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Normal martingales and polynomial families

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Abstract. Wiener and compensated Poisson processes, as normal martingales, are associated to classical sequences of polynomials, namely Hermite polynomials for the first one and Charlier polynomials for the second. The problem studied in this paper is to find if there exist other normal martingales which are associated to classical sequences of polynomials. Privault, Solé and Vives [5] solved this problem via the quantum Kabanov formula under some assumptions on the normal martingales considered. We solve the problem without these assumptions and we give a complete study of this subject in Section 2. In Section 3 we introduce the notion of algebraic process and we prove that Azéma martingales are infinitely algebraic.

1. Introduction. Let $\{X_t, t \geq 0\}$ be a semimartingale such that $X_{0^-} = 0$. By induction on n we define the semimartingales

$$P_{0^-}^{(n)} = 0, \qquad P_t^{(0)} = 1,$$
$$P_t^{(1)} = X_t, \quad P_t^{(n)} = \int_0^t P_{s^-}^{(n-1)}\, dX_s.$$

We also define the nth coefficient C_t^n by

$$C_t^1 = X_t, \quad C_t^2 = [X,X]_t,$$
$$C_t^n = \sum_{s\leq t} (\Delta X_s)^n \quad (\forall n > 2).$$

The Kailath–Segall formula (Meyer [4]) is a relationship of convolution type which links the semimartingales $P_t^{(n)}$ to the coefficients C_t^n:

$$P_t^{(n)} = \frac{1}{n}\Big[\sum_{k=1}^{n} (-1)^{k+1} P_t^{(n-k)} C_t^k\Big]. \tag{1}$$

If X is continuous all the coefficients $C_t^n = \sum_{s\leq t}(\Delta X_s)^n$ vanish for every

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$n > 2$ and (1) can be written simply as

$$P_t^n = \frac{1}{n}[X_t P_t^{(n-1)} - a_t P_t^{(n-2)}], \tag{2}$$

where a_t is the angle bracket of X. This form resembles the Hermite polynomials whose definition is:

$$H_0(x) = 1, \quad H_1(x) = x, \quad H_2(x) = \frac{1}{2}(x^2 - 1),$$
$$H_n(x) = \frac{1}{n}[xH_{n-1}(x) - H_{n-2}(x)].$$

If X is continuous and in particular if X is a Wiener process then the Kailath–Segall formula can be written formally as

$$\frac{P^{(n)}}{a^{n/2}} = H_n\left(\frac{X}{a^{1/2}}\right). \tag{3}$$

If X is a compensated Poisson process (i.e. $X_t = \alpha N_t - t/\alpha$, where N is a standard Poisson process) we recognize the Charlier polynomials which are defined by

$$\mathcal{C}_0(x, y) = 1,$$
$$\mathcal{C}_n(x, y) = \frac{1}{n}\Big[x\mathcal{C}_{n-1} + (x + y)\Big(\sum_{k=2}^{n}(-1)^{k+1}\mathcal{C}_{n-k}\Big)\Big].$$

$P^{(n)}/\alpha^n$ is a Charlier sequence in $(X/\alpha, b)$, where $\alpha = \pm 1/\sqrt{\lambda}$, $b_t = \lambda t$, and λ is the intensity of N. So we can write

$$P_t^{(n)}/\alpha^n = \mathcal{C}_n(X_t/\alpha, b_t) \quad \text{for each } t \geq 0. \tag{4}$$

2. Normal martingales. We will say that X is a *normal martingale* if its angle bracket $\langle X, X\rangle$ is deterministic and $\langle X, X\rangle_t = t$. For such a martingale, the iterated integral $\int_{C_n} f(s_1, \ldots, s_n)\, dX_1 \cdots dX_n$ can be defined for each f in $L^2(C_n)$, where $C_n = \{(s_1, \ldots, s_n) \in \mathbb{R}_+^n : 0 \leq s_1 < \cdots < s_n\}$ (Meyer [4]). The nth *chaos subspace* of X is the set $H_n(X) = \{\int_{C_n} f(s_1, \ldots, s_n)\, dX_1 \cdots dX_n : f \in L^2(C_n)\}$. We denote by $\mathcal{F}_t^X$ the σ-algebra $\sigma\{X_s,\ 0 \leq s \leq t\}$ and by $\mathcal{F}_\infty^X$ the σ-algebra $\sigma\{X_t,\ 0 \leq t\}$, so the chaos subspaces of X are orthogonal subspaces of $L^2(\mathcal{F}_\infty^X)$. A random variable $F \in L^2(\mathcal{F}_\infty^X)$ has a *chaos decomposition* if it is in the Hilbertian sum $\overline{\bigoplus H_n(X)}$. Finally, recall three important facts which are often used in our study:

