Translation equation on monoids

by Andrzej Mach (Kielce) and Zenon Moszner (Kraków)

Abstract. We give large classes of solutions of the translation equation on a monoid satisfying the identity condition.

Let $X$ be a nonempty set and let $(G, \cdot)$ be a groupoid. By $F : X \times G \to X$ we denote an arbitrary solution of the translation equation:

\[ F(F(\alpha, k), l) = F(\alpha, k \cdot l), \quad \alpha \in X; \; k, l \in G. \]

This equation appears in several mathematical domains: abstract geometric and algebraic objects, abstract automata, groups of transformations, iterations, representations of groups, dynamical systems and others (see [5]) and therefore has at present a general theory (see [7]).

János Aczél of the University of Waterloo, in a letter to the second author, posed the following problem: what can we say about solutions $F : X \times \mathbb{N} \to X$ of the translation equation (1) for which $F(\alpha, 1) = \alpha$ (the identity condition), where $X$ is an interval and $(\mathbb{N}, \cdot)$ is the monoid of natural numbers?

We give large classes of solutions of the translation equation on monoids $G$ satisfying the identity condition $F(\alpha, 1) = \alpha$, where 1 denotes the unit element of $(G, \cdot)$.

The problem of finding the general solution of the translation equation for $(G, \cdot) = (\mathbb{N}, \cdot)$ is still open.

Remark 1. If $F$ is a solution of (1), then $G$ acts on $X$ by means of the mapping $k \mapsto F(\cdot, k): X \to X$.

Definition 1. A family \{${E_j}_{j \in J}$\} of nonempty pairwise disjoint subsets of $G$ is called an invariant decomposition of the groupoid $(G, \cdot)$ if $G = \bigcup_{j \in J} E_j$ and

\[ \forall j \in J \; \forall k \in G \; \exists l \in J : \; (E_j \cdot k \subseteq E_l). \]

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Theorem 1. Let $X$ be a nonempty set. Let $X = \bigcup_{s \in S} X_s$ be a decomposition of $X$ into a disjoint union of nonempty sets such that for every $s \in S$ there exists an invariant decomposition $\{E_{js}\}_{j \in J_s}$ of the monoid $(G, \cdot)$ with $\text{card } X_s = \text{card } J_s$. Let $\bar{g}_s : \{E_{js}\}_{j \in J_s} \rightarrow X_s$ be an arbitrary bijection and set $g_s(k) := \bar{g}_s(E_{js})$ for $k \in E_{js}$. Then the function $F : X \times G \rightarrow X$ defined by

$$F(\alpha, k) = \{g_s(g_s^{-1}(\{\alpha\}) \cdot k)\}, \quad \alpha \in X_s, \ k \in G,$$

is a solution of the translation equation (1) for which $F(\alpha, 1) = \alpha$.

The symbol $\{\cdot\}A$ in (3) denotes the element of a set $A$ when card $A = 1$. The proof of Theorem 1 is a simple verification, so it can be omitted.

Remark 2. The decomposition $\{E_j\}_{j \in J}$ of $G$ is invariant if and only if the relation $a \equiv b \iff \exists j \in J : a, b \in E_j$

is right-compatible with the groupoid operation, i.e.

$$\forall a, b, c \in G : \ [a \equiv b \Rightarrow a \cdot c \equiv b \cdot c].$$

If the groupoid $G$ is Abelian, then every equivalence relation $\equiv$ right-compatible with the groupoid operation is a congruence relation, that is,

$$\forall a, b, c, d \in G : \ [(a \equiv b \land c \equiv d) \Rightarrow a \cdot c \equiv b \cdot d].$$

An equivalence relation $\equiv$ on a groupoid $G$ is a congruence (respectively: is right-compatible with the groupoid operation) if and only if there exists a function $h : G \rightarrow G$ such that

$$a \equiv b \iff h(a) = h(b) \ 	ext{and} \ h(a \cdot b) = h[h(a) \cdot h(b)]$$

(respectively: $h(a \cdot b) = h[h(a) \cdot b]$) for $a, b \in G$.

In the case of a congruence relation, the function $h$ is a homomorphism of $G$ onto the groupoid $h(G)$ with the operation $c \# d = h(c \cdot d)$. This means that the equivalence relation $\equiv$ is a congruence in the groupoid $(G, \cdot)$ if and only if there exists a homomorphism $H$ of $G$ into a groupoid $T$ such that $a \equiv b \iff H(a) = H(b)$ (see [2, pp. 35–37]). This yields a method of constructing invariant decompositions (see Remark 4, due to Andrzej Schinzel).

Remark 3. If the groupoid $G$ is a group, then its invariant decompositions are sets of right cosets of some subgroup (see [1, pp. 34–35]). Moreover, if the group $G$ is Abelian then invariant decompositions are determined by quotient groups.

Remark 4 (by A. Schinzel). By Remark 2 all congruences $\equiv$ in the monoid $(\mathbb{N}, \cdot)$ are obtained by the following
Construction $C_1$

1° Take an arbitrary Abelian semigroup $(T, +)$ with neutral element $0$.

2° Take an arbitrary function $\phi : P \to T$, where $P$ is the set of all prime numbers. Define a homomorphism $H : (\mathbb{N}, \cdot) \to (T, +)$ by setting, for $a = \prod_{p \in P} p^{\alpha(p)} \in \mathbb{N}$, where $\alpha(p)$ are nonnegative integers,

$$H(a) := \sum_{p \in P} \alpha(p)\phi(p).$$

3° For $a, b \in \mathbb{N}$ define: $a \equiv b \iff H(a) = H(b)$, that is,

$$\prod_{p \in P} p^{\alpha(p)} \equiv \prod_{p \in P} p^{\beta(p)} \iff \sum_{p \in P} \alpha(p)\phi(p) = \sum_{p \in P} \beta(p)\phi(p),$$

where $\alpha(p)$, $\beta(p)$ are nonnegative integers.

To describe all congruence relations means to describe all semigroups and, in consequence, to solve the association equation

$$F(F(a, b), c) = F(a, F(b, c)), \quad \text{where} \quad F : G \times G \to G.$$

Example 1 (by A. Schinzel). Let $T := 2^\mathbb{N}$ be the monoid with the union operation. If we define $\phi(p) := \{p\}$ we get the congruence relation

$$a \equiv b \iff a \text{ and } b \text{ have the same prime factors};$$

this means that components of the invariant decomposition of $\mathbb{N}$ are sets of natural numbers having the same prime factors.

Remark 5. To obtain the same invariant decomposition as in Example 1, it is possible to take $\mathbb{N}$ with a suitable operation in place of $2^\mathbb{N}$. The function $h : \mathbb{N} \to \mathbb{N}$ such that $h(a)$ equals the product of the prime factors of $a$ for $a > 1$ and $h(1) = 1$ satisfies (4), hence $h$ is a homomorphism of $(\mathbb{N}, \cdot)$ into $\mathbb{N}$ with the operation $a \# b = h(a \cdot b)$.

Remark 6. When the monoid $(G, \cdot)$ is the group then Theorem 1 yields all solutions of the translation equation (1) satisfying $F(\alpha, 1) = \alpha$. In this case the invariant decompositions consist of right cosets of some subgroup $G_s$ of $G$ (see Remark 3) and $\overline{g}_s$ is equal to $g_s$ and $g_s : G / G_s \to X_s$. Also the general solution of the translation equation (1) satisfying $F(\alpha, 1) = \alpha$ has been given in [4] by the following

Construction $C_2$

1° Let $X = \bigcup_{s \in S} X_s$ be a disjoint union of nonempty sets (fibres) $X_s$ such that for every $s \in S$ there exists a subgroup $G_s \leq G$ and a bijection $g_s : G / G_s \to X_s$, where $G / G_s$ is the set of right cosets of $G_s$ in $G$.

