ANNALES
POLONICI MATHEMATICI
84.2 (2004)

Translation equation on monoids

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Abstract. We give large classes of solutions of the translation equation on a monoid satisfying the identity condition.

Let X be a nonempty set and let $(G,\cdot)$ be a groupoid. By $F : X\times G \to X$ we denote an arbitrary solution of the *translation equation*:

$$F(F(\alpha,k),l) = F(\alpha,k\cdot l), \quad \alpha\in X;\ k,l\in G. \tag{1}$$

This equation appears in several mathematical domains: abstract geometric and algebraic objects, abstract automata, groups of transformations, iterations, representations of groups, dynamical systems and others (see [5]) and therefore has at present a general theory (see [7]).

János Aczél of the University of Waterloo, in a letter to the second author, posed the following problem: what can we say about solutions $F : X\times\mathbb{N}\to X$ of the translation equation (1) for which $F(\alpha,1)=\alpha$ (the *identity condition*), where X is an interval and $(\mathbb{N},\cdot)$ is the monoid of natural numbers?

We give large classes of solutions of the translation equation on monoids G satisfying the identity condition $F(\alpha,1)=\alpha$, where 1 denotes the unit element of $(G,\cdot)$.

The problem of finding the general solution of the translation equation for $(G,\cdot)=(\mathbb{N},\cdot)$ is still open.

Remark 1. If F is a solution of (1), then G acts on X by means of the mapping $k\mapsto F(\cdot,k)$: $X\to X$.

Definition 1. A family $\{E_j\}_{j\in J}$ of nonempty pairwise disjoint subsets of G is called an *invariant decomposition* of the groupoid $(G,\cdot)$ if $G=\bigcup_{j\in J}E_j$ and

$$\forall j\in J\ \forall k\in G\ \exists l\in J: \quad (E_j\cdot k\subseteq E_l). \tag{2}$$

2000 *Mathematics Subject Classification*: Primary 39B82; Secondary 39B12, 39B52.
Key words and phrases: translation equation, grupoid, monoid, invariant decomposition.

Theorem 1. *Let X be a nonempty set. Let $X = \bigcup_{s\in S} X_s$ be a decomposition of X into a disjoint union of nonempty sets such that for every $s \in S$ there exists an invariant decomposition $\{E_{js}\}_{j\in J_s}$ of the monoid $(G,\cdot)$ with* $\operatorname{card} X_s = \operatorname{card} J_s$. *Let $\overline{g}_s$: $\{E_{js}\}_{j\in J_s} \to X_s$ be an arbitrary bijection and set $g_s(k) := \overline{g}_s(E_{js})$ for $k \in E_{js}$. Then the function $F : X \times G \to X$ defined by*

$$F(\alpha,k) = \}g_s(g_s^{-1}(\{\alpha\})\cdot k)\{,\qquad \alpha\in X_s,\ k\in G, \tag{3}$$

is a solution of the translation equation (1) *for which* $F(\alpha,1)=\alpha$.

The symbol $\}A\{$ in (3) denotes the element of a set A when $\operatorname{card} A = 1$.

The proof of Theorem 1 is a simple verification, so it can be omitted.

Remark 2. The decomposition $\{E_j\}_{j\in J}$ of G is invariant if and only if the relation

$$a\equiv b \ \Leftrightarrow\ \exists j\in J : a,b\in E_j$$

is *right-compatible* with the groupoid operation, i.e.

$$\forall a,b,c\in G:\quad [a\equiv b \Rightarrow a\cdot c\equiv b\cdot c].$$

If the groupoid G is Abelian, then every equivalence relation $\equiv$ right-compatible with the groupoid operation is a *congruence* relation, that is,

$$\forall a,b,c,d\in G:\quad [(a\equiv b\ \wedge\ c\equiv d)\Rightarrow a\cdot c\equiv b\cdot d].$$

An equivalence relation $\equiv$ on a groupoid G is a congruence (respectively: is right-compatible with the groupoid operation) if and only if there exists a function $h : G\to G$ such that

$$a\equiv b\ \Leftrightarrow\ h(a)=h(b) \text{ and } h(a\cdot b)=h[h(a)\cdot h(b)] \tag{4}$$

(respectively: $h(a\cdot b)=h[h(a)\cdot b]$) for $a,b\in G$.

In the case of a congruence relation, the function h is a homomorphism of G onto the groupoid $h(G)$ with the operation $c \# d = h(c\cdot d)$. This means that the equivalence relation $\equiv$ is a congruence in the groupoid $(G,\cdot)$ if and only if there exists a homomorphism H of G into a groupoid T such that $a\equiv b \Leftrightarrow H(a)=H(b)$ (see [2, pp. 35–37]). This yields a method of constructing invariant decompositions (see Remark 4, due to Andrzej Schinzel).

Remark 3. If the groupoid G is a group, then its invariant decompositions are sets of right cosets of some subgroup (see [1, pp. 34–35]). Moreover, if the group G is Abelian then invariant decompositions are determined by quotient groups.

Remark 4 (by A. Schinzel). By Remark 2 all congruences $\equiv$ in the monoid $(\mathbb{N},\cdot)$ are obtained by the following

Construction C_1

1° *Take an arbitrary Abelian semigroup* $(T,+)$ *with neutral element* 0.

2° *Take an arbitrary function* $\phi : P \to T$, *where* P *is the set of all prime numbers. Define a homomorphism* $H : (\mathbb{N},\cdot) \to (T,+)$ *by setting, for* $a = \prod_{p\in P} p^{\alpha(p)} \in \mathbb{N}$, *where* $\alpha(p)$ *are nonnegative integers,*

$$H(a) := \sum_{p\in P} \alpha(p)\phi(p).$$

3° *For* $a,b \in \mathbb{N}$ *define:* $a \equiv b \Leftrightarrow H(a) = H(b)$, *that is,*

$$\prod_{p\in P} p^{\alpha(p)} \equiv \prod_{p\in P} p^{\beta(p)} \;\Leftrightarrow\; \sum_{p\in P} \alpha(p)\phi(p) = \sum_{p\in P} \beta(p)\phi(p),$$

where $\alpha(p)$, $\beta(p)$ *are nonnegative integers.*

To describe all congruence relations means to describe all semigroups and, in consequence, to solve the association equation

$$F(F(a,b),c) = F(a,F(b,c)), \quad \text{where} \quad F : G \times G \to G.$$

Example 1 (by A. Schinzel). Let $T := 2^{\mathbb{N}}$ be the monoid with the union operation. If we define $\phi(p) := \{p\}$ we get the congruence relation

$$a \equiv b \;\Leftrightarrow\; a \text{ and } b \text{ have the same prime factors;}$$

this means that components of the invariant decomposition of $\mathbb{N}$ are sets of natural numbers having the same prime factors.

Remark 5. To obtain the same invariant decomposition as in Example 1, it is possible to take $\mathbb{N}$ with a suitable operation in place of $2^{\mathbb{N}}$. The function $h : \mathbb{N} \to \mathbb{N}$ such that $h(a)$ equals the product of the prime factors of a for $a > 1$ and $h(1) = 1$ satisfies (4), hence h is a homomorphism of $(\mathbb{N},\cdot)$ into $\mathbb{N}$ with the operation $a \# b = h(a\cdot b)$.

Remark 6. When the monoid $(G,\cdot)$ is the group then Theorem 1 yields all solutions of the translation equation (1) satisfying $F(\alpha,1) = \alpha$. In this case the invariant decompositions consist of right cosets of some subgroup G_s of G (see Remark 3) and $\overline{g}_s$ is equal to g_s and $g_s : G/G_s \to X_s$. Also the general solution of the translation equation (1) satisfying $F(\alpha,1) = \alpha$ has been given in [4] by the following

Construction C_2

1° *Let* $X = \bigcup_{s\in S} X_s$ *be a disjoint union of nonempty sets (fibres)* X_s *such that for every* $s \in S$ *there exists a subgroup* $G_s \leq G$ *and a bijection* $g_s : G/G_s \to X_s$, *where* G/G_s *is the set of right cosets of* G_s *in* G.

2° *Then* $F(\alpha,k) = g_s(g_s^{-1}(\alpha)\cdot k)$, $\alpha \in X_s$, $k \in G$.

REMARK 7. Construction C_3 mentioned below, quoted from [3], includes the general form of invariant decompositions for the subsemigroup G^+ of positive elements of the group $(G, +, \leq)$, linearly ordered and Abelian.

Before the presentation of the construction of the decompositions, we need two definitions.

DEFINITION 2. A subset A of G is called *bounded* if $\exists z \in G^+ \ \forall a \in A$: $(a < z$ and $a > -z)$, and *unbounded* if it is not bounded.

DEFINITION 3. Let A, B be subsets of G and $A \subseteq B \subseteq G$. We say that:

(a) A is an *initial interval* of B if

$$\forall a_0 \in A : \quad \{a \in B : \ a \leq a_0\} \subseteq A,$$

(b) A is a *final interval* of B if

$$\forall a_0 \in A : \quad \{a \in B : \ a_0 \leq a\} \subseteq A.$$

All G^+-invariant decompositions of the semigroup G^+ of positive elements of a linearly ordered, Abelian group G are obtained by

CONSTRUCTION C_3

1° *Take a family $\{G_s\}_{s \in S}$ of distinct, bounded subgroups of G forming a chain, i.e. $G_s \subset G_t$ or $G_t \subset G_s$ for $s, t \in S$, and an unbounded subgroup G^* such that $G^* \supseteq \bigcup_{s \in S} G_s$.*

2° *Let Φ be a function from the family $\{G_s\}_{s \in S}$ onto a family of initial intervals of G^+ such that*

(a) *$\Phi(G_s)$ is a union of intersections with G^+ of cosets of $C(G_s)$ in G, where $C(G_s)$ denotes the smallest convex subgroup containing G_s (the convexity of $C(G_s)$ means that together with every positive element a the subgroup $C(G_s)$ contains all elements $x \in G^+$ with $x \leq a$),*

(b) *if $G_s \subset G_t$, then $\Phi(G_s) \subset \Phi(G_t)$.*

3° *Every nonempty set*

$$W \cap \Big[\Phi(G_s) \setminus \bigcup_{G_t \subsetneq G_s} \Phi(G_t)\Big], \qquad W \in G_s, \ s \in S,$$

is a component of the decomposition.

4° *The sets*

$$V \cap \Big[G^+ \setminus \bigcup_{s \in S} \Phi(G_s)\Big], \qquad V \in G/G^*,$$

are the remaining components.

If we assume additionally that $(G, +, \leq)$ is an Archimedian group, then Construction C_3 is reduced to the following result from the paper [6].

Every G^+-invariant decomposition of the semigroup G^+ of positive elements of a linearly ordered, Archimedian group is of the following form:

(a) *there exists a right-closed or right-open interval $[0, x_0|$ such that every element belonging to $[0, x_0|$ is a component of the decomposition (this interval may be empty),*

(b) *the remaining components are the intersections with $G^+ \setminus [0, x_0|$ of cosets of some subgroup G^* in G.*

Using Construction C_3 and Theorem 1 we can obtain examples of solutions for the semigroup G^+ of positive elements of a linearly ordered, Abelian group $(G, +, \leq)$.

Example 2. Let $\mathbb{Z}$ denote the set of integes and $G := \{ax + b : a, b \in \mathbb{Z}\}$ be the group of linear polynomials with ordinary addition and with linear order defined as follows:

$$(ax + b \leq cx + d) \;\Leftrightarrow\; (a < c) \text{ or } (a = c \text{ and } b \leq d).$$

The semigroup of positive elements is

$$G^+ := \{ax + b : a > 0,\, b \in \mathbb{Z}\} \cup \mathbb{Z}^+,$$

where $\mathbb{Z}^+ := \{a \in \mathbb{Z} : a \geq 0\}$. According to Construction C_3, take the chain of bounded subgroups $\{0\} \subset \mathbb{Z}$ and the unbounded subgroup $G^* := G$. Define $\Phi(\{0\}) := \mathbb{Z}^+$ and $\Phi(\mathbb{Z}) := \mathbb{Z}^+ \cup (\mathbb{Z} + x)$, where $\mathbb{Z} + x \in G/\mathbb{Z}$. Every element of $\mathbb{Z}^+$ is a component of the decomposition. The sets $\mathbb{Z} + x$, $G^+ \setminus (\mathbb{Z}^+ \cup (\mathbb{Z} + x))$ are also components.