1. If X is a normal martingale then the process $\{X_t^2 - t,\ 0 \leq t\}$ is a martingale.

2. $L^2(\mathcal{F}_\infty^X)$ and $L^2(C_n)$ are isometric spaces, in particular

$$E\Big(\int_{C_n} f(s_1,\ldots,s_n)\,dX_1\cdots dX_n\Big)^2 = \int_{C_n} f^2(s_1,\ldots,s_n)\,ds_1\cdots ds_n.$$

3. If K_t is an integrable process with respect to X then

$$E\Big(\int K_s\,dX_s\Big)^2 = \int E(K_s)^2\,ds.$$

As we mentioned in the abstract, Privault, Solé and Vives [5] studied the question of whether there exist other normal martingales X whose iterated integrals $P^{(n)}$ can be expressed as polynomials in X according to the following definition:

Definition 1. We will say that a normal martingale X has an *associated family of polynomials* $(Q_n^{[y]}(x))_{n\in\mathbb{N}}$, where $Q_n^{[y]}$ is a polynomial of degree n in x for each $y\in\mathbb{R}$, if

$$P_t^{(n)} = Q_n^{[t]}(X_t). \tag{5}$$

They considered normal martingales X satisfying the following assumptions:

1) X is in $L^6(\Omega)$.
2) X is a solution of a structure equation [3] (i.e. $d[X,X]_t = dt + \phi_t dX_t$ and $X_0 = x_0$).
3) ϕ has a chaotic decomposition.

They proved, by applying the quantum Kabanov formula [5], that only Wiener and compensated Poisson processes can satisfy Definition 1.

We study this problem without the above hypotheses. We show that the second and third hypotheses are automatically satisfied since we just assume that $P_t^{(2)}$ is a polynomial of degree 2 in X_t. In this way, we prove that ϕ is exactly in $\bigoplus_{k=0}^2 H_k(X)$. The study of $P_t^{(3)}$ permits us to conclude that X is a Wiener or a compensated Poisson process. This is the subject of the following theorem:

Theorem 1. *Let $\{X_t, t\ge 0\}$ be a normal martingale starting at x_0. Then*

(i) *X is associated to a sequence of polynomials if and only if for each integer k the coefficient C_t^k can be written as $C_t^k = P_k^{[t]}(X_t)$, where $P_k^{[t]}(x)$ is a polynomial in x of degree at least k for all $t\ge 0$.*
(ii) *If $C_t^2 = [X,X]_t$ is a polynomial in X_t of degree at least two, then there exist two real numbers β and γ such that*

$$d[X,X]_t = dt + (\beta X_{t-} + \gamma)dX_t. \tag{6}$$

(iii) *If X satisfies* (6) *with $\beta \neq -2$, then C_t^2 is a polynomial of degree two in X_t.*

(iv) *If $\beta = -2$, then $P_t^{(2)}$ is not a polynomial in X_t.*

(v) *$P_t^{(2)}$ and $P_t^{(3)}$ are polynomials in X_t, of degrees respectively at least two and three*, *if and only if X is a Wiener or a compensated Poisson process.*

Proof. (i) A slight transformation of the Kailath–Segall formula (1) allows us to link the nth coefficient C_t^n with the $n-1$ others:

$$C_t^n = (-1)^{n+1}\Big[nP_t^{(n)} + \sum_{k=1}^{n-1}(-1)^k P_t^{(n-k)} C_t^k\Big]. \tag{7}$$

By induction on n we can easily see that the coefficients C_t^n are polynomials in X_t if and only if $P_t^{(n)}$ are.

