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A pseudo-trigonometry related to Ptolemy's theorem and the hyperbolic geometry of punctured spheres

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Abstract. A hyperbolic geodesic joining two punctures on a Riemann surface has infinite length. To obtain a useful distance-like quantity we define a finite *pseudo-length* of such a geodesic in terms of the hyperbolic length of its surrounding geodesic loop. There is a well defined angle between two geodesics meeting at a puncture, and our pseudo-trigonometry connects these angles with pseudo-lengths. We state and prove a theorem resembling Ptolemy's classical theorem on cyclic quadrilaterals and three general lemmas on intersections of shortest (in the sense of pseudo-length) geodesic joins. These ideas are then applied to the description of an optimal fundamental region for the covering Fuchsian group of a five-punctured sphere, effectively also giving a fundamental region for the modular group $M(0,5)$.

1. Introduction. Suppose that G is a torsion-free Fuchsian group of genus zero acting on the upper half-plane H, and having at least three distinct conjugacy classes of parabolic elements. Since the genus is zero, the field of functions automorphic with respect to G is the field of rational functions of just one such function λ, determined to within composition with a Möbius transformation. The range of λ is a Riemann surface which in this case is a region Ω in the extended complex plane, whose boundary $\partial\Omega$ has at least three single-point components (punctures). We say that λ is a *conformal universal covering map* of Ω. The group G is a free group freely generated by those parabolic elements which correspond to simple loops in $\pi_1(\Omega)$ each surrounding just one puncture. The reader is referred to the author's survey paper [5] for a treatment of the background here, which goes back in its elements to the work of Poincaré [7]. See also [9].

Let $\triangle(\alpha_1,\alpha_2,\alpha_3)$ be a zero-angle hyperbolic triangle in H, with vertices α_i which are fixed points of parabolic elements T_i of G, and with the property that no two points in its interior are equivalent. Then λ maps $\triangle(\alpha_1,\alpha_2,\alpha_3)$ onto a curvilinear triangle $\triangle(p_1,p_2,p_3)$ in Ω, whose sides are geodesic arcs.

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These arcs meet at the vertices p_i at interior angles ϕ_i which, unlike the angles in $\triangle(\alpha_1, \alpha_2, \alpha_3)$, are not zero.

To the side $p_j p_k$ of our triangle we assign not its hyperbolic length, which is infinite, but a *finite* pseudo-length ψ_{jk} related to (but not equal to) the hyperbolic length of the geodesic loop surrounding $p_j p_k$. Our pseudo-trigonometry is based on the relation

$$\phi_1 = \frac{\psi_{23}}{\psi_{12}\psi_{13}}. \tag{1.1}$$

This will be developed in Section 3. In Section 4 we deduce the equality for quadrilaterals

$$\psi_{13}\psi_{24} = \psi_{12}\psi_{34} + \psi_{14}\psi_{23}, \tag{1.2}$$

which resembles the well-known theorem of Ptolemy ([8, p. 50]) on cyclic quadrilaterals.

Our principal application is to n-punctured spheres. An *n-punctured sphere* is an open set $\mathbb{C}^* \setminus \{p_1, \ldots, p_n\}$. When we do not need to specify the punctures we shall denote an n-punctured sphere simply by Ω_n.

Since punctures are removable singularities, two n-punctured spheres are conformally equivalent if and only if there exists a Möbius transformation which maps the punctures of one onto the punctures of the other. Thus, for $n \geq 4$, two n-punctured spheres are unlikely to be conformally equivalent, whereas for $n \leq 3$ they are always equivalent. In this paper we consider only the cases $n \geq 3$, since Ω_n has no hyperbolic structure when $n < 3$.

The space $\mathcal{T}_n$ of conformal equivalence classes of n-punctured spheres can be regarded as an $(n-3)$-dimensional complex manifold. In rough terms, three of the punctures can be fixed at $0, 1$, and ∞, and the remaining $n-3$ serve as local complex coordinates.

We can also regard $\mathcal{T}_n$ as a $(2n-6)$-dimensional real manifold with local coordinates chosen from the parameters involved in the uniformization of the n-punctured sphere. These parameters are intimately related to the hyperbolic structure of Ω_n.

In Section 2 we present the Poincaré construction [7] of a fundamental domain for the Fuchsian group of covering transformations. The real boundary points of this domain are parabolic cusp points for the covering group. They are convenient real coordinates for the space of n-punctured spheres.

In Section 5 we introduce the concept of *shortest geodesic join*, which will play a major role in what follows, and we prove three key lemmas about the intersections of these shortest geodesic joins.

In Sections 6 and 7 we consider the problem of *normalizing* our choice of group parameters, or equivalently, of coordinates for the space of conformal equivalence classes. For a given Ω_n there are infinitely many equivalence classes of Poincaré fundamental regions and it is desirable to have a decision process of selection, based on "sensible" criteria. We present solutions to this problem in the cases $n = 4, 5$. In the simpler case $n = 4$ we relate our normalization to the configuration of the punctures and their shortest geodesic joins. In the case $n = 5$ this relationship is only partially understood. The problem here is part of a larger one, namely, that of finding the covering group when the punctures are known. The paper [5] contains further discussion of this and related problems.

The three-punctured sphere is the subject of an extensive classical literature, the essentials of which are very lucidly presented in L. V. Ahlfors [1]. In [4] the author established some monotonicity properties of the hyperbolic density function for Ω_3 and used these to obtain sharp bounds in two classical theorems.

The theory in the present paper does not add to what is known in the case $n = 3$. The reason for this is that the ambient geodesic of every simple geodesic join collapses to the remaining puncture. The space $\mathcal{T}_3$ reduces to a singleton, since all three-punctured spheres are conformally equivalent. The covering group G is (conjugate to) the modular subgroup Γ_2, and the conformal universal covering map λ is the classical Legendre elliptic modular function.

We remark that all the geodesics in Figure 1 of Section 7 are *calculated*, as are the coordinates h_i. Numerical approximation methods using the Matlab package were exploited to calculate the accessory parameters and to solve the Fuchsian differential equations describing the universal covering.