2° Then $F(\alpha, k) = g_s(g_s^{-1}(\alpha) \cdot k)$, $\alpha \in X_s$, $k \in G$. 
Remark 7. Construction $C_3$ mentioned below, quoted from [3], includes the general form of invariant decompositions for the subsemigroup $G^+$ of positive elements of the group $(G, +, \leq)$, linearly ordered and Abelian.

Before the presentation of the construction of the decompositions, we need two definitions.

Definition 2. A subset $A$ of $G$ is called bounded if $\exists z \in G^+ \ \forall a \in A : (a < z \text{ and } a > -z)$, and unbounded if it is not bounded.

Definition 3. Let $A, B$ be subsets of $G$ and $A \subseteq B \subseteq G$. We say that:
(a) $A$ is an initial interval of $B$ if
$$\forall a_0 \in A : \{a \in B : a \leq a_0\} \subseteq A,$$
(b) $A$ is a final interval of $B$ if
$$\forall a_0 \in A : \{a \in B : a_0 \leq a\} \subseteq A.$$

All $G^+$-invariant decompositions of the semigroup $G^+$ of positive elements of a linearly ordered, Abelian group $G$ are obtained by

Construction $C_3$

1° Take a family $\{G_s\}_{s \in S}$ of distinct, bounded subgroups of $G$ forming a chain, i.e. $G_s \subseteq G_t$ or $G_t \subseteq G_s$ for $s, t \in S$, and an unbounded subgroup $G^*$ such that $G^* \supseteq \bigcup_{s \in S} G_s$.

2° Let $\Phi$ be a function from the family $\{G_s\}_{s \in S}$ onto a family of initial intervals of $G^+$ such that
(a) $\Phi(G_s)$ is a union of intersections with $G^+$ of cosets of $C(G_s)$ in $G$, where $C(G_s)$ denotes the smallest convex subgroup containing $G_s$ (the convexity of $C(G_s)$ means that together with every positive element a the subgroup $C(G_s)$ contains all elements $x \in G^+$ with $x \leq a$),
(b) if $G_s \subseteq G_t$, then $\Phi(G_s) \subseteq \Phi(G_t)$.

3° Every nonempty set
$$W \cap \left[ \Phi(G_s) \setminus \bigcup_{G_t \subsetneq G_s} \Phi(G_t) \right], \quad W \in G_s, \ s \in S,$$
is a component of the decomposition.

4° The sets
$$V \cap \left[ G^+ \setminus \bigcup_{s \in S} \Phi(G_s) \right], \quad V \in G/G^*,$$
are the remaining components.

If we assume additionally that $(G, +, \leq)$ is an Archimedean group, then Construction $C_3$ is reduced to the following result from the paper [6].
Every $G^+$-invariant decomposition of the semigroup $G^+$ of positive elements of a linearly ordered, Archimedean group is of the following form:

(a) there exists a right-closed or right-open interval $[0, x_0]$ such that every element belonging to $[0, x_0]$ is a component of the decomposition (this interval may be empty),

(b) the remaining components are the intersections with $G^+ \setminus [0, x_0]$ of cosets of some subgroup $G^*$ in $G$.

Using Construction $C_3$ and Theorem 1 we can obtain examples of solutions for the semigroup $G^+$ of positive elements of a linearly ordered, Abelian group $(G, +, \leq)$.