(ii) We will first establish that if $C_t^2 = [X,X]_t$ is a polynomial in X_t of degree at least two, then X satisfies a structure equation

$$d[X,X]_t = dt + \phi_t dX_t.$$

Assume that $[X,X]_t$ is a polynomial in X_t of degree at least two. Then there exist deterministic functions $a(t)$, $b(t)$ and $c(t)$ such that

$$X_t^2 - [X,X]_t = a(t)X_t^2 + b(t)X_t + c(t).$$

For simplicity we assume here that $x_0 = 0$. We have

$$X_t^2 - [X,X]_t = 2\int_0^t X_{s^-}\, dX_s = 2\int_0^t\int_0^{s^-} dX_u\, dX_s \in H_2(X).$$

Now $a(t) \neq 0$, since otherwise the above formulas imply that $X_t^2 - [X,X]_t \in [H_0(X) \oplus H_1(X)] \cap H_2(X) = \{0\}$ and then $\int_0^t X_{s^-}\, dX_s = 0$, therefore $\int_0^t E(X_{s^-}^2)\, ds = 0$, which is impossible since X is a normal martingale and $E(X_{s^-}^2) = s$.

Thus $X_t^2 - [X,X]_t = a(t)X_t^2 + b(t)X_t + c(t) \in H_0(X) \oplus H_1(X) \oplus H_2(X)$, whence $X_t^2 \in H_0(X) \oplus H_1(X) \oplus H_2(X)$, so its chaos decomposition can be written as

$$X_t^2 = t + \int_0^t f(s_1)\, dX_{s_1} + \int_0^t\int_0^{s_2} g(s_1,s_2)\, dX_{s_1}\, dX_{s_2} = 2\int_0^t X_{s^-}\, dX_s + [X,X]_t,$$

where $f \in L^2(\mathbb{R}^+)$ and $g \in L^2(C_2)$, and f and g are independent of t. In fact by a formula due to C. Dellacherie, B. Maisonneuve and P. A. Meyer [2] the nth chaotic coefficient of a r.v. F is

$$f_n(s_1,\ldots,s_n) = L^2\text{-}\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon^n} E[F(X_{s_1+\varepsilon} - X_{s_1}) \cdots (X_{s_n+\varepsilon} - X_{s_n})]$$

Then with $F = X_t^2$ we have

$$f(s_1) = L^2\text{-}\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon} E[X_t^2(X_{s_1+\varepsilon} - X_{s_1})]$$
$$= L^2\text{-}\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon} E[(X_t^2 - t + t)(X_{s_1+\varepsilon} - X_{s_1})]$$
$$= L^2\text{-}\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon} E[(X_{s_1+\varepsilon}^2 - s_1 - \varepsilon)(X_{s_1+\varepsilon} - X_{s_1})]$$

for $s_1 < t$, and for a sufficiently small ε we have $s_1 < s_1 + \varepsilon \leq t$. Then $X_{s_1+\varepsilon} - X_{s_1}$ is $\mathcal{F}^X_{s_1+\varepsilon}$-measurable. Since X is a normal martingale we obtain the last equality. The same argument leads to

$$g(s_1, s_2) = L^2\text{-}\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon^2} E[X_t^2(X_{s_1+\varepsilon} - X_{s_1})(X_{s_2+\varepsilon} - X_{s_2})]$$
$$= L^2\text{-}\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon^2} E[(X_{s_2+\varepsilon}^2 - s_2 - \varepsilon)(X_{s_1+\varepsilon} - X_{s_1})(X_{s_2+\varepsilon} - X_{s_2})],$$

whence f and g are independent of t and then

$$d[X,X]_t = dt + f(t)dX_t + \left(\int_0^t [g(t,s) - 2]\, dX_s\right) dX_t,$$

which can be written as

$$d[X,X]_t = dt + \phi_t dX_t \quad \text{with} \quad \phi_t = f(t) + \int_0^t [g(t,s) - 2]\, dX_s.$$