2. Fundamental domains. The Koebe–Poincaré Uniformization Theorem (see [6]), applied to the n-punctured sphere, has the following consequences. The surface Ω_n is conformally equivalent to the quotient of the upper half-plane H by a Fuchsian group G, generated by n parabolic Möbius transformations $T_1, \ldots, T_n$ which satisfy the relation

$$T_1 \cdots T_n = I. \tag{2.1}$$

To be more specific, though informal, we connect $p_1, \ldots, p_n$, in this order, by a Jordan arc Γ. Then the complement in $\mathbb{C}^*$ of Γ is the conformal image by the mapping λ of a subregion P of H, whose boundary meets the extended real line at $2n-2$ points $\alpha_1, \ldots, \alpha_{2n-2}$, corresponding respectively to $p_1, \ldots, p_{n-1}, p_n$, followed by $p_{n-1}, \ldots, p_3, p_2$. Now we replace the boundary segments of P by hyperbolic lines, or Euclidean semicircles in H, orthogonal to the real line and meeting it at the same points. We continue to call the modified region P. We replace Γ by the image of the boundary of the new region P, to obtain another Jordan arc, which we continue to call Γ. Chapter 1 of the classic text [6] by R. Nevanlinna has a detailed treatment of this construction. Such an arc Γ will be referred to as a *marking* for Ω_n.

Equipping Ω_n with a marking is equivalent to a choice of generators for its fundamental group. Let z_0 be a point not on Γ, and let τ_0 be its unique preimage in P. Then $\pi_1(\Omega_n, z_0)$ is generated by the equivalence classes of positively oriented loops γ_i surrounding p_i and crossing Γ at most twice, in the subarcs $p_{i-1}p_i$ and $p_i p_{i+1}$ if they exist, in this order. The γ_i satisfy the relation $\gamma_1 \dots \gamma_n = \mathrm{id}$. Analytic continuation of λ^{-1} around γ_i results in τ_0 being replaced by $T_i(\tau_0)$, where T_i is a parabolic transformation with fixed point α_i.

Here and subsequently, when there is no risk of ambiguity, we shall denote by (α, β) the hyperbolic line with ideal end-points α and β.

For $k = 1, \dots, n-1$ the transformation $S_k = T_1 \cdots T_k$ maps the side (α_k, α_{k+1}) of P onto the side $(\alpha_{2n-k}, \alpha_{2n-k-1})$. Here, and later, we adopt the convention that $\alpha_{2n-1} = \alpha_1$. The following lemma is stated several times, but without proof, by H. Poincaré in [7].

LEMMA 2.1. *The points α_k satisfy the relationship*

$$\prod_{k=1}^{n-1} \frac{\alpha_{2k+1} - \alpha_{2k}}{\alpha_{2k} - \alpha_{2k-1}} = -1,$$

where we assume the usual limiting value to be taken when any of the α_i is infinite.

We present a proof at the end of the next section.

If M is any Möbius transformation of H onto itself, then the composite map $\lambda \circ M$ is also a universal covering map, and all conformal universal covering maps of Ω_n by H are obtained in this way. This enables us to fix three of the α_k. We frequently find it convenient to do so by setting

$$\alpha_1 = \infty; \quad \alpha_2 = 0; \quad \alpha_{2n-2} = 1. \tag{2.2}$$

Then we have

$$0 < \alpha_3 < \cdots < \alpha_n < \cdots < \alpha_{2n-3} < 1,$$

and, when we take account of the relationship between the cusp points α_k, described in Lemma 2.1, we have essentially $2n-6$ real coordinates for the space of *marked* n-punctured spheres.

For $1 \le k \le n$, the cusp α_k is the unique fixed point of the Möbius transformation T_k. With our normalization (2.2) we have the following formulae for the T_k:

$$T_1 \tau = \tau + 1,$$

and, for $k > 1$,

$$\frac{1}{T_k \tau - \alpha_k} = \frac{1}{\tau - \alpha_k} - c_k \tag{2.3}$$

for some positive numbers c_k. Instead of working with the α_k, we shall often prefer to work with what we call the *angle* coordinates h_k, defined by

$$h_k = \alpha_{k+2} - \alpha_{k+1} \tag{2.4}$$

for $k = 1, \ldots, 2n-4$. See Remark 3.4 in the next section for the reason for the name. For future reference we note the relations

$$\sum_{k=1}^{2n-4} h_k = 1, \tag{2.5}$$

$$\prod_{k=1}^{n-2} \frac{h_{2k}}{h_{2k-1}} = 1, \tag{2.6}$$

the second of which is a restatement of Lemma 2.1.

The reader is referred to Figures 1 and 2 in Section 7 for an example of a marking and a corresponding fundamental domain.

Remark 2.2. If all the punctures are replaced by their complex conjugates, we obtain an n-punctured sphere whose angle coordinates h_k are simply related to those of the original n-punctured sphere. The correspondence is given by

$$h_k \mapsto h_{2n-3-k}.$$

In particular, if $h_k = h_{2n-3-k}$ for all k, then the punctures are concyclic, and the marking consists of the obvious consecutive arcs of the circle.

Remark 2.3. Different markings for the same n-punctured sphere, or, equivalently, different choices of parabolic generators T_i satisfying (2.1), are related to each other through *braid* transformations. We refer the interested reader to J. Birman's book [3]. Braid transformations can be described transparently and simply in terms of the *pseudo-lengths* of the consecutive joins of a marking, to be defined in the next section. However the description in terms of the angle coordinates h_i is not lucid.

3. The pseudo-trigonometry. In this section we describe pseudo-lengths and angles and the relationships between them, and then we use these to prove Lemma 2.1.

As in Section 1, we consider a hyperbolic triangle $\triangle(p_1, p_2, p_3)$ with vertices which are punctures, on a general hyperbolic Riemann surface Ω. We assume vertices of triangles to be described anticlockwise.

Let $\lambda : H \to \Omega$ be a universal covering map, and let $\triangle(\alpha_1, \alpha_2, \alpha_3)$ be a zero-angle hyperbolic triangle mapping onto $\triangle(p_1, p_2, p_3)$. If G is the automorphism group of λ, there exist well defined parabolic elements $T_k \in G$, with fixed points α_k, corresponding to homotopy classes of positive simple loops around p_k (see [6, p. 16]). If α_k is finite, then T_k is given by (2.3) and lifts to the matrix $\widetilde{T}_k \in \mathrm{SL}_2(C)$ given by

$$\widetilde{T}_k = I - c_k \begin{pmatrix} \alpha_k & -\alpha_k^2 \\ 1 & -\alpha_k \end{pmatrix}. \tag{3.1}$$

If $\alpha_k = \infty$, then T_k lifts to $\widetilde{T}_k \in \mathrm{SL}_2(C)$ given by

$$\widetilde{T}_k = I + c_k \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{3.2}$$

In both cases c_k is positive. Replacing c_k by $-c_k$ gives the inverse transformation or matrix.

Let J be a hyperbolic geodesic connecting two punctures, not necessarily distinct. The *ambient geodesic loop* γ *of* J is the unique geodesic in the homotopy class of closed curves surrounding J once. Let $L(\gamma)$ be the hyperbolic length of γ.

Definition 3.1. The *pseudo-length* $\psi(J)$ of a geodesic join J is defined by

$$\psi(J) = 2\cosh\left(\frac{L(\gamma)}{4}\right). \tag{3.3}$$

It follows that a pseudo-length is greater than or equal to two, with equality in the limiting case where J collapses to one puncture.

Suppose J joins p_i to p_j, and a preimage of J under λ is the hyperbolic line (α_i, α_j). These points are the fixed points of elements T_i, T_j of G with descriptions given by (3.1) or (3.2). We relate pseudo-length to the group parameters through the following lemma.

Lemma 3.2. *In the situation described above we have the conjugacy-invariant relation*

$$\psi(J) = \sqrt{c_i c_j}\,|\alpha_i - \alpha_j| \tag{3.4}$$

if both α_i, α_j *are finite*, *and*

$$\psi(J) = \sqrt{c_i c_j} \tag{3.5}$$

if one of the fixed points is infinite.

Proof. The length of the ambient geodesic loop γ around J is the translation length of the hyperbolic transformation $T_i T_j$. The formula (see [2, p. 173]) connecting this to trace is

$$2\cosh(L(\gamma)/2) = |\mathrm{trace}(\widetilde{T_i T_j})|.$$

From (3.1) and (3.2) we calculate that

$$\mathrm{trace}(\widetilde{T_i T_j}) = 2 - c_i c_j (\alpha_i - \alpha_j)^2$$

if both α_i, α_j are finite, and

$$\mathrm{trace}(\widetilde{T_i T_j}) = 2 - c_i c_j$$

if one of the fixed points is infinite. Since $|\mathrm{trace}(\widetilde{T_iT_j})| > 2$ it follows, when both fixed points are finite, that

$$2\cosh(L(\gamma)/2) = c_ic_j(\alpha_i - \alpha_j)^2 - 2,$$

and (3.4) follows from the duplication formula for cosh. Similarly we have (3.5) when one fixed point is infinite. This completes the proof. ∎

We remark that, if we keep α_i fixed and let α_j go through all possible transforms $T\alpha_j$ for $T \in G$, we actually go through all possible geodesic joins J of p_i to p_j. *Nevertheless* we shall frequently write ψ_{ij} instead of $\psi(J)$ in order to specify the points p_i and p_j which are being joined, but, for the sake of economy in notation, allowing the particular join from p_i to p_j to be determined by the context.

We next consider the angles at p_i made by the geodesic joins which meet there. For the sake of convenience we measure these not in radians but in *revolutions*. Thus our angle of measure 1/2 is the usual π radians. In our next lemma we establish (1.1).

LEMMA 3.3. *Let p_1, p_2, p_3 be the vertices of a simple geodesic triangle, and let ϕ_1 be the interior angle at p_1, measured in revolutions. Then*

$$\phi_1 = \frac{\psi_{23}}{\psi_{12}\psi_{13}},$$

where ψ_{ij} is the pseudo-length of the side joining p_i to p_j.

Proof. We assume for simplicity that the punctures p_i and their preimages α_i are finite. From (2.3) it follows that the function λ has an expansion of the following form near α_1:

$$\lambda(\tau) = p_1 + \sum_{k=1}^{\infty} a_k \exp\left[\frac{-2k\pi i}{c_1(\tau - \alpha_1)}\right],$$

where, by local univalence, $a_1 \neq 0$. For τ on (α_1,α_2), we have

$$\mathrm{Im}\left[\frac{-2\pi i}{c_1(\tau - \alpha_1)}\right] = \frac{-2\pi}{c_1(\alpha_2 - \alpha_1)},$$

and we deduce that for such τ,

$$\lim_{\tau\to\alpha_1} \arg(\lambda(\tau) - p_1) = \arg a_1 - \frac{2\pi}{c_1(\alpha_2 - \alpha_1)}.$$

It follows that the angle ϕ_1 at p_1 (measured in revolutions) between the images of (α_1, α_2) and (α_1, α_3) is given by

$$\phi_1 = \left|\frac{1}{c_1(\alpha_2 - \alpha_1)} - \frac{1}{c_1(\alpha_3 - \alpha_1)}\right| = \left|\frac{\alpha_3 - \alpha_2}{c_1(\alpha_2 - \alpha_1)(\alpha_3 - \alpha_1)}\right|.$$

Now according to Lemma 3.2, $\psi_{jk} = \sqrt{c_jc_k}\,|\alpha_k - \alpha_j|$; the result follows. ∎

REMARK 3.4. If $\alpha_1 = \infty$ the above expression for ϕ_1 is replaced by

$$\phi_1 = \left|\frac{\alpha_3 - \alpha_2}{c_1}\right|,$$

and the lemma also holds in this case. If $p_1 = \infty$ we interpret the angle ϕ_1 in the usual inversive way. In a normalization described in Section 2, we had $c_1 = 1$, and this explains why we referred to the h_k (defined to be equal to $\alpha_{k+2} - \alpha_{k+1}$) of that section as *angle* coordinates.

COROLLARY 3.5. *In the situation of the lemma, let ϕ_i, $i = 1, 2$, be the interior angles at p_i, $i = 1, 2$, respectively. Then*

$$\psi_{23}^2 = \phi_2\phi_3.$$

Proof. This follows immediately from the two equations obtained by cyclic reordering from (1.1). ■

We remark that knowledge of ϕ_2, ϕ_3, and ψ_{23} does not enable us to solve the triangle, as in ordinary trigonometry.

COROLLARY 3.6. *In the situation of the lemma, let ψ_{23}^* be the pseudo-length of the join γ_{23}^* of p_2 to p_3, obtained by "flipping" γ_{23} over p_1. Then*

$$\psi_{23}^* = \psi_{12}\psi_{13} - \psi_{23}.$$

Furthermore, $\psi_{23} < \psi_{23}^*$ if and only if $\phi_1 < 1/2$.

Proof. Corresponding to (1.1), there is the equality

$$\phi_1^* = \frac{\psi_{23}^*}{\psi_{12}\psi_{13}},$$

with $\phi_1^* = 1 - \phi_1$, and the corollary follows. ■

We conclude this account of our pseudo-trigonometry with an application to conformal universal coverings of n-punctured spheres.

Proof of Lemma 2.1. The equality we have to establish is

$$\prod_{k=1}^{n-1} \frac{\alpha_{2k+1} - \alpha_{2k}}{\alpha_{2k} - \alpha_{2k-1}} = -1, \tag{3.6}$$

where, for the sake of uniformity, we can assume all the $2n - 2$ fixed points α_i are finite. In any case, the expression on the left is invariant under the simultaneous action of Möbius transformations. This follows easily from the fact, to be used again shortly, that if

$$M\tau = \frac{a\tau + b}{c\tau + d} \tag{3.7}$$

with $ad - bc = 1$, then

$$M\tau_1 - M\tau_2 = \frac{\tau_1 - \tau_2}{Q(\tau_1)Q(\tau_2)}, \tag{3.8}$$

where $Q(\tau) = c\tau + d$.

We can arrange that $\alpha_1, \ldots, \alpha_{2n-2}$ is an increasing sequence, so, since $\alpha_{2n-1} = \alpha_1$, (3.6) is equivalent to

$$\prod_{k=1}^{n-1} \frac{|\alpha_{2k+1} - \alpha_{2k}|}{|\alpha_{2k} - \alpha_{2k-1}|} = 1,$$

or, with deliberate obfuscation,

$$\prod_{k=1}^{n-1} \frac{\sqrt{c_{2k+1}c_{2k}}\,|\alpha_{2k+1} - \alpha_{2k}|}{\sqrt{c_{2k}c_{2k-1}}\,|\alpha_{2k} - \alpha_{2k-1}|} = 1,$$

where $c_{2n-1} = c_1$. We use Lemma 3.2 and recall the fact that the sides (α_k, α_{k+1}) of P are mapped onto the sides $(\alpha_{2n-k}, \alpha_{2n-k-1})$ by the transformation $S_k = T_1 \cdots T_k$. It follows that

$$\sqrt{c_{k+1}c_k}\,|\alpha_{k+1} - \alpha_k| = \sqrt{c_{2n-k}c_{2n-k-1}}\,|\alpha_{2n-k} - \alpha_{2n-k-1}|,$$

and the expressions on the left in the previous display cancel in pairs. Thus Lemma 2.1 is established. ∎

4. A theorem like that of Ptolemy. The next theorem bears an uncanny similarity to Ptolemy's famous theorem on cyclic quadrilaterals. A simple proof makes use of Lemma 3.3, and we leave this to the reader. However we give another proof, which more clearly exhibits the similarity to Ptolemy's Theorem.

THEOREM 4.1. *Let* p_i, $i = 1, \ldots, 4$, *be the vertices*, *in order*, *of a simple geodesic quadrilateral*, *and let* ψ_{ij} *be the pseudo-lengths of the sides and diagonals. Then*

$$\psi_{12}\psi_{34} + \psi_{23}\psi_{14} = \psi_{13}\psi_{24}.$$

Proof. The following identity is central to the proofs of both theorems:

$$(x_4 - x_3)(x_2 - x_1) + (x_4 - x_1)(x_3 - x_2) = (x_4 - x_2)(x_3 - x_1). \tag{4.1}$$

(Quick check: both sides are affine functions of x_4 which agree when $x_4 = x_3$ and when $x_4 = x_1$.)

Suppose now that $x_i = \alpha_i$, where the α_i are finite fixed points of parabolic transformations $T_i \in G$, as defined by (2.3), or (3.1), with α_i increasing and having the property that the geodesic quadrilateral is the conformal image under λ of the zero-angle hyperbolic quadrilateral with the α_i as vertices. Then we deduce, from (4.1),

$$|\alpha_4 - \alpha_3|\,|\alpha_2 - \alpha_1| + |\alpha_4 - \alpha_1|\,|\alpha_3 - \alpha_2| = |\alpha_4 - \alpha_2|\,|\alpha_3 - \alpha_1|.$$

On multiplying both sides by $\sqrt{c_1c_2c_3c_4}$ and referring to (3.4), we obtain the theorem. ∎

We make the important remark, which we will refer to later.

Remark 4.2. The assumption that the original quadrilateral with vertices p_i is simple, that is, has sides and diagonals that do not intersect, except for the two diagonals, is unnecessary. The preimage quadrilateral is simple, but may contain pairs of points in its interior equivalent under G, and some of the vertices α_i, though distinct, may themselves be equivalent, that is, map to the same puncture.

For the sake of completeness of our presentation, we give a proof of the original Ptolemy Theorem.

Theorem 4.3 (Ptolemy). *For $i = 1, \ldots, 4$, let β_i be points on a circle, following each other in cyclic numerical order. Then*

$$|\beta_4 - \beta_3|\,|\beta_2 - \beta_1| + |\beta_4 - \beta_1|\,|\beta_3 - \beta_2| = |\beta_4 - \beta_2|\,|\beta_3 - \beta_1|.$$

Proof. Let M be a Möbius transformation which takes the unit circle onto the real line, in such a way that $\alpha_i = M\beta_i$ form an increasing sequence. Using (3.7) and (3.8), we see that the statement of the theorem follows from

$$|\alpha_4 - \alpha_3|\,|\alpha_2 - \alpha_1| + |\alpha_4 - \alpha_1|\,|\alpha_3 - \alpha_2| - |\alpha_4 - \alpha_2|\,|\alpha_3 - \alpha_1| = 0,$$

where we can clearly omit the absolute value signs. The theorem thus again follows from (4.1). ∎

A better-known statement and proof are essentially the same as Ptolemy's original ones in [8, p. 50], and of course different from the above.

5. Shortest geodesic joins. As we remarked earlier, for a fixed pair of punctures p_i, p_j, there are infinitely many geodesic joins γ_{ij}. In the case of a finitely generated group, it is well known (see [2, p. 265]) that there are finitely many geodesics of any given maximum length. It follows that there exists at least one geodesic join γ_{ij}^* whose pseudo-length ψ_{ij}^* has the least possible value. Such a geodesic join will be called a *shortest geodesic join.*

The following three lemmas will be used in Sections 6 and 7.

Lemma 5.1. *Every shortest geodesic join is simple.*

Proof. Suppose a certain shortest geodesic join of p_1 to p_3 has a point of self-intersection z_0, and that $\lambda(\tau_0) = z_0$. Then τ_0 is the point of intersection of two hyperbolic lines (α_1, α_3) and (α_2, α_4), where $\alpha_2 = M(\alpha_1)$ and $\alpha_4 = M(\alpha_3)$ for some $M \in G$. Then, by Theorem 4.1 and Remark 4.2, we have, since $\psi_{13} = \psi_{24}$,

$$\psi_{13}^2 = \psi_{12}\psi_{34} + \psi_{23}\psi_{14} > 4 + \psi_{23}\psi_{14}.$$

But then $\min(\psi_{23}, \psi_{14}) < \psi_{13}$. Since both (α_2, α_3) and (α_1, α_4) are preimages under λ of geodesics joining p_1 to p_3 this contradicts the assumption made at the beginning of this proof. Thus the lemma is proved. ∎

In a similar vein, we have the following lemma. We use the term "internal" for points of geodesic joins other than the end-point punctures.

LEMMA 5.2. *Two shortest geodesic joins having a common end-point puncture have no internal point in common.*

Proof. Suppose that shortest geodesic joins of p_1 to p_2 and p_3 have a point of intersection z_0, and that $\lambda(\tau_0) = z_0$. Then τ_0 is the point of intersection of two hyperbolic lines (α_1, α_3) and (α_2, α_4), where $p_i = \lambda(\alpha_i)$ for $i = 1, 2, 3$, and $p_1 = \lambda(\alpha_4)$. Then, from Theorem 4.1 and Remark 4.2, we obtain

$$\psi_{13}\psi_{24} = \psi_{12}\psi_{34} + \psi_{23}\psi_{14} > \psi_{12}\psi_{34} + 4.$$

But then we must have either $\psi_{12} < \psi_{24}$ or $\psi_{34} < \psi_{13}$. This contradicts the minimality of at least one of the geodesics described at the beginning of this proof. Thus Lemma 5.2 is established. ∎

Our next lemma goes further:

LEMMA 5.3. *Two shortest geodesic joins have at most one internal point in common.*

Proof. Suppose that γ_{12} and γ_{34} are shortest geodesic joins p_1 to p_2 and p_3 to p_4 respectively which have consecutive points of intersection z_1 and z_2, which on γ_{12} appear in the order p_1, z_1, z_2, p_2. There are two cases to consider:

CASE (i): On each geodesic join the crossings at z_1, z_2 are in opposite directions.

CASE (ii): On each geodesic join the crossings at z_1, z_2 are in the same direction.

Though similar, the proofs in the two cases are different and need to be presented separately. We emphasize that in neither case do we *a priori* exclude the possibility of more than two intersections. This exclusion *follows* from the present proof: if there are no consecutive intersections, there is no more than one intersection.

CASE (i): Suppose that on γ_{34} the points z_1, z_2 appear in the order p_3, z_2, z_1, p_4 and that the loop consisting of the segments z_1 to z_2 along γ_{12} and then z_2 to z_1 along γ_{34} is described anticlockwise. All other subcases of Case (i) can be reduced to this by relabelling.

Let γ'_{12} be the geodesic join in the homotopy class of the arc consisting of the segments p_1 to z_1 along γ_{12}, z_1 to z_2 along γ_{43}, and z_2 to p_2 along γ_{12}. Let γ'_{23} be the geodesic join in the homotopy class of the arc consisting of the segments p_2 to z_1 along γ_{21}, and z_2 to p_3 along γ_{43}. Let γ'_{34} be the geodesic join in the homotopy class of the arc consisting of the segments p_3

to z_2 along γ_{34}, z_2 to z_1 along γ_{21}, and z_1 to p_4 along γ_{34}. Let γ'_{41} be the geodesic join in the homotopy class of the arc consisting of the segments p_4 to z_1 along γ_{43}, and z_1 to p_3 along γ_{21}.

Our proof in Case (i) consists in proving that either $\psi(\gamma_{12}) > \psi(\gamma'_{12})$, or $\psi(\gamma_{34}) > \psi(\gamma'_{34})$, thus showing that γ_{12} and γ_{34} cannot both be shortest geodesic joins. As before, we go to the conformal universal cover.

There is a sequence of real points $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha'_1$ such that λ maps the hyperbolic lines $(\alpha_1, \alpha_2), (\alpha_2, \alpha_3), (\alpha_3, \alpha_4)$ onto $\gamma'_{12}, \gamma'_{23}, \gamma'_{34}$ respectively, and the line (α_4, α'_1) onto γ'_{41}. Since $\lambda(\alpha'_1) = \lambda(\alpha_1) = p_1$ there is an element $M \in G$ such that $\alpha'_1 = M\alpha_1$. In fact M corresponds to the element of the fundamental group $\pi_1(\Omega_n)$ represented by the closed loop z_1 to z_2 along γ_{12} followed by z_2 to z_1 along γ_{34}.

The hyperbolic line $(M\alpha_1, \alpha_2)$ is a preimage under λ of γ_{12}, and the hyperbolic line $(M\alpha_3, \alpha_4)$ is a preimage under λ of γ_{34}.

By replacing G by a conjugate subgroup of $\mathrm{PSL}(2, \mathbb{R})$ we can arrange that $0 < \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 < \alpha'_1$, and that M is given by $M\tau = m^2\tau$ for some $m > 1$ if M is hyperbolic, or that $M\tau = \tau + 1$ if M is parabolic.

We calculate $\psi(\gamma_{12})$ using (2.3) and (3.4). If T_1 is the parabolic transformation in G with fixed point α_1 then the conjugate transformation $T'_1 = MT_1M^{-1}$ has fixed point $M\alpha_1$.

In the case where M is hyperbolic and $M\tau = m^2\tau$ we write (2.3) for T_1 and T'_1:

$$\frac{1}{T_1\tau - \alpha_1} = \frac{1}{\tau - \alpha_1} - c_1,$$

and

$$\frac{1}{MT_1M^{-1}\tau - m^2\alpha_1} = \frac{1}{\tau - m^2\alpha_1} - c'_1, \tag{5.1}$$

from which we see that $c'_1 = c_1 m^{-2}$. Formula (3.4) then gives

$$\psi(\gamma_{12}) = \sqrt{c'_1 c_2}\,(m^2\alpha_1 - \alpha_2) = \sqrt{c_1 c_2}\,(m\alpha_1 - m^{-1}\alpha_2).$$

Similarly $\psi(\gamma_{34}) = \sqrt{c_3 c_4}\,(m\alpha_3 - m^{-1}\alpha_4)$. Since $\psi(\gamma'_{12}) = \sqrt{c_1 c_2}\,(\alpha_2 - \alpha_1)$ and $\psi(\gamma'_{34}) = \sqrt{c_3 c_4}\,(\alpha_4 - \alpha_3)$, the assumption that γ_{12} and γ_{34} are shortest geodesic joins is thus equivalent to the pair of inequalities

$$m\alpha_1 - m^{-1}\alpha_2 < \alpha_2 - \alpha_1,$$
$$m\alpha_3 - m^{-1}\alpha_4 < \alpha_4 - \alpha_3,$$

which reduce to $m\alpha_1 < \alpha_2$ and $m\alpha_3 < \alpha_4$. But then we have the impossible inequality sequence

$$m^2\alpha_1 < m\alpha_2 < m\alpha_3 < \alpha_4 < m^2\alpha_1.$$

We still have to consider Case (i) in the subcase where M is parabolic. Since this corresponds to the loop z_1 to z_2 to z_1 surrounding just one puncture, the result is a simple consequence of Corollary 3.6, but we complete the proof with the present method. With $M\tau = \tau + 1$, and $\alpha_1' = \alpha_1 + 1$, (5.1) now reads

$$\frac{1}{MT_1M^{-1}\tau - \alpha_1 - 1} = \frac{1}{\tau - \alpha_1 - 1} - c_1',$$

which means that $c_1' = c_1$, and $\psi(\gamma_{12}) = \sqrt{c_1c_2}\,(\alpha_1 + 1 - \alpha_2)$. In a similar way $\psi(\gamma_{34}) = \sqrt{c_3c_4}\,(\alpha_3 + 1 - \alpha_4)$, so the assumption that γ_{12} and γ_{34} are shortest geodesic joins leads to the pair of inequalities

$$\alpha_1 + 1 - \alpha_2 < \alpha_2 - \alpha_1,$$
$$\alpha_3 + 1 - \alpha_4 < \alpha_4 - \alpha_3,$$

or $2\alpha_1 + 1 < 2\alpha_2$ and $2\alpha_3 + 1 < 2\alpha_4$. This leads to the contradiction

$$2\alpha_1 < 2\alpha_2 - 1 < 2\alpha_3 - 1 < 2\alpha_4 - 2 < 2\alpha_1.$$

This completes the proof of the lemma in Case (i).

Case (ii). Suppose that on γ_{34} the points z_1, z_2 appear in the order p_3, z_1, z_2, p_4 and that, to a traveller along γ_{12}, the geodesic join γ_{34} appears to cross from right to left firstly at z_1 and then at z_2. All other subcases of Case (ii) can be reduced to this.

Let γ_{12}' be the geodesic join in the homotopy class of the arc consisting of the segments p_1 to z_1 along γ_{12}, z_1 to z_2 along γ_{34}, and z_2 to p_2 along γ_{12}. Let γ_{34}' be the geodesic join in the homotopy class of the arc consisting of the segments p_3 to z_1 along γ_{34}, z_1 to z_2 along γ_{12}, and z_2 to p_4 along γ_{34}.

As in Case (i), our proof in Case (ii) will be accomplished by proving that either $\psi(\gamma_{12}) > \psi(\gamma_{12}')$, or $\psi(\gamma_{34}) > \psi(\gamma_{34}')$.

Suppose the hyperbolic line (α_1, α_2) is a preimage under λ of γ_{12}. There are two hyperbolic lines (α_3', α_4') and (α_3, α_4) which are preimages under λ of γ_{34}, crossing (α_1, α_2) at preimages of z_1 and z_2. By Lemma 5.1, (α_3', α_4') and (α_3, α_4) do not intersect. Hence the α_i appear in the cyclic order $(\alpha_1, \alpha_3', \alpha_3, \alpha_2, \alpha_4, \alpha_4')$.

There is an element $M \in G$ such that $\alpha_3' = M\alpha_3$ and $\alpha_4' = M\alpha_4$. Since M has fixed points between α_3 and α_4 and between α_4' and α_3' in the above cyclic order, M is hyperbolic. Therefore we can arrange that M is given by $M\tau = m^2\tau$ for some $m > 1$. In this situation we have the inequalities

$$\alpha_1 < m^2\alpha_3 < \alpha_3 < 0 < \alpha_2 < \alpha_4 < m^2\alpha_2.$$

Denoting by $\psi(\alpha_i, \alpha_j)$ the pseudo-length of (α_i, α_j), we find that, as in the proof of Case (i),

$$\begin{aligned}
\psi(\gamma_{12}) &= \psi(\alpha_1, \alpha_2)\sqrt{c_1 c_2}\,(\alpha_2 - \alpha_1), \\
\psi(\gamma_{34}) &= \psi(\alpha_3, \alpha_4)\sqrt{c_3 c_4}\,(\alpha_4 - \alpha_3), \\
\psi(\gamma'_{12}) &= \psi(\alpha_1, m^2\alpha_2) = \sqrt{c_1 c_2}\,(m\alpha_2 - m^{-1}\alpha_1), \\
\psi(\gamma'_{34}) &= \psi(m^2\alpha_3, \alpha_4) = \sqrt{c_3 c_4}\,(m^{-1}\alpha_4 - m\alpha_3).
\end{aligned}$$

The assumption that the first two of the above quantities are respectively smaller than the second two simplifies to the pair of inequalities $m\alpha_2 > -\alpha_1$ and $-m\alpha_3 > \alpha_4$. But then we have the chain of inequalities

$$-m^2\alpha_3 > m\alpha_4 > m\alpha_2 > -\alpha_1,$$

which is inconsistent with the beginning of the previous chain. Thus γ_{12} and γ_{34} cannot both be shortest geodesic joins.

This completes the proof of Case (ii), and thus of Lemma 5.3. ∎

6. The four-punctured sphere. In the case $n = 4$ we prove the following theorem for the angle coordinates h_i, introduced in Section 2. Since $h_1 + h_2 + h_3 + h_4 = 1$, and $h_1 h_3 = h_2 h_4$, it is easiest to work in terms of h_1 and h_2 alone.

THEOREM 6.1. *The angle coordinates h_1, h_2 for the space of 4-punctured spheres can be chosen so that they satisfy the inequalities*

$$2h_1 + h_2 \le 1, \tag{6.1}$$

$$h_1 + 2h_2 \le 1, \tag{6.2}$$

$$2h_1 + 2h_2 \ge 1. \tag{6.3}$$

Two distinct points (h_1, h_2) and (h'_1, h'_2) in this triangle represent the same element of $\mathcal{T}_4$ if and only if equality holds for (h_1, h_2) in at least one of the inequalities (6.1), (6.2), *and* (6.3), *and then $h_1 = h'_2$ and $h_2 = h'_1$.*

Proof. Consider the six shortest geodesic joins γ_{ij}, $1 \le i < j \le 4$, joining the punctures p_i to p_j respectively. We assume, for the moment, that these are uniquely determined. According to Lemma 5.1 they are all simple, and, according to Lemma 5.2, no two with a common end-point puncture intersect again.

Every 4-punctured sphere has a non-trivial group of conformal automorphisms, isomorphic to the Klein 4-group (see [9, p. 150]). The ambient geodesic loop (Definition 3.1) of γ_{12} is also the ambient geodesic loop for a geodesic join γ^*_{34} joining p_3 and p_4. By minimality, we have $\gamma^*_{34} = \gamma_{34}$. Similarly for the other pairs. Thus the six joins form a tetrahedral pattern, without any intersections except at the end-point punctures, and with opposite edges equal in pseudo-length.

We set $\psi_{ij} = \psi(\gamma_{ij})$, and suppose, after possible relabelling, that $\psi_{13} > \psi_{12}$ and $\psi_{13} > \psi_{14}$, and that the triangle $\triangle(p_2, p_3, p_4)$ has its vertices described in the negative (clockwise) direction. Select as marking Γ the succession of edges $\gamma_{12}, \gamma_{23}, \gamma_{34}$. We assert that the coordinates h_i associated with this marking through the construction of Section 2 satisfy the conditions of the theorem.

To see this, we let $h = 1 - h_1 - h_2$ be the angle at p_1 in $\triangle(p_1, p_2, p_4)$. We remark that in terms of the construction of Section 2, $h = h_3 + h_4$. Then by Lemma 3.3,

$$h = \frac{\psi_{24}}{\psi_{12}\psi_{14}} = \frac{\psi_{13}}{\psi_{12}\psi_{14}}.$$

Since also

$$h_1 = \frac{\psi_{23}}{\psi_{12}\psi_{13}} = \frac{\psi_{14}}{\psi_{12}\psi_{13}}, \qquad h_2 = \frac{\psi_{34}}{\psi_{13}\psi_{14}} = \frac{\psi_{12}}{\psi_{13}\psi_{14}},$$

we have $h > h_1$, which implies strict inequality in (6.1), and $h > h_2$, which implies the same in (6.2). To see that (6.3) also holds strictly, we apply Corollary 3.6 to $\triangle(p_2, p_3, p_4)$. Since $\psi(\gamma_{24}) < \psi(\gamma^*_{24})$ it follows that $h < 1/2$, which is equivalent to the strict case of (6.3).

The limiting case of equality in (6.1) corresponds to $\psi_{13} = \psi_{14}$ and, after permuting p_2, p_3, p_4 to p_3, p_4, p_2, or equivalently h_1, h_2, h to h_2, h, h_1, we see that this corresponds to the limiting case of equality in (6.2), with h_1 and h_2 interchanged.

The limiting case of equality in (6.3) corresponds to non-uniqueness in the choice of γ_{13} as shortest join. In this case, in terms of Corollary 3.6, γ_{24} can be replaced by its "flipped-over" counterpart γ^*_{24}, which is equivalent to interchanging h_1 and h_2, as before.

This completes the proof of Theorem 6.1. ∎

It is interesting to describe the configuration of Theorem 6.1 in the further normalized situation $p_1 = \infty$, $p_2 = 0$, $p_4 = 1$. Then $p_3 = p$ is confined to the curvilinear triangle described by $\mathrm{Im}(p) \geq 0$, $|p| \leq 1$, $|p - 1| \leq 1$. Equality in (6.1) holds on the boundary segment $|p| = 1$, equality in (6.2) holds on the boundary segment $|p - 1| = 1$, and equality in (6.3) holds on the boundary segment $\mathrm{Im}(p) = 0$. The marking Γ is the most natural sequence of geodesic joins of the points in their numerical order. Thus, even though we cannot solve the deeper problem: given the punctures find the angle coordinates, we can at least find the setwise image of the boundary given in Theorem 6.1.

In the next section we present a theorem similar to Theorem 6.1 for the case of five punctures, but the corresponding configurations on the sphere are not easy to obtain.

7. The five-punctured sphere. The reader will find it convenient to refer to the following figures throughout this section. The angles h_i are angles at the puncture at infinity.

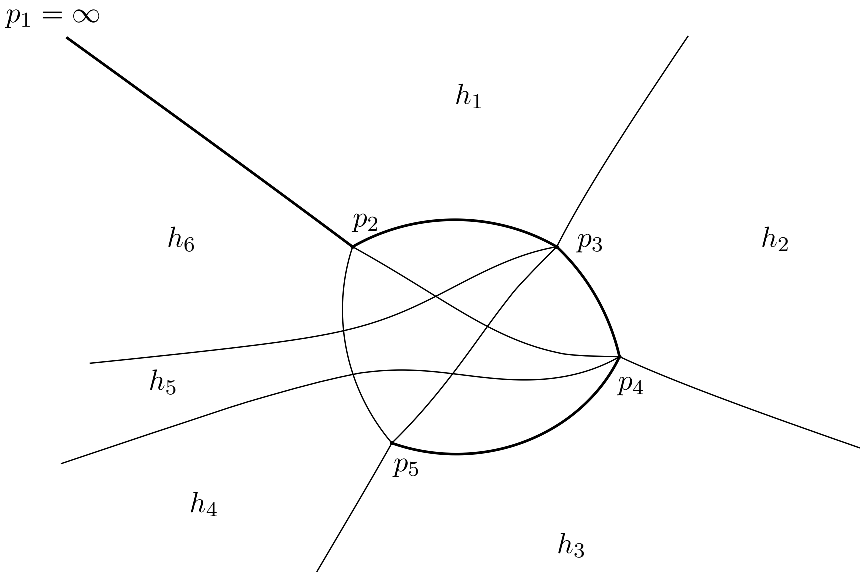

Fig. 1. A marking in the case $n = 5$

In the next figure, we show part of the tessellation of the upper half-plane by fundamental domains, associated with the above marking.

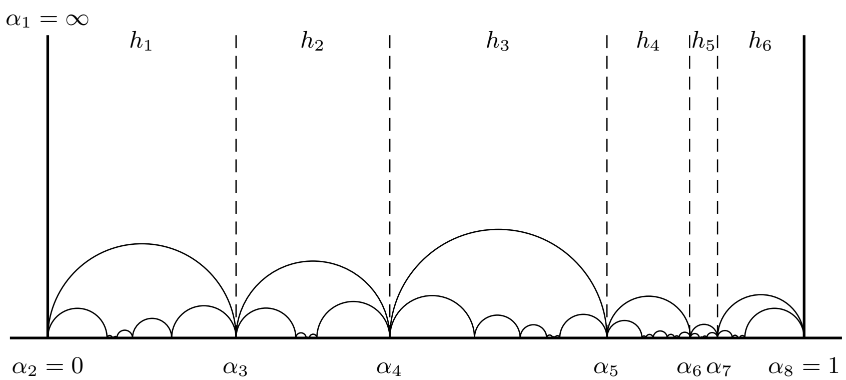

Fig. 2. A fundamental domain with neighbours

Of central importance here is the following lemma, special to the case $n = 5$.

LEMMA 7.1. *Let Γ be any marking (sequence of geodesic joins, not necessarily shortest) on Ω_5, and suppose that the angles made by the segments of Γ at three consecutive punctures, and on the same side, have measures all less than or equal to* $1/2$. *Then these measures are in fact all equal to* $1/2$, *and the punctures are concyclic.*

Proof. Referring to Figure 1 for convenience, we take the three punctures at p_5, p_1, and p_2, modifying the marking shown in bold type by including the join form p_5 to p_1 and removing the join form p_3 to p_4. The marking Γ joins p_4, p_5, p_1, p_2, p_3 in this order.

Let us denote by k_5 and k_2 the angles at the punctures p_5 and p_2 on the *left* side of the marking just referred to. Then the assumption of the lemma is that

$$k_5 \leq 1/2, \tag{7.1}$$

$$h_1 + h_2 + h_3 \leq 1/2, \tag{7.2}$$

$$k_2 \leq 1/2. \tag{7.3}$$

A simple exercise in the pseudo-trigonometry of Section 3 yields the values

$$k_5 = \frac{h_4}{h_3 + h_4}, \qquad k_2 = \frac{h_6}{h_1 + h_6}.$$

The inequalities (7.1) and (7.3) can therefore be replaced by

$$h_4 \leq h_3, \qquad h_6 \leq h_1 \tag{7.4}$$

respectively. In the case $n = 5$, (2.5) and (2.6) state that

$$h_1 + h_2 + h_3 + h_4 + h_5 + h_6 = 1, \tag{7.5}$$

and that

$$h_1 h_3 h_5 = h_2 h_4 h_6. \tag{7.6}$$

From (7.4) and (7.6), we deduce that $h_5 \leq h_2$. Using (7.2) and (7.5), we obtain

$$1/2 \leq h_4 + h_5 + h_6 \leq h_1 + h_2 + h_3 \leq 1/2.$$

We deduce that all of the inequalities appearing in this proof are equalities, and hence that $h_1 + h_2 + h_3 = k_2 = k_5 = 1/2$.

Now we apply what we have so far to the consecutive angles at p_1, p_2 and p_3 and see that the angle at p_3 is also equal to $1/2$. Similarly the angle at p_4 is also equal to $1/2$.

Thus the lemma is established. ∎

To prove our theorem concerning normalizations of the coordinates for the space of 5-punctured spheres, we need the following corollary.

Corollary 7.2. *Three shortest geodesic joins involving all five punctures cannot form a* crossed V *pattern, except when the punctures are concyclic. To explain this, suppose that a geodesic join from p_2 to p_4 crosses the shortest geodesic join from p_1 to p_3, and then, in the same direction, the shortest geodesic join from p_5 to p_3. Then the geodesic join from p_2 to p_4 is not shortest, unless the p_i are concyclic* (*in the numerical order of their labelling*).

Proof. Suppose that the join p_2 to p_4 is shortest. Then, by Corollary 3.6, the marking obtained by joining p_1 to p_2 to p_3 to p_4 to p_5 without crossing any of the three given joins has three consecutive angles on the same side less than $1/2$. This contradicts Lemma 7.1. ∎

We are now ready to state and prove

Theorem 7.3. *The angle coordinates h_i, $i = 1, \ldots, 6$, for the space of* 5*-punctured spheres, which we know are positive and satisfy* (7.5) *and* (7.6), *can be chosen so that they also satisfy*

$$h_1 + h_2 \leq 1/2, \tag{7.7}$$

$$h_2 + h_3 \leq 1/2, \tag{7.8}$$

$$h_3(h_5 + h_6) \leq h_4(h_1 + h_2), \tag{7.9}$$

$$h_1(h_4 + h_5) \leq h_6(h_2 + h_3). \tag{7.10}$$

If two distinct sets of coordinates $[h_i]$ and $[h_i']$ represent conformally equivalent 5*-punctured spheres and satisfy the above inequalities then in the case of both sets equality holds in at least one of the inequalities. More precisely if equality holds for $[h_i]$ in* (7.7), *then equality holds for $[h_i']$ in* (7.8), *and vice versa. If equality holds for $[h_i]$ in* (7.9), *then equality holds for $[h_i']$ in either* (7.9) *or* (7.10). *If equality holds for $[h_i]$ in* (7.10), *then equality holds for $[h_i']$ in either* (7.9) *or* (7.10).

The reader should recognize here a resemblance to the side-pairing which occurs for fundamental regions of Fuchsian groups. The group here is the modular group $M(0,5)$ acting on the Teichmüller space $T(0,5)$, and whose orbits are conformal equivalence classes of 5-punctured spheres.

Proof. We consider the ten shortest geodesic joins in the case where these are uniquely determined. We first prove that only one pair intersect. Our proof makes repeated use of Lemmas 5.1–5.3, as well as Corollary 7.2, but to avoid being tedious we make explicit reference only to the corollary, leaving the simpler explanations based on the lemmas to the reader.

Fix one puncture, say p_1, at infinity. Then the four shortest joins from p_1 are simple, and do not intersect each other. Label the punctures in such a way that p_2, p_3, p_4, p_5 appear in anticlockwise order (relative to p_1) as the extremities of the four joins, which we conveniently label as $\gamma_2, \gamma_3, \gamma_4, \gamma_5$ respectively.

If the shortest geodesic join J from p_2 to p_4 intersects γ_3, then it also intersects γ_5 or it does not. We prove that it does not. If it does, then the trio composed of J, the shortest geodesic join from p_2 to p_5, and γ_3, would form a crossed V, contradicting Corollary 7.2.

If J intersects γ_3 we now show that the shortest geodesic join K from p_5 to p_3 does not intersect J, $\gamma_2, \gamma_3, \gamma_4$ or γ_5. Certainly K does not intersect γ_3 or γ_5. By an application to K of what has already been proved about J, K intersects J if and only if it also intersects one of γ_2 or γ_4. Suppose it is γ_2. Then J, K, and γ_3 form a crossed V, contradicting Corollary 7.2.

Still assuming J intersects γ_3, we next show that the shortest geodesic join from p_2 to p_5 intersects none of the other ones already considered. Certainly it does not intersect γ_2, γ_5, J, or K. If it intersects γ_3 then together with J and γ_3 it forms a crossed V; