_1,n_2}$-object $\mathbb{R}^{m_1,m_2,n_1,n_2}$.

Given two fibered manifolds $Z_1 \to M$ and $Z_2 \to M$ over the same base M, we denote the space of all base preserving fibered manifold morphisms of Z_1 into Z_2 by $\mathcal{C}^\infty_M(Z_1, Z_2)$. In [3], [4], the authors studied the rth order Helmholtz morphism $H(B)$ of variational calculus on an (m,n)-dimensional fibered manifold $p : X \to X_0$ as the Helmholtz operator

$$H : \mathcal{C}^\infty_X(J^rX, V^*X \otimes \bigwedge^m T^*X_0) \to \mathcal{C}^\infty_{J^rX}(J^{2r}X, V^*J^rX \otimes V^*X \otimes \bigwedge^m T^*X_0).$$

They deduced the following classification theorem:

THEOREM 1 ([3], [4]). *Any $\mathcal{FM}_{m,n}$-natural operator (in the sense of [2]) of the type of the Helmholtz operator is of the form cH, $c \in \mathbb{R}$, provided $n \geq 2$.*

The purpose of the present section is to obtain a similar result in the fibered-fibered manifold case. Namely, we study the Helmholtz morphism $\mathcal{H}(B)$ of variational calculus on an (m_1, m_2, n_1, n_2)-dimensional fibered-fibered manifold $\pi : Y \to X$ as the Helmholtz operator

$$\mathcal{H} : \mathcal{C}^\infty_Y(J^{r,s,q}Y, \mathcal{V}^*Y \otimes \bigwedge^m T^*X) \\ \to \mathcal{C}^\infty_{J^{p,p,p}Y}(J^{2p,2p,2p}Y, \mathcal{V}^*J^{p,p,p}Y \otimes \mathcal{V}^*Y \otimes \bigwedge^m T^*X),$$

where $s \geq r \leq q$ are natural numbers, $p = \max(s,q)$ and $m = m_1 + m_2 = \dim X$. We prove the following classification theorem.

THEOREM 2. *Any $\mathcal{FM}_{m_1,m_2,n_1,n_2}$-natural operator (in the sense of [2]) of the type of the Helmholtz operator is of the form $c\mathcal{H}$, $c \in \mathbb{R}$, provided $n_2 \geq 2$.*

REMARK 5. In view of Remark 3 the assertion of Theorem 2 also holds for natural operators

$$D : \mathcal{C}^\infty_Y(J^{r,s,q}Y, \mathcal{V}^*Y \otimes \bigwedge^m T^*X) \\ \to \mathcal{C}^\infty_{J^{r,s,r}Y}(J^{s+p,s+p,2p}Y, \mathcal{V}^*J^{r,s,r}Y \otimes \mathcal{V}^*Y \otimes \bigwedge^m T^*X).$$

REMARK 6. The assumption of the last theorem means that for any $\mathcal{FM}_{m_1,m_2,n_1,n_2}$-morphism $f : Y \to Y'$ and any morphisms

$$B \in \mathcal{C}^\infty_Y(J^{r,s,q}Y, \mathcal{V}^*Y \otimes \bigwedge^m T^*X)$$

and

$$B' \in \mathcal{C}^\infty_{Y'}(J^{r,s,q}Y', \mathcal{V}^*Y' \otimes \bigwedge^m T^*X'),$$

if B and B' are f-related then so are $D(B)$ and $D(B')$. Moreover D is regular and local. The regularity means that D transforms a smoothly parametrized family of appriopriate type morphisms into a smoothly parametrized family of appriopriate type morphisms. The locality means that $D(B)_u$ depends on the germ of B at $\pi^{p,p,p}_{r,s,q}(u)$.

Proof of Theorem 2. Let D be an operator in question.

Let (x^i, X^I, y^k, Y^K) be the usual fibered-fibered coordinate system on $\mathbb{R}^{m_1,m_2,n_1,n_2}$, $i = 1, \dots, m_1$, $I = 1, \dots, m_2$, $k = 1, \dots, n_1$, $K = 1, \dots, n_2$.

Since an $\mathcal{FM}_{m_1,m_2,n_1,n_2}$-map

$$(x^i, X^I, y^k - \sigma^k(x^i, X^I), Y^K - \Sigma^K(x^i, X^I))$$

sends $j^{2p,2p,2p}_{(0,0)}(x^i, X^I, \sigma^k, \Sigma^K)$ to

$$\Theta = j^{2p,2p,2p}_{(0,0)}(x^i, X^I, 0, 0) \in (J^{2p,2p,2p}(\mathbb{R}^{m_1,m_2,n_1,n_2}))_{(0,0,0,0)},$$

$J^{2p,2p,2p}(\mathbb{R}^{m_1,m_2,n_1,n_2})$ is the $\mathcal{FM}_{m_1,m_2,n_1,n_2}$-orbit of Θ. Then D is uniquely determined by the evaluations

$$\langle D(B)_\Theta, w \otimes v\rangle \in \bigwedge^m T^*_0\mathbb{R}^m$$

for all

$$B \in \mathcal{C}^\infty_{\mathbb{R}^{m_1,m_2,n_1,n_2}}(J^{r,s,q}(\mathbb{R}^{m_1,m_2,n_1,n_2}), \mathcal{V}^*\mathbb{R}^{m_1,m_2,n_1,n_2} \otimes \bigwedge^m T^*\mathbb{R}^m),$$
$$w \in \mathcal{V}_{\pi^{2p,2p,2p}_{p,p,p}(\Theta)} J^{p,p,p}(\mathbb{R}^{m_1,m_2,n_1,n_2}), \qquad v \in T_0\mathbb{R}^{n_2} = \mathcal{V}_{(0,0,0,0)}\mathbb{R}^{m_1,m_2,n_1,n_2}.$$

Using the invariance of D with respect to $\mathcal{FM}_{m_1 m_2,n_1,n_2}$-maps of the form $\mathrm{id}_{\mathbb{R}^m} \times \psi$ for appriopriate linear ψ (since $n_2 \geq 2$) we find that D is uniquely determined by the evaluations

$$\left\langle D(B)_\Theta, \frac{d}{dt}_0 (t j^{p,p,p}_{(0,0)}(x^i, X^I, 0, \dots, 0, f(x^i, X^I), 0, \dots, 0)) \otimes \frac{\partial}{\partial Y^2}_0 \right\rangle$$

for all

$$B \in \mathcal{C}^\infty_{\mathbb{R}^{m_1,m_2,n_1,n_2}}(J^{r,s,q}(\mathbb{R}^{m_1,m_2,n_1,n_2}), \mathcal{V}^*\mathbb{R}^{m_1,m_2,n_1,n_2} \otimes \bigwedge^m T^*\mathbb{R}^m)$$

and all $f : \mathbb{R}^m \to \mathbb{R}$, where $f(x^i, X^I)$ is at position Y^1.

Using the invariance of D with respect to the $\mathcal{FM}_{m_1 m_2,n_1,n_2}$-map

$$(x^1, \dots, x^{m_1}, X^1, \dots, X^{m_2}, y^1, \dots, y^{n_1}, Y^1 + f(x^i, X^I)Y^1, Y^2, \dots, Y^{n_2})$$

preserving Θ we can assume $f = 1$, i.e. D is uniquely determined by the evaluations

$$\left\langle D(B)_\Theta, \frac{d}{dt}_0 (t j^{p,p,p}_{(0,0)}(x^i, X^I, 0, \dots, 0, 1, 0, \dots, 0)) \otimes \frac{\partial}{\partial Y^2}_0 \right\rangle \in \bigwedge^m T^*_0\mathbb{R}^m$$

for all

$$B \in \mathcal{C}^\infty_{\mathbb{R}^{m_1,m_2,n_1,n_2}}(J^{r,s,q}(\mathbb{R}^{m_1,m_2,n_1,n_2}), \mathcal{V}^*\mathbb{R}^{m_1,m_2,n_1,n_2} \otimes \bigwedge^m T^*\mathbb{R}^m),$$

where 1 is at position Y^1.