Let now $X := [0, \infty[$, $S := [0, 1[$, $X_s := \{s + j : j = 0, 1, 2, \ldots\}$ and $J_s := \mathbb{N} \cup \{0\}$ for $s \in S$. Moreover, $E_{0s} := \mathbb{Z} + x$, $E_{1s} := G^+ \setminus (\mathbb{Z}^+ \cup (\mathbb{Z} + x))$ and $E_{js} := \{j - 2\}$ for $j \in \{2, 3, 4, \ldots\}$ and $s \in S$. Assume that $\overline{g}_s(E_{js}) := s + j$ for $j \in \mathbb{N} \cup \{0\}$ and for $s \in S$. We get the following solution $F : X \times G^+ \to X$ of the translation equation:

$$F(\alpha, w) = \begin{cases} \alpha & \text{if } \alpha \in [0, 1[\text{ and } w \in \mathbb{Z}^+ \text{ or } \alpha \in [1, 2[\text{ and } w \in G^+, \\ \alpha + 1 & \text{if } \alpha \in [0, 1[\text{ and } w \in G^+ \setminus \mathbb{Z}^+, \\ \alpha - E(\alpha) & \text{if } \alpha \in X \setminus [0, 2[\text{ and } w \in \mathbb{Z} + x, \\ \alpha - E(\alpha) + 1 & \text{if } \alpha \in X \setminus [0, 2[\text{ and } w \in G^+ \setminus (\mathbb{Z}^+ \cup (\mathbb{Z} + x)), \\ \alpha + w & \text{if } \alpha \in X \setminus [0, 2[\text{ and } w \in \mathbb{Z}^+, \end{cases}$$

where $E(\alpha)$ denotes the integer part of α.

In what follows $(\mathbb{N}, \cdot)$ and $(\mathbb{Q}_+, \cdot)$ denote the monoid of natural numbers and the group of positive rational numbers respectively.

Using Theorem 1 we can obtain examples of solutions for $(G, \cdot) = (\mathbb{N}, \cdot)$.

Example 3. Let $X :=]1/4, 1]$ and take $S :=]1/2, 1]$, $X_s :=]s/2, s]$, $J_s := \{1, 2\}$ for $s \in S$. Moreover, $E_{1s} := \{1, 3, 5, \ldots\}$, $E_{2s} := \{2, 4, 6, \ldots\}$ for

$s \in S$. Define $\overline{g}_s(E_{1s}) := s/2$, $\overline{g}_s(E_{2s}) := s$ for $s \in S$. We get the following solution:

$$F(\alpha,k) = \begin{cases} 2\alpha & \text{for } \alpha \in \,]1/4,1/2],\ k \in \{2,4,6,\ldots\}, \\ \alpha & \text{for } \alpha \in \,]1/2,1],\ k \in \mathbb{N} \text{ or } \alpha \in \,]1/4,1/2],\ k \in \{1,3,5,\ldots\}. \end{cases}$$

EXAMPLE 4. Let X, S, $\{X_s\}$, J_s for $s \in S$ be as in Example 3. We take $E_{1s} := \{1\}$, $E_{2s} := \mathbb{N} \setminus \{1\}$ for $s \in S$. The functions $\overline{g}_s$ are defined as in Example 2. We get the following solution:

$$F(\alpha,k) = \begin{cases} 2\alpha & \text{for } \alpha \in \,]1/4,1/2],\ k \in \mathbb{N} \setminus \{1\}, \\ \alpha & \text{for } \alpha \in \,]1/2,1],\ k \in \mathbb{N} \text{ or } \alpha \in \,]1/4,1/2],\ k = 1. \end{cases}$$

EXAMPLE 5. Let $X := [0,\infty[$, $S := [0,1[$, $X_s := \{s+j : j = 0,1,2,\ldots\}$, $J_s := \mathbb{N} \cup \{0\}$ for $s \in S$. Moreover, $E_{0s} := \{2,4,6,\ldots\}$ and $E_{js} := \{2j-1\}$ for $j \in \mathbb{N}$ and $s \in S$. Define $\overline{g}_s(E_{js}) := s + j$ for $j \in \mathbb{N} \cup \{0\}$ and $s \in S$. We get the following solution:

$$F(\alpha,k) = \begin{cases} \alpha - E(\alpha) & \text{for } \alpha \in X \setminus [0,1[\text{ and } k \in \{2,4,6,\ldots\}, \\ & \text{or } \alpha \in [0,1[\text{ and } k \in \mathbb{N}, \\ \alpha + E(\alpha)(k-1) - (k-1)/2 & \text{for } \alpha \in X \setminus [0,1[\text{ and } k \in \{1,3,5,\ldots\}, \end{cases}$$

where $E(\alpha)$ denotes the integer part of α.