**Example 2.** Let $\mathbb{Z}$ denote the set of integers and $G := \{ax + b : a, b \in \mathbb{Z}\}$ be the group of linear polynomials with ordinary addition and with linear order defined as follows:

$$(ax + b \leq cx + d) \Leftrightarrow (a < c) \text{ or } (a = c \text{ and } b \leq d).$$

The semigroup of positive elements is

$$G^+ := \{ax + b : a > 0, b \in \mathbb{Z}\} \cup \mathbb{Z}^+,$$

where $\mathbb{Z}^+ := \{a \in \mathbb{Z} : a \geq 0\}$. According to Construction $C_3$, take the chain of bounded subgroups $\{0\} \subset \mathbb{Z}$ and the unbounded subgroup $G^* := G$. Define $\Phi(\{0\}) := \mathbb{Z}^+$ and $\Phi(\mathbb{Z}) := \mathbb{Z}^+ \cup (\mathbb{Z} + x)$, where $\mathbb{Z} + x \in G/\mathbb{Z}$. Every element of $\mathbb{Z}^+$ is a component of the decomposition. The sets $\mathbb{Z} + x$, $G^+ \setminus (\mathbb{Z}^+ \cup (\mathbb{Z} + x))$ are also components.

Let now $X := [0, \infty[$, $S := [0, 1[, X_s := \{s + j : j = 0, 1, 2, \ldots\}$ and $J_s := \mathbb{N}\cup\{0\}$ for $s \in S$. Moreover, $E_{0s} := \mathbb{Z} + x$, $E_{1s} := G^+ \setminus (\mathbb{Z}^+ \cup (\mathbb{Z} + x))$ and $E_{js} := \{j - 2\}$ for $j \in \{2, 3, 4, \ldots\}$ and $s \in S$. Assume that $\overline{g}_s(E_{js}) := s + j$ for $j \in \mathbb{N}\cup\{0\}$ and for $s \in S$. We get the following solution $F : X \times G^+ \to X$ of the translation equation:

$$F(\alpha, w) = \begin{cases} 
\alpha & \text{if } \alpha \in [0, 1] \text{ and } w \in \mathbb{Z}^+ \text{ or } \alpha \in [1, 2] \text{ and } w \in G^+, \\
\alpha + 1 & \text{if } \alpha \in [0, 1] \text{ and } w \in G^+ \setminus \mathbb{Z}^+, \\
\alpha - E(\alpha) & \text{if } \alpha \in X \setminus [0, 2] \text{ and } w \in \mathbb{Z} + x, \\
\alpha - E(\alpha) + 1 & \text{if } \alpha \in X \setminus [0, 2] \text{ and } w \in G^+ \setminus (\mathbb{Z}^+ \cup (\mathbb{Z} + x)), \\
\alpha + w & \text{if } \alpha \in X \setminus [0, 2] \text{ and } w \in \mathbb{Z}^+,
\end{cases}$$

where $E(\alpha)$ denotes the integer part of $\alpha$.

In what follows $(\mathbb{N}, \cdot)$ and $(\mathbb{Q}_+, \cdot)$ denote the monoid of natural numbers and the group of positive rational numbers respectively.

Using Theorem 1 we can obtain examples of solutions for $(G, \cdot) = (\mathbb{N}, \cdot)$.

**Example 3.** Let $X := ]1/4, 1]$ and take $S := ]1/2, 1]$, $X_s := ]s/2, s]$, $J_s := \{1, 2\}$ for $s \in S$. Moreover, $E_{1s} := \{1, 3, 5, \ldots\}$, $E_{2s} := \{2, 4, 6, \ldots\}$ for
s ∈ S. Define \( \overline{g}_s(E_{1s}) := s/2 \), \( \overline{g}_s(E_{2s}) := s \) for \( s ∈ S \). We get the following solution:

\[
F(\alpha, k) = \begin{cases} 
2\alpha & \text{for } \alpha ∈ [1/4, 1/2], k ∈ \{2, 4, 6, \ldots\}, \\
\alpha & \text{for } \alpha ∈ [1/2, 1], k ∈ \mathbb{N} \text{ or } \alpha ∈ [1/4, 1/2], k ∈ \{1, 3, 5, \ldots\}.
\end{cases}
\]