Consider a morphism

$$B \in \mathcal{C}^\infty_{\mathbb{R}^{m_1,m_2,n_1,n_2}}(J^{r,s,q}(\mathbb{R}^{m_1,m_2,n_1,n_2}), \mathcal{V}^*\mathbb{R}^{m_1,m_2,n_1,n_2} \otimes \bigwedge^m T^*\mathbb{R}^m).$$

Using the invariance of D with respect to the $\mathcal{FM}_{m_1,m_2,n_1,n_2}$-maps

$$\psi_{\tau,\mathcal{T}} = \left(x^i, X^I, \frac{1}{\tau^k}\, y^k, \frac{1}{\mathcal{T}^K}\, Y^K\right)$$

for $\tau^k \neq 0$ and $\tau^K \neq 0$ we get the homogeneity condition

$$\Big\langle D((\psi_{\tau,\mathcal{T}})_* B)_\Theta, \frac{d}{dt}_0 (tj^{p,p,p}_{(0,0)}(x^i, X^I, 0, \ldots, 0, 1, 0, \ldots, 0)) \otimes \frac{\partial}{\partial Y^2}_0 \Big\rangle$$
$$= \mathcal{T}^1 \mathcal{T}^2 \Big\langle D(B)_\Theta, \frac{d}{dt}_0 (tj^{p,p,p}_{(0,0)}(x^i, X^I, 0, \ldots, 0, 1, 0, \ldots, 0)) \otimes \frac{\partial}{\partial Y^2}_0 \Big\rangle$$

for $\tau = (\tau^k)$ and $\mathcal{T} = (\mathcal{T}^K)$. By Corollary 19.8 in [1] of the non-linear Peetre theorem we can assume that B is a polynomial (of arbitrary degree). The regularity of D implies that

$$\Big\langle D(B)_\Theta, \frac{d}{dt}_0 (tj^{p,p,p}_{(0,0)}(x^i, X^I, 0, \ldots, 0, 1, 0, \ldots, 0)) \otimes \frac{\partial}{\partial Y^2}_0 \Big\rangle$$

is smooth with respect to the coordinates of B. Then by the homogeneous function theorem (and the above type of homogeneity) we deduce that

$$\Big\langle D(B)_\Theta, \frac{d}{dt}_0 (tj^{p,p,p}_{(0,0)}(x^i, X^I, 0, \ldots, 0, 1, 0, \ldots, 0)) \otimes \frac{\partial}{\partial Y^2}_0 \Big\rangle$$

depends linearly on the coordinates of B on all $x^\varrho X^\sigma Y^1_{(\beta,\gamma)} dY^2 \otimes d^{m_1}x \wedge d^{m_2}X$ and $x^\varrho X^\sigma Y^2_{(\beta,\gamma)} dY^1 \otimes d^{m_1}x \wedge d^{m_2}X$, it depends bilinearly on the coordinates of B on all $x^\varrho X^\sigma dY^1 \otimes d^{m_1}x \wedge d^{m_2}X$ and $x^\varrho X^\sigma dY^2 \otimes d^{m_1}x \wedge d^{m_2}X$, and it is independent of the other coordinates of B, where (of course) $(x^i, X^I, y^k_\alpha, Y^K_{(\beta,\gamma)})$ is the induced coordinate system on the prolongation $J^{r,s,q}(\mathbb{R}^{m_1,m_2,n_1,n_2})$ and $d^{m_1}x = dx^1 \wedge \cdots \wedge dx^{m_1}$ and $d^{m_2}X = dX^1 \wedge \cdots \wedge dX^{m_2}$. (Here and in what follows, α, β are arbitrary m_1-tuples and γ is an arbitrary m_2-tuple with $|\alpha| \leq q$, $|\beta| + |\gamma| \leq r$ or $|\gamma| \leq s$ if $\beta = (0)$).

In other words (and more precisely),

$$\Big\langle D(B)_\Theta, \frac{d}{dt}_0 (tj^{r,s,q}_{(0,0)}(x^i, X^I, 0, \ldots, 0, 1, 0, \ldots, 0)) \otimes \frac{\partial}{\partial Y^2}_0 \Big\rangle$$

is determined by the values

$$\Big\langle D(x^\varrho X^\sigma Y^2_{(\beta,\gamma)} dY^1 \otimes d^{m_1}x \wedge d^{m_2}X)_\Theta,$$
$$\frac{d}{dt}_0 (tj^{r,s,q}_{(0,0)}(x^i, X^I, 0, \ldots, 0, 1, 0, \ldots, 0)) \otimes \frac{\partial}{\partial Y^2}_0 \Big\rangle,$$
$$\Big\langle D(x^\varrho X^\sigma Y^1_{(\beta,\gamma)} dY^2 \otimes d^{m_1}x \wedge d^{m_2}X)_\Theta,$$
$$\frac{d}{dt}_0 (tj^{r,s,q}_{(0,0)}(x^i, X^I, 0, \ldots, 0, 1, 0, \ldots, 0)) \otimes \frac{\partial}{\partial Y^2}_0 \Big\rangle,$$

$$\Big\langle D(x^\varrho X^\sigma dY^1 \otimes d^{m_1}x \wedge d^{m_2}X + x^{\tilde\varrho}X^{\tilde\sigma}dY^2 \otimes d^{m_1}x \wedge d^{m_2}X)_\Theta,$$
$$\frac{d}{dt}_0(tj^{r,s,q}_{(0,0)}(x^i, X^I, 0, \ldots, 0, 1, 0, \ldots, 0)) \otimes \frac{\partial}{\partial Y^2}_0\Big\rangle.$$

Furthermore, $\langle D(B)_\Theta, \frac{d}{dt}_0(tj^{r,s,q}_{(0,0)}(x^i, X^I, 0, \ldots, 0, 1, 0, \ldots, 0)) \otimes \frac{\partial}{\partial Y^2}_0\rangle$ is linear in B for B from the $\mathbb{R}$-vector subspace spanned by all elements $x^\varrho X^\sigma Y^1_{(\beta,\gamma)}dY^2 \otimes d^{m_1}x \wedge d^{m_2}X$ and $x^{\tilde\varrho}X^{\tilde\sigma}Y^2_{(\beta,\gamma)}dY^1 \otimes d^{m_1}x \wedge d^{m_2}X$; moreover,

$$\Big\langle D(dY^1 \otimes d^{m_1}x \wedge d^{m_2}X + B)_\Theta,$$
$$\frac{d}{dt}_0(tj^{r,s,q}_{(0,0)}(x^i, X^I, 0, \ldots, 0, 1, 0, \ldots, 0)) \otimes \frac{\partial}{\partial Y^2}_0\Big\rangle$$
$$= \Big\langle D(B)_\Theta, \frac{d}{dt}_0(tj^{r,s,q}_{(0,0)}(x^i, X^I, 0, \ldots, 0, 1, 0, \ldots, 0)) \otimes \frac{\partial}{\partial Y^2}_0\Big\rangle$$

for B from the vector subspace (over $\mathbb{R}$) spanned by all $x^\varrho X^\sigma Y^1_{(\beta,\gamma)}dY^2 \otimes d^{m_1}x \wedge d^{m_2}X$ and $x^{\tilde\varrho}X^{\tilde\sigma}Y^2_{(\beta,\gamma)}dY^1 \otimes d^{m_1}x \wedge d^{m_2}X$; and

$$(1)\quad \Big\langle D(ax^\varrho X^\sigma dY^1 \otimes d^{m_1}x \wedge d^{m_2}X + bx^{\tilde\varrho}X^{\tilde\sigma}dY^2 \otimes d^{m_1}x \wedge d^{m_2}X)_\Theta,$$
$$\frac{d}{dt}_0(tj^{r,s,q}_{(0,0)}(x^i, X^I, 0, \ldots, 0, 1, 0, \ldots, 0)) \otimes \frac{\partial}{\partial Y^2}_0\Big\rangle$$
$$= ab\Big\langle D(x^\varrho X^\sigma dY^1 \otimes d^{m_1}x \wedge d^{m_2}X + x^{\tilde\varrho}X^{\tilde\sigma}dY^2 \otimes d^{m_1}x \wedge d^{m_2}X)_\Theta,$$
$$\frac{d}{dt}_0(tj^{r,s,q}_{(0,0)}(x^i, X^I, 0, \ldots, 0, 1, 0, \ldots, 0)) \otimes \frac{\partial}{\partial Y^2}_0\Big\rangle$$

for all real numbers a and b.

Then by the invariance of D with respect to $(\tau^i x^i, \mathcal{T}^I X^I, y^k, Y^K)$ for $\tau^i \neq 0$ and $\mathcal{T}^I \neq 0$ we get

$$\Big\langle D(x^\varrho X^\sigma Y^2_{(\beta,\gamma)}dY^1 \otimes d^{m_1}x \wedge d^{m_2}X)_\Theta,$$
$$\frac{d}{dt}_0(tj^{r,s,q}_{(0,0)}(x^i, X^I, 0, \ldots, 0, 1, 0, \ldots, 0)) \otimes \frac{\partial}{\partial Y^2}_0\Big\rangle$$
$$= \Big\langle D(x^\varrho X^\sigma Y^1_{(\beta,\gamma)}dY^2 \otimes d^{m_1}x \wedge d^{m_2}X)_\Theta,$$
$$\frac{d}{dt}_0(tj^{r,s,q}_{(0,0)}(x^i, X^I, 0, \ldots, 0, 1, 0, \ldots, 0)) \otimes \frac{\partial}{\partial Y^2}_0\Big\rangle = 0$$

for $(\beta,\gamma)\neq(\varrho,\sigma)$, and

$$\Big\langle D(x^{\varrho}X^{\sigma}dY^1\otimes d^{m_1}x\wedge d^{m_2}X+x^{\tilde\varrho}X^{\tilde\sigma}dY^2\otimes d^{m_1}x\wedge d^{m_2}X)_{\Theta},$$
$$\frac{d}{dt}_0(tj^{r,s,q}_{(0,0)}(x^i,X^I,0,\dots,0,1,0,\dots,0))\otimes\frac{\partial}{\partial Y^2}_0\Big\rangle=0$$

for all ϱ, $\tilde\varrho$, σ and $\tilde\sigma$.

Hence D is determined by the evaluations

$$(2)\quad \Big\langle D(x^{\beta}X^{\gamma}Y^1_{(\beta,\gamma)}dY^2\otimes d^{m_1}x\wedge d^{m_2}X)_{\Theta},$$
$$\frac{d}{dt}_0(tj^{r,s,q}_{(0,0)}(x^i,X^I,0,\dots,0,1,0,\dots,0))\otimes\frac{\partial}{\partial Y_2}_0\Big\rangle,$$

$$(3)\quad \Big\langle D(x^{\beta}X^{\gamma}Y^2_{(\beta,\gamma)}dY^1\otimes d^{m_1}x\wedge d^{m_2}X)_{\Theta},$$
$$\frac{d}{dt}_0(tj^{r,s,q}_{(0,0)}(x^i,X^I,0,\dots,0,1,\dots,0))\otimes\frac{\partial}{\partial Y^2}_0\Big\rangle.$$

Suppose $\beta^{i_0}\neq 0$ for some $i_0=1,\dots,m_1$. We use the invariance of D with respect to the locally defined $\mathcal{FM}_{m_1,m_2,n_1,n_2}$-map

$$\psi^{i_0}=(x^i,X^I,y^k,Y^1,Y^2+x^{i_0}Y^2,Y^3,\dots,Y^{n_2})^{-1}$$

preserving x^i, X^I, , Θ, Y^1, $j^{r,s,q}_{(0,0)}(x^i,X^I,0,\dots,0,1,0,\dots,0)$, $\frac{\partial}{\partial Y^2}_0$ and sending $Y^2_{(\beta,\gamma)}$ to $Y^2_{(\beta,\gamma)}+x^{i_0}Y^2_{(\beta,\gamma)}+Y^2_{(\beta-1_{i_0},\gamma)}$. Applying this invariance to

$$\Big\langle D(x^{\beta-1_{i_0}}X^{\gamma}Y^2_{(\beta,\gamma)}dY^1\otimes d^{m_1}x\wedge d^{m_2}X)_{\Theta},$$
$$\frac{d}{dt}_0(tj^{r,s,q}_{(0,0)}(x^i,X^I,0,\dots,0,1,0,\dots,0))\otimes\frac{\partial}{\partial Y^2}_0\Big\rangle$$