The case $x_0 \neq 0$ is similar, since if $Y_t = X_t - x_0$ then $[X,X]_t$ is a polynomial in X_t if and only if $[Y,Y]_t$ is a polynomial in Y_t. Now ϕ_t is of the form $\beta X_{t^-} + \gamma$. Indeed, the equality of expectations in

$$X_t^2 - [X,X]_t = a(t)X_t^2 + b(t)X_t + c(t)$$

gives

$$a(t)(x_0^2 + t) + b(t)x_0 + c(t) = 0,$$

whence

$$X_t^2 - [X,X]_t = a(t)(X_t^2 - x_0^2 - t) + b(t)(X_t - x_0).$$

On the other hand, the structure equation and the Itô formula

$$[X,X]_t = x_0^2 + t + \int_0^t \phi_s\, dX_s, \qquad X_t^2 - [X,X]_t = 2\int_0^t X_{s^-}\, dX_s$$

permit us to write

$$\int_0^t [2X_{s^-} - a(t)(\phi_s + 2X_{s^-}) - b(t)]\, dX_s = 0.$$

The expectation of its square also vanishes:

$$\int_0^t E[2X_{s^-} - a(t)(\phi_s + 2X_{s^-}) - b(t)]^2\, ds = 0$$

and thus ds-a.e. on $[0,t]$,

$$2X_{s^-} = a(t)(\phi_s + 2X_{s^-}) + b(t) \quad \text{a.s.}$$

Since $a(t)$ does not vanish we obtain

$$\phi_s = \frac{2(1-a(t))}{a(t)}\, X_{s^-} - \frac{b(t)}{a(t)} \quad \text{a.s. and } ds\text{-a.e. on } [0,t].$$

Define now on $\mathbb{R}_+^*$ two functions

$$\beta_t = \frac{2(1-a(t))}{a(t)}, \qquad \gamma_t = -\frac{b(t)}{a(t)}.$$

Then β_t and γ_t are constant because if $t_1 < t_2$ then for almost every s in $[0,t_1]$,

$$\beta_{t_1} X_{s^-} + \gamma_{t_1} = \beta_{t_2} X_{s^-} + \gamma_{t_2} \quad \text{a.s.,}$$

whence $\beta_t = \beta = c^{te}$ and $\gamma_t = \gamma = c^{te}$.

(iii) Conversely, if X satisfies

$$d[X,X]_t = dt + (\beta X_{t-} + \gamma) dX_t \quad \text{a.s.}$$

with $\beta \neq -2$, then

$$(8) \qquad \begin{aligned}[X,X]_t &= x_0^2 + t + \beta \int_0^t X_{t^-}\, dX_t + \gamma X_t \\ &= x_0^2 + t + \frac{\beta}{2}\,(X_t^2 - [X,X]_t) + \gamma X_t\end{aligned}$$

and

$$(9) \qquad \frac{\beta+2}{2}\,[X,X]_t = \frac{\beta}{2}\, X_t^2 + \gamma X_t + x_0^2 + t,$$

so $[X,X]_t$ is a polynomial of degree two in X_t.

(iv) Assume now that $\beta = -2$. Then

$$[X,X]_t = x_0^2 + t + \int_0^t (-2X_{s-} + \gamma)\, dX_s, \qquad X_t^2 = 2\int_0^t X_{s^-}\, dX_s + [X,X]_t.$$

Therefore

$$X_t^2 = x_0^2 + t + \gamma(X_t - x_0) = \lambda_2(t,x_0) + \mu_2(t,x_0) X_t,$$

where $\lambda_2(t,x_0) = x_0^2 + t - \gamma x_0$ and $\mu_2(t,x_0) = \gamma$. A simple induction on n shows us that there exist two deterministic functions $\lambda_n(t,x_0)$ and $\mu_n(t,x_0)$ such that

$$X_t^n = \lambda_n(t,x_0) + \mu_n(t,x_0) X_t.$$

Therefore if $P_t^{(2)}$ is a polynomial in X_t, its degree cannot exceed 1 and $P_t^{(2)}$ can be written as $P_t^{(2)} = \lambda(t, x_0) + \mu(t, x_0)X_t$. Then it belongs to $H_0(X) \oplus H_1(X)$. On the other hand,

$$P_t^{(2)} = 2\int_0^t X_{s^-}\, dX_s = 2\int_0^t \int_0^{s^-} dX_u\, dX_s,$$

so $P_t^{(2)}$ is in $H_2(X) - \{0\}$, which is impossible since $[H_0(X) \oplus H_1(X)] \cap (H_2(X) - \{0\}) = \emptyset$. Therefore $P_t^{(2)}$ is not a polynomial in X_t.