REMARK 8. If we define $\phi : P \to T = 2^{\mathbb{N}}$ by $\phi(p) := \emptyset$ for $p \neq 2$ and $\phi(2) := \{1\}$, where $T = 2^{\mathbb{N}}$ denotes the monoid described in Example 1, then by Construction $C_1(3^{\text{o}})$ in Remark 4 we get the congruence equivalent to the invariant decomposition from Example 3, which means that $E_1 := \{1,3,5,\ldots\}$, $E_2 := \{2,4,6,\ldots\}$.

Similarly, if we define $\phi : P \to T = 2^{\mathbb{N}}$ by $\phi(p) := \mathbb{N}$ for all $p \in P$, then by Construction $C_1(3^{\text{o}})$ we get the congruence equivalent to the invariant decomposition from Example 4, which means that $E_1 := \{1\}$, $E_2 := \mathbb{N} \setminus \{1\}$.

To obtain the invariant decomposition from Example 5, it is sufficient to consider the semigroup $(T,\cdot) := (2^{\mathbb{R}\setminus\{0\}},\cdot)$, where the operation is defined by $A \cdot B := \{a \cdot b : a \in A, b \in B\}$ for $A, B \in 2^{\mathbb{R}\setminus\{0\}}$, and to define $\phi : P \to T = 2^{\mathbb{R}\setminus\{0\}}$ by $\phi(p) := \{p\}$ for $p \neq 2$ and $\phi(2) := \mathbb{R} \setminus \{0\}$.

REMARK 9. If the solution of equation (1) is trivial, that is, $F(\alpha,k) := \alpha$ for every $(\alpha,k) \in X \times \mathbb{N}$, where X denotes an arbitrary nonempty set, then the invariant decomposition of $\mathbb{N}$ has exactly one element $\{\mathbb{N}\}$, the set X is decomposed into singletons and $\overline{g}_s(\mathbb{N}) := s$.

REMARK 10. The function $F(\alpha,k) := k \cdot \alpha$ for $(\alpha,k) \in X \times \mathbb{N}$ and $X := \,]0,\infty[$ is a solution of the translation equation (1). This solution is not of the form (3) (see Remark 11).

THEOREM 2. *Let $X \subset \mathbb{R}$ be an arbitrary interval. Suppose that a solution $F : X \times \mathbb{N} \to X$ of the translation equation* (1) *satisfying $F(\alpha, 1) = \alpha$ for $\alpha \in X$ can be extended to a solution $\overline{F} : X \times \mathbb{Q}_+ \to X$ of this equation. Then there exists a family $\{X_s\}_{s \in S}$ of disjoint sets such that $\bigcup_{s \in S} X_s = X$ and for every $s \in S$ there exists a subgroup $\mathbb{Q}_s \leq \mathbb{Q}_+$ and a bijection $g_s : \mathbb{Q}_+/\mathbb{Q}_s \to X_s$ for which*

$$F(\alpha, k) = g_s(g_s^{-1}(\alpha) \cdot k), \quad \alpha \in X_s,\ k \in \mathbb{N}. \tag{5}$$

Proof. This follows immediately from Construction C_2.

THEOREM 3. *Let $X \subset \mathbb{R}$ be an arbitrary interval. A function $F : X \times \mathbb{N} \to X$ is a solution of the translation equation* (1) *such that for every $\alpha \in X$ the function $F(\alpha, \cdot)$ is increasing and for every $k \in \mathbb{N}$ the function $F(\cdot, k)$ is increasing and surjective if and only if there exists a family $\{X_s\}_{s \in S}$ of disjoint sets such that $\bigcup_{s \in S} X_s = X$ and there exists a family of increasing bijections $g_s : \mathbb{Q}_+ \to X_s$, $s \in S$, such that*

$$F(\alpha, k) = g_s(g_s^{-1}(\alpha) \cdot k), \qquad \alpha \in X_s,\ k \in \mathbb{N}. \tag{6}$$

We present two proofs of this theorem. The first one is a corollary from Theorem 2 and the other proof is direct.