it follows that the value (3) is zero if it is zero for $\beta-1_{i_0}$ instead of β. Continuing this process and a similar one for the $\mathcal{FM}_{m_1,m_2,n_1,n_2}$-morphism

$$\Psi^{I_0}=(x^i,X^I,y^k,Y^1,Y^2+X^{I_0}Y^2,Y^3,\dots,Y^{n_0})^{-1}$$

instead of ψ^{i_0} we see that (3) is zero if it is zero for $(\beta,\gamma)=((0),(0))$.

By similar arguments (since ψ^{i_0} sends dY^2 to $dY^2+x^{i_0}dY^2$ and Ψ^{I_0} sends dY^2 to $dY^2+X^{I_0}dY^2$), from the equality

$$\Big\langle D(x^{\beta-1_{i_0}}X^{\gamma}Y^1_{(\beta,\gamma)}dY^2\otimes d^{m_1}x\wedge d^{m_2}X)_{\Theta},$$
$$\frac{d}{dt}(tj^{r,s,q}_{(0,0)}(x^i,X^I,0,\dots,0,1,0,\dots,0))\otimes\frac{\partial}{\partial Y^2}_0\Big\rangle=0$$

for $\beta_{i_0} \neq 0$ (or a similar equality for $\gamma_{I_0} \neq 0$) we find that (2) is zero if $(\beta,\gamma) \neq ((0),(0))$.

In other words, D is uniquely determined by the values (2) and (3) for $(\beta,\gamma) = ((0),(0))$.

Using the invariance of D with respect to the (local) $\mathcal{FM}_{m_1,m_2,n_1,n_2}$-map

$$(x^i, X^I, y^k, Y^1 + Y^1Y^2, Y^2, \dots, Y^{n_1})^{-1}$$

preserving Θ, $j^{r,s,q}_{(0,0)}(x^i, X^I, 0, \dots, 0, 1, 0, \dots, 0)$ and $\frac{\partial}{\partial Y^2}{}_0$, from the equality

$$\Big\langle D(dY^1 \otimes d^{m_1}x \wedge d^{m_2}X)_\Theta, \\ \frac{d}{dt}{}_0 (tj^{r,s,q}_{(0,0)}(x^i, X^I, 0, \dots, 0, 1, 0, \dots, 0)) \otimes \frac{\partial}{\partial Y^2}{}_0 \Big\rangle = 0$$

(see (1)) we deduce that

$$\Big\langle D(Y^2_{((0),(0))} dY^1 \otimes d^{m_1}x \wedge d^{m_2}X)_\Theta, \\ \frac{d}{dt}{}_0 (tj^{r,s,q}_{(0,0)}(x^i, X^I, 0, \dots, 0, 1, 0, \dots, 0)) \otimes \frac{\partial}{\partial Y^2}{}_0 \Big\rangle \\ = -\Big\langle D(Y^1_{((0),(0))} dY^2 \otimes d^{m_1}x \wedge d^{m_2}X)_\Theta, \\ \frac{d}{dt}{}_0 (tj^{r,s,q}_{(0,0)}(x^i, X^I, 0, \dots, 0, 1, 0, \dots, 0)) \otimes \frac{\partial}{\partial Y^2}{}_0 \Big\rangle.$$

Thus D is uniquely determined by

$$\Big\langle D(Y^2_{((0),(0))} dY^1 \otimes d^{m_1}x \wedge d^{m_2}X)_\Theta, \\ \frac{d}{dt}{}_0 (tj^{r,s,q}_{(0,0)}(x^i, X^I, 0, \dots, 0, 1, 0, \dots, 0)) \otimes \frac{\partial}{\partial Y^2}{}_0 \Big\rangle \in \bigwedge{}^m T^*_0\mathbb{R}^m = \mathbb{R}.$$

So the vector space of all D in question is of dimension less than or equal to 1. Hence $D = c\mathcal{H}$ for some $c \in \mathbb{R}$. ■

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