**Example 4.** Let \( X, S, \{X_s\}, J_s \) for \( s ∈ S \) be as in Example 3. We take \( E_{1s} := \{1\} \), \( E_{2s} := \mathbb{N} \setminus \{1\} \) for \( s ∈ S \). The functions \( \overline{g}_s \) are defined as in Example 2. We get the following solution:

\[
F(\alpha, k) = \begin{cases} 
2\alpha & \text{for } \alpha ∈ [1/4, 1/2], k ∈ \mathbb{N} \setminus \{1\}, \\
\alpha & \text{for } \alpha ∈ [1/2, 1], k ∈ \mathbb{N} \text{ or } \alpha ∈ [1/4, 1/2], k = 1.
\end{cases}
\]

**Example 5.** Let \( X := [0, \infty[ \), \( S := [0, 1[ \), \( X_s := \{s + j : j = 0, 1, 2, \ldots\} \), \( J_s := \mathbb{N} \cup \{0\} \) for \( s ∈ S \). Moreover, \( E_{0s} := \{2, 4, 6, \ldots\} \) and \( E_{js} := \{2j - 1\} \) for \( j ∈ \mathbb{N} \) and \( s ∈ S \). Define \( \overline{g}_s(E_{js}) := s + j \) for \( j ∈ \mathbb{N} \cup \{0\} \) and \( s ∈ S \). We get the following solution:

\[
F(\alpha, k) = \begin{cases} 
\alpha - E(\alpha) & \text{for } \alpha ∈ X \setminus [0, 1[ \text{ and } k ∈ \{2, 4, 6, \ldots\}, \\
\alpha + E(\alpha)(k - 1) - (k - 1)/2 & \text{for } \alpha ∈ X \setminus [0, 1[ \text{ and } k ∈ \{1, 3, 5, \ldots\}, \\
\alpha & \text{or } \alpha ∈ [0, 1[ \text{ and } k ∈ \mathbb{N},
\end{cases}
\]

where \( E(\alpha) \) denotes the integer part of \( \alpha \).

**Remark 8.** If we define \( \phi : P → T = 2^\mathbb{N} \) by \( \phi(p) := \emptyset \) for \( p ≠ 2 \) and \( \phi(2) := \{1\} \), where \( T = 2^\mathbb{N} \) denotes the monoid described in Example 1, then by Construction \( C_1(3^o) \) in Remark 4 we get the congruence equivalent to the invariant decomposition from Example 3, which means that \( E_1 := \{1, 3, 5, \ldots\}, E_2 := \{2, 4, 6, \ldots\} \).

Similarly, if we define \( \phi : P → T = 2^\mathbb{N} \) by \( \phi(p) := \mathbb{N} \) for all \( p ∈ P \), then by Construction \( C_1(3^o) \) we get the congruence equivalent to the invariant decomposition from Example 4, which means that \( E_1 := \{1\}, E_2 := \mathbb{N} \setminus \{1\} \).

To obtain the invariant decomposition from Example 5, it is sufficient to consider the semigroup \( (T, \cdot) := (2^{\mathbb{R}} \setminus \{0\}, \cdot) \), where the operation is defined by \( A \cdot B := \{a \cdot b : a ∈ A, b ∈ B\} \) for \( A, B ∈ 2^{\mathbb{R}} \setminus \{0\} \), and to define \( \phi : P → T = 2^{\mathbb{R}} \setminus \{0\} \) by \( \phi(p) := \{p\} \) for \( p ≠ 2 \) and \( \phi(2) := \mathbb{R} \setminus \{0\} \).

**Remark 9.** If the solution of equation (1) is trivial, that is, \( F(\alpha, k) := \alpha \) for every \((\alpha, k) ∈ X × \mathbb{N}\), where \( X \) denotes an arbitrary nonempty set, then the invariant decomposition of \( \mathbb{N} \) has exactly one element \( \{\mathbb{N}\} \), the set \( X \) is decomposed into singletons and \( \overline{g}_s(\mathbb{N}) := s \).

**Remark 10.** The function \( F(\alpha, k) := k \cdot \alpha \) for \((\alpha, k) ∈ X × \mathbb{N}\) and \( X := [0, \infty[ \) is a solution of the translation equation (1). This solution is not of the form (3) (see Remark 11).
Theorem 2. Let $X \subset \mathbb{R}$ be an arbitrary interval. Suppose that a solution $F : X \times \mathbb{N} \to X$ of the translation equation (1) satisfying $F(\alpha, 1) = \alpha$ for $\alpha \in X$ can be extended to a solution $\overline{F} : X \times \mathbb{Q}_+ \to X$ of this equation. Then there exists a family $\{X_s\}_{s \in S}$ of disjoint sets such that $\bigcup_{s \in S} X_s = X$ and for every $s \in S$ there exists a subgroup $\mathbb{Q}_s \leq \mathbb{Q}_+$ and a bijection $g_s : \mathbb{Q}_+/\mathbb{Q}_s \to X_s$ for which

$$F(\alpha, k) = g_s(g_s^{-1}(\alpha) \cdot k), \quad \alpha \in X_s, \ k \in \mathbb{N}. \quad (5)$$

Proof. This follows immediately from Construction $C_2$.

Theorem 3. Let $X \subset \mathbb{R}$ be an arbitrary interval. A function $F : X \times \mathbb{N} \to X$ is a solution of the translation equation (1) such that for every $\alpha \in X$ the function $F(\alpha, \cdot)$ is increasing and for every $k \in \mathbb{N}$ the function $F(\cdot, k)$ is increasing and surjective if and only if there exists a family $\{X_s\}_{s \in S}$ of disjoint sets such that $\bigcup_{s \in S} X_s = X$ and there exists a family of increasing bijections $g_s : \mathbb{Q}_+ \to X_s$, $s \in S$, such that

$$F(\alpha, k) = g_s(g_s^{-1}(\alpha) \cdot k), \quad \alpha \in X_s, \ k \in \mathbb{N}. \quad (6)$$

We present two proofs of this theorem. The first one is a corollary from Theorem 2 and the other proof is direct.

Proof I (of the “only if” part of Theorem 3, using Theorem 2). Note that the assumptions about $F : X \times \mathbb{N} \to X$ imply that $F(\alpha, 1) = \alpha$ and $F$ can be extended to a solution $\overline{F} : X \times \mathbb{Q}_+ \to X$ of (1). Indeed, since $F(F(\alpha, 1), 1) = F(\alpha, 1)$, by injectivity of $F(\cdot, 1)$ we get $F(\alpha, 1) = \alpha$. We can put

$$\overline{F}(\alpha, k/l) := \beta \quad \text{such that} \quad F(\alpha, k) = F(\beta, l), \quad (7)$$

for every $\alpha \in X$ and $k/l \in \mathbb{Q}_+$. The existence and uniqueness of $\beta$ result from the assumption that $F(\cdot, l)$ is surjective and injective, so $\overline{F}$ is correctly defined. One can verify easily that $\overline{F}$ is a solution of the translation equation. Indeed, let

$$\overline{F}\left(\overline{F}\left(\alpha, \frac{k}{l}\right), \frac{m}{n}\right) =: \gamma \quad \text{and} \quad \overline{F}\left(\alpha, \frac{k \cdot m}{l \cdot n}\right) =: \delta.$$

If we set $\overline{F}(\alpha, k/l) =: \beta$, then by definition (7), $F(\alpha, k) = F(\beta, l)$ and $F(\beta, m) = F(\gamma, n)$ and $F(\alpha, k \cdot m) = F(\delta, l \cdot n)$. Hence

$$F(\gamma, n \cdot l) = F(\beta, m \cdot l) = F(\alpha, k \cdot m) = F(\delta, n \cdot l),$$

so $\gamma = \delta$.

Therefore, by Theorem 2, we have the form (5) of the solution $F : X \times \mathbb{N} \to X$. Since the functions $F(\alpha, \cdot)$ are injective for every $\alpha \in X$, by Construction $C_2$ we get $\mathbb{Q}_s = \{1\} \leq \mathbb{Q}_+$ for every $s \in S$, which yields (6).
We will verify that the bijections \( \{g_s\}_{s \in S} \) are increasing. Let \( k/l < k_1/l_1 \) and 
\[
g_s(k/l) =: \alpha, \quad g_s(k_1/l_1) =: \beta.
\]
By (6) and by definition of \( F \) we get 
\[
F\left(\alpha, \frac{k_1l}{kl_1}\right) = g_s\left(\frac{k}{l}, \frac{k_1l}{kl_1}\right) = g_s\left(\frac{k_1}{l_1}\right) = \beta, \quad \text{whence} \quad F(\alpha, k_1l) = F(\beta, kl_1).
\]
Since \( kl_1 < lk_1 \), by assumptions we have 
\[
F(\alpha, k_1l) < F(\alpha, k_1l) = F(\beta, kl_1) \quad \text{and} \quad \alpha < \beta.
\]
Since the “if” part is evident, the first proof is complete.

Proof II (of the “only if” part of Theorem 3). We define the following relation in \( X \):
\[
\forall \alpha, \beta \in X : \quad \alpha \sim_F \beta \iff \exists k, l \in \mathbb{N} : F(\alpha, k) = F(\beta, l).
\]
It is to verify that it is an equivalence relation. Indeed, evidently it is symmetric and reflexive. Let now \( \alpha \sim_F \beta \) and \( \beta \sim_F \gamma \). Then 
\[
\exists k, l, k_1, l_1 \in \mathbb{N} : \quad F(\alpha, k) = F(\beta, l) \quad \text{and} \quad F(\beta, k_1) = F(\gamma, l_1).
\]
Hence 
\[
F(\alpha, k \cdot k_1) = F(\beta, l \cdot k_1) = F(\gamma, l \cdot l_1), \quad \text{so} \quad \alpha \sim_F \gamma.
\]
We denote by \( \{X_s\}_{s \in S} \) the set of equivalence classes. Fix \( s \in S \) and \( \alpha_0 \in X_s \). We define \( h_s : X_s \to \mathbb{Q}_+ \) by 
\[
h_s(\alpha) := k/l, \quad \text{where} \quad F(\alpha_0, k) = F(\alpha, l).
\]
The function \( h_s \) is correctly defined. Indeed, if 
\[
F(\alpha_0, k) = F(\alpha, l) \quad \text{and} \quad F(\alpha_0, k_1) = F(\alpha, l_1),
\]
then 
\[
F(\alpha_0, l \cdot k_1) = F(\alpha, l \cdot l_1) = F(\alpha_0, l_1 \cdot k).
\]
Since \( F(\alpha_0, \cdot) \) is injective, \( l \cdot k_1 = l_1 \cdot k \), whence \( k_1/l_1 = k/l \).

We will show that \( h_s : X_s \to \mathbb{Q}_+ \) is a bijection. If \( h_s(\alpha) = h_s(\beta) = k/l \) then \( F(\alpha, l) = F(\alpha_0, k) = F(\beta, l) \) and by injectivity of \( F(\cdot, l) \) we get \( \alpha = \beta \). To prove the surjectivity, take \( m/n \in \mathbb{Q}_+ \). Let \( F(\alpha_0, m) = \beta \). By the surjectivity of \( F(\cdot, n) \), we have \( F(\alpha_0, m) = \beta = F(\alpha, n) \) for some \( \alpha \), so \( h_s(\alpha) = m/n \).

Now, we will show that \( h_s \) is an increasing function. Let \( \alpha < \beta \) and 
\[
h_s(\alpha) = k/l, \quad h_s(\beta) = k_1/l_1.
\]
We have \( F(\alpha_0, k) = F(\alpha, l) \) and \( F(\alpha_0, k_1) = F(\beta, l_1) \). Since \( F(\cdot, ll_1) \) is increasing, we obtain 
\[
F(\alpha_0, kl_1) = F(\alpha, ll_1) < F(\beta, ll_1) = F(\alpha_0, k_1l),
\]
therefore \( kl_1 < k_1l \) and \( k/l < k_1/l_1 \).
Let now $\alpha \in X_s$, $k \in \mathbb{N}$. Let $h_s(\alpha) = K/L$ and $\beta := F(\alpha, k)$. Hence
\[h_s(\alpha) \cdot k = K \cdot k/L.\]
We will show
\[h_s(\beta) = K \cdot k/L.\]
Indeed, we have $F(\beta, L) = F(\alpha, kL)$, $F(\alpha_0, K) = F(\alpha, L)$ and
\[F(\alpha_0, Kk) = F(\alpha, Lk) = F(\beta, L),\]
therefore $h_s(\beta) = K \cdot k/L$, and so
\[F(\alpha, k) = \beta = h_s^{-1}(K \cdot k/L) = h_s^{-1}(h_s(\alpha) \cdot k).\]
Putting $g_s = h_s^{-1}$ we have the form (6), which was to be shown.

**Remark 11.** If $F : X \times \mathbb{N} \to X$ satisfies the assumptions of Theorem 3, then $F$ cannot be obtained by means of Theorem 1.

Indeed, otherwise let $g_s(1) =: \alpha_0$ for some $s \in S$. Then $X_s = \{F(\alpha_0, k) : k \in \mathbb{N}\} = g_s(\mathbb{N})$ and $\alpha_0 \in X_s$. Let $\bar{F} : X \times \mathbb{Q}_+ \to X$ be an extension of the solution $F$. Since $\bar{F}(\alpha_0, 1/2) < F(\alpha_0, k)$ for $k \in \mathbb{N}$, we have $\bar{F}(\alpha_0, 1/2) \notin X_s$. Let $\bar{F}(\alpha_0, 1/2) \in X_t$, $t \neq s$. Hence
\[F(\bar{F}(\alpha_0, 1/2), 2) = F(\alpha_0, 1) = \alpha_0,\]
so $\alpha_0 \in X_t$, which contradicts the relation $X_t \cap X_s = \emptyset$.

**Remark 12.** Let $X := [0, \infty[$ and define $F : X \times \mathbb{N} \to X$ by
\[F(\alpha, k) = \begin{cases} 
\alpha, & \alpha \in X, \ k = 1, \\
1, & \alpha \in [0, 1], \ k \in \mathbb{N} \setminus \{1\}, \\
k\alpha, & \alpha \in X \setminus [0, 1], \ k \in \mathbb{N} \setminus \{1\}.
\end{cases}\]
Then $F$ is a solution of (1) which cannot be extended to a solution $\bar{F} : X \times \mathbb{Q}_+ \to X$ and is not of the form (3).

Indeed, for every solution $\bar{F} : X \times \mathbb{Q}_+ \to X$ of (1) satisfying $\bar{F}(\alpha, 1) = \alpha$, all functions $\bar{F}(\cdot, k)$ ought to be bijections. But
\[F(1/2, 2) = 1 = F(3/4, 2),\]
therefore $F$ cannot be extended to a solution $\bar{F} : X \times \mathbb{Q}_+ \to X$.

Moreover, by Theorem 1, $\text{card } X_s = \text{card } J_s$ for $s \in S$. It is easy to see that for the solution $F$ one of the elements of the family $\{X_s\}_{s \in S}$ is the set $X_n = [0, 1]$ for some $n \in S$. This implies the following contradiction:
\[c = \text{card } [0, 1] = \text{card } J_n \leq \text{card } \mathbb{N} = \aleph_0.\]

**References**


Institute of Mathematics
Świętokrzyska Academy
Świętokrzyska 15
25-406 Kielce, Poland
E-mail: amach@pu.kielce.pl

Institute of Mathematics
Pedagogical Academy
Pochorzących 2
30-084 Kraków, Poland
E-mail: zmoszner@ap.krakow.pl

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