Proof I (of the "only if" part of Theorem 3, using Theorem 2). Note that the assumptions about $F : X \times \mathbb{N} \to X$ imply that $F(\alpha, 1) = \alpha$ and F can be extended to a solution $\overline{F} : X \times \mathbb{Q}_+ \to X$ of (1). Indeed, since $F(F(\alpha, 1), 1) = F(\alpha, 1)$, by injectivity of $F(\cdot, 1)$ we get $F(\alpha, 1) = \alpha$. We can put

$$\overline{F}(\alpha, k/l) := \beta \quad \text{such that} \quad F(\alpha, k) = F(\beta, l), \tag{7}$$

for every $\alpha \in X$ and $k/l \in \mathbb{Q}_+$. The existence and uniqueness of β result from the assumption that $F(\cdot, l)$ is surjective and injective, so $\overline{F}$ is correctly defined. One can verify easily that $\overline{F}$ is a solution of the translation equation. Indeed, let

$$\overline{F}\left(\overline{F}\left(\alpha, \frac{k}{l}\right), \frac{m}{n}\right) =: \gamma \quad \text{and} \quad \overline{F}\left(\alpha, \frac{k \cdot m}{l \cdot n}\right) =: \delta.$$

If we set $\overline{F}(\alpha, k/l) =: \beta$, then by definition (7), $F(\alpha, k) = F(\beta, l)$ and $F(\beta, m) = F(\gamma, n)$ and $F(\alpha, k \cdot m) = F(\delta, l \cdot n)$. Hence

$$F(\gamma, n \cdot l) = F(\beta, m \cdot l) = F(\alpha, k \cdot m) = F(\delta, n \cdot l),$$

so $\gamma = \delta$.

Therefore, by Theorem 2, we have the form (5) of the solution $F : X \times \mathbb{N} \to X$. Since the functions $F(\alpha, \cdot)$ are injective for every $\alpha \in X$, by Construction C_2 we get $\mathbb{Q}_s = \{1\} \leq \mathbb{Q}_+$ for every $s \in S$, which yields (6).

We will verify that the bijections $\{g_s\}_{s\in S}$ are increasing. Let $k/l < k_1/l_1$ and

$$g_s(k/l) =: \alpha, \quad g_s(k_1/l_1) =: \beta.$$

By (6) and by definition of $\bar{F}$ we get

$$\bar{F}\left(\alpha, \frac{k_1 l}{k l_1}\right) = g_s\left(\frac{k}{l}\cdot\frac{k_1 l}{k l_1}\right) = g_s\left(\frac{k_1}{l_1}\right) = \beta, \quad \text{whence} \quad F(\alpha, k_1 l) = F(\beta, k l_1).$$

Since $kl_1 < lk_1$, by assumptions we have

$$F(\alpha, kl_1) < F(\alpha, k_1 l) = F(\beta, kl_1) \quad \text{and} \quad \alpha < \beta.$$

Since the "if" part is evident, the first proof is complete.

Proof II (of the "only if" part of Theorem 3). We define the following relation in X:

$$\forall \alpha, \beta \in X: \quad \alpha \sim_F \beta \Leftrightarrow \exists k, l \in \mathbb{N} : F(\alpha, k) = F(\beta, l).$$

It is to verify that it is an equivalence relation. Indeed, evidently it is symmetric and reflexive. Let now $\alpha \sim_F \beta$ and $\beta \sim_F \gamma$. Then

$$\exists k, l, k_1, l_1 \in \mathbb{N}: \quad F(\alpha, k) = F(\beta, l) \text{ and } F(\beta, k_1) = F(\gamma, l_1).$$

Hence

$$F(\alpha, k\cdot k_1) = F(\beta, l\cdot k_1) = F(\gamma, l\cdot l_1), \quad \text{so} \quad \alpha \sim_F \gamma.$$

We denote by $\{X_s\}_{s\in S}$ the set of equivalence classes. Fix $s \in S$ and $\alpha_0 \in X_s$. We define $h_s : X_s \to \mathbb{Q}_+$ by

$$h_s(\alpha) := k/l, \quad \text{where} \quad F(\alpha_0, k) = F(\alpha, l).$$

The function h_s is correctly defined. Indeed, if

$$F(\alpha_0, k) = F(\alpha, l) \quad \text{and} \quad F(\alpha_0, k_1) = F(\alpha, l_1),$$

then

$$F(\alpha_0, l\cdot k_1) = F(\alpha, l\cdot l_1) = F(\alpha_0, l_1\cdot k).$$

Since $F(\alpha_0, \cdot)$ is injective, $l\cdot k_1 = l_1 \cdot k$, whence $k_1/l_1 = k/l$.

We will show that $h_s : X_s \to \mathbb{Q}_+$ is a bijection. If $h_s(\alpha) = h_s(\beta) = k/l$ then $F(\alpha, l) = F(\alpha_0, k) = F(\beta, l)$ and by injectivity of $F(\cdot, l)$ we get $\alpha = \beta$. To prove the surjectivity, take $m/n \in \mathbb{Q}_+$. Let $F(\alpha_0, m) = \beta$. By the surjectivity of $F(\cdot, n)$, we have $F(\alpha_0, m) = \beta = F(\alpha, n)$ for some α, so $h_s(\alpha) = m/n$.

Now, we will show that h_s is an increasing function. Let $\alpha < \beta$ and

$$h_s(\alpha) = k/l, \quad h_s(\beta) = k_1/l_1.$$

We have $F(\alpha_0, k) = F(\alpha, l)$ and $F(\alpha_0, k_1) = F(\beta, l_1)$. Since $F(\cdot, ll_1)$ is increasing, we obtain

$$F(\alpha_0, kl_1) = F(\alpha, ll_1) < F(\beta, ll_1) = F(\alpha_0, k_1 l),$$

therefore $kl_1 < k_1 l$ and $k/l < k_1/l_1$.

Let now $\alpha \in X_s,\ k \in \mathbb{N}$. Let $h_s(\alpha) = K/L$ and $\beta := F(\alpha, k)$. Hence

$$h_s(\alpha) \cdot k = K \cdot k/L.$$

We will show

$$h_s(\beta) = K \cdot k/L.$$

Indeed, we have $F(\beta, L) = F(\alpha, kL)$, $F(\alpha_0, K) = F(\alpha, L)$ and

$$F(\alpha_0, Kk) = F(\alpha, Lk) = F(\beta, L),$$

therefore $h_s(\beta) = K \cdot k/L$, and so

$$F(\alpha, k) = \beta = h_s^{-1}(K \cdot k/L) = h_s^{-1}(h_s(\alpha) \cdot k).$$

Putting $g_s = h_s^{-1}$ we have the form (6), which was to be shown.

Remark 11. If $F : X \times \mathbb{N} \to X$ satisfies the assumptions of Theorem 3, then F cannot be obtained by means of Theorem 1.

Indeed, otherwise let $g_s(1) =: \alpha_0$ for some $s \in S$. Then $X_s = \{F(\alpha_0, k) : k \in \mathbb{N}\} = g_s(\mathbb{N})$ and $\alpha_0 \in X_s$. Let $\overline{F} : X \times \mathbb{Q}_+ \to X$ be an extension of the solution F. Since $\overline{F}(\alpha_0, 1/2) < F(\alpha_0, k)$ for $k \in \mathbb{N}$, we have $\overline{F}(\alpha_0, 1/2) \notin X_s$. Let $\overline{F}(\alpha_0, 1/2) \in X_t$, $t \neq s$. Hence

$$F(\overline{F}(\alpha_0, 1/2), 2) = F(\alpha_0, 1) = \alpha_0,$$

so $\alpha_0 \in X_t$, which contradicts the relation $X_t \cap X_s = \emptyset$.

Remark 12. Let $X := [0, \infty[$ and define $F : X \times \mathbb{N} \to X$ by

$$F(\alpha, k) = \begin{cases} \alpha, & \alpha \in X,\ k = 1, \\ 1, & \alpha \in [0,1],\ k \in \mathbb{N} \setminus \{1\}, \\ k\alpha, & \alpha \in X \setminus [0,1],\ k \in \mathbb{N} \setminus \{1\}. \end{cases}$$

Then F is a solution of (1) which cannot be extended to a solution $\overline{F} : X \times \mathbb{Q}_+ \to X$ and is not of the form (3).

Indeed, for every solution $\overline{F} : X \times \mathbb{Q}_+ \to X$ of (1) satisfying $\overline{F}(\alpha, 1) = \alpha$, all functions $\overline{F}(\cdot, k)$ ought to be bijections. But

$$F(1/2, 2) = 1 = F(3/4, 2),$$

therefore F cannot be extended to a solution $\overline{F} : X \times \mathbb{Q}_+ \to X$.

Moreover, by Theorem 1, $\operatorname{card} X_s = \operatorname{card} J_s$ for $s \in S$. It is easy to see that for the solution F one of the elements of the family $\{X_s\}_{s\in S}$ is the set $X_n = [0,1]$ for some $n \in S$. This implies the following contradiction:

$$\mathfrak{c} = \operatorname{card}[0,1] = \operatorname{card} J_n \leq \operatorname{card} \mathbb{N} = \aleph_0.$$

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Reçu par la Rédaction le 11.3.2004
Révisé le 21.10.2004 (1508)