(v) Assume now (6) with $\beta \in \mathbb{R}^* - \{-2\}$. From (8) and (9) we obtain

$$P_t^{(2)} = a_0(t) + a_1(t)X_t + a_2(t)X_t^2,$$

where

$$a_0(t) = -\frac{x_0^2 + t - \gamma x_0}{\beta + 2}, \qquad a_1(t) = -\frac{\gamma}{\beta + 2}, \qquad a_2(t) = \frac{1}{\beta + 2}.$$

If $P_t^{(3)}$ is a polynomial in X_t of degree at least three, then

$$P_t^{(3)} = b_0(t) + b_1(t)X_t + b_2(t)X_t^2 + b_3(t)X_t^3.$$

Since $P_t^{(3)} = \int_0^t P_{s^-}^{(2)} dX_s$, we will first write $P_t^{(3)}$ as a stochastic integral $\int_0^t K_s\, dX_s$ with respect to X, where K_s is a polynomial in X_{s^-} of degree two, and then identify K_s and $P_{s^-}^{(2)}$. Indeed, in the expression $P_t^{(3)} = b_0(t) + b_1(t)X_t + b_2(t)X_t^2 + b_3(t)X_t^3$, we have $X_t = x_0 + \int_0^t dX_s$. The structure equation (6) gives

$$X_t^2 = x_0^2 + t + \int_0^t [(\beta + 2)X_{s-} + \gamma]\, dX_s.$$

The change of variable formula due to Emery [3] permits us to write

$$\begin{aligned} X_t^3 &= x_0^3 + \lambda_\beta \int_0^t X_{s^-}^2 dX_s + \mu_\beta \int_0^t X_{s^-}\, ds \\ &= x_0^3 + \mu_\beta t x_0 + \int_0^t [\lambda_\beta X_{s^-}^2 + \mu_\beta(t - s)]\, dX_s \end{aligned}$$

where

$$\lambda_\beta = \frac{(\beta + 1)^3 - 1}{\beta}, \qquad \mu_\beta = \frac{(\beta + 1)^3 - 1 - 3\beta}{\beta^2}.$$

From the above equalities we deduce the expression of K_s. Its identification with $P_{s^-}^{(2)}$ leads us in particular to

$$\lambda_\beta = \mu_\beta,$$

but this equation has no solution in $\mathbb{R}^* - \{-2\}$.

In the case $\beta = -2$, we showed that X cannot be associated to polynomials. We can also see that if $\gamma = 0$, this martingale satisfies $X_t^2 = x_0^2 + t$ and is called *parabolic*.

In the case $\beta = 0$ the structure equation becomes

$$d[X,X]_t = dt + \gamma dX_t.$$

If $\gamma = 0$ this is a Wiener process and if $\gamma \neq 0$ it is a compensated Poisson process.

We say that a normal martingale is an *Azéma martingale* [3] if its structure equation is of the form $d[X,X]_t = dt + \beta X_{t-} dX_t$. From the above study we can obtain

COROLLARY. *A normal martingale is a non-parabolic Azéma martingale if and only if $[X,X]_t$ is a polynomial of degree two in X_t.*

3. Algebraic processes. We saw that Azéma martingales are not associated to polynomials according to Definition 1. The aim of this section is to define a context where they are. In this context we give explicitly a new sequence of polynomials which is linked to them.

DEFINITION 2. Let $\mathcal{C}$ be a class of processes and $X \in \mathcal{C}$. We say that X is *algebraic* on $\mathcal{C}$ if there exists a polynomial $Q(t,x) = \sum_{k=0}^n a_k(t) x^k$ in x of degree higher than two such that the process $\{Q(t,X_t),\, t \geq 0\}$ is still in $\mathcal{C}$. The deterministic functions a_k are in $C^\infty(\mathbb{R}^+)$.

The *algebraic degree* of X is the integer $d_{\mathrm{al}}(X) = \inf\{\deg_x Q : Q(\cdot, X_\cdot) \in \mathcal{C}\}$. If the set $\{\deg_x Q : Q(\cdot, X_\cdot) \in \mathcal{C}\}$ is infinite we say that X is *infinitely algebraic* on $\mathcal{C}$; if $\operatorname{card}\{\deg_x Q : (\cdot, X_\cdot) \in \mathcal{C}\} < 2$ we say that X is *transcendent* on $\mathcal{C}$.

For example, Wiener processes are algebraic, as are all normal martingales; for those process we have $d_{\mathrm{al}}(X) = 2$. Azéma martingales are algebraic; in the next theorem we show that they are infinitely algebraic.

For each integer $k \geq 2$ define

$$\lambda_k = \frac{(\beta+1)^k - 1}{\beta}, \qquad \mu_k = \frac{(\beta+1)^k - 1 - k\beta}{\beta^2},$$

$$r_0 = r_1 = 1, \qquad r_k = \begin{cases} \dfrac{1}{\mu_2\mu_4\cdots\mu_k} & \text{if } k \text{ is even,} \\[2ex] \dfrac{1}{\mu_3\mu_5\cdots\mu_k} & \text{if } k \text{ is odd.} \end{cases}$$

Finally, we define the sequence of polynomials S_n^β by

$$S_n^\beta(t,x) = t^{n/2} R_n^\beta\left(\frac{x}{\sqrt{t}}\right), \quad \text{where} \quad R_n^\beta(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k r_{n-2k}}{k!}\, x^{n-2k}.$$

THEOREM 2. *Let $\{X_t, t \geq 0\}$ be an Azéma martingale satisfying the structure equation*

$$d[X,X]_t = dt + \beta X_{t-} dX_t, \qquad X_0 = 0.$$

(i) *There exists no martingale of the form $Q(X_t) + h(t)$, where Q is a polynomial of degree higher than three and h is a deterministic function on $\mathbb{R}^+$. If $\deg Q = 1$ or 2 the only martingales of this form are $aX_t + b$ or $a(X_t^2 - t) + bX_t + c$.*

(ii) *Azéma martingales are infinitely algebraic, more precisely the process $\{S_n^\beta(t, X_t),\ t \geq 0\}$ is a martingale for each even integer n.*

Proof. (i) and (ii) are immediately derived from a change of variable formula due to Yor [7]. Indeed, for a function $\phi \in \mathcal{C}^{1,2}(\mathbb{R}^+ \times \mathbb{R})$, the process $\phi(t, X_t)$ is a martingale if and only if

$$L^\beta \phi + \frac{\partial \phi}{\partial t} = 0, \tag{10}$$

where the operator $L^\beta \phi$ is defined by

$$L^\beta \phi(t,x) = \frac{\phi(t,(1+\beta)x) - \phi(t,x) - \beta x \phi'_x(t,x)}{\beta^2 x^2}$$

and $L^\beta \phi(t,0)$ is the limit of $L^\beta \phi(t,x)$ at zero. Then the application of the condition (10) with $\phi(t,x) = \sum_{j=1}^n a_j(t) x^j$ leads us to a simple differential system satisfied by the n functions a_j on $\mathbb{R}^+$, $j = 1, \ldots, n$. For $n = 2p$, if we choose $a_{2p}(t) = 1$ and $a_{2p-1}(t) = 0$, then one solution of this system is given by

$$a_{2j}(t) = (-1)^{p-j} \Bigg(\prod_{l=0}^{p-j-1} \frac{1}{\mu_{2(p-l)}} \Bigg) \frac{t^{p-j}}{(p-j)!}, \qquad j = 1, \ldots, p-1,$$

$$a_{2j-1}(t) = 0,$$

and we obtain a martingale of the form $M_t = \phi(t, X_t) = \sum_{j=1}^p a_{2j}(t) X_t^{2j}$; we see clearly that $M_t = S_{2p}^\beta(t, X_t)$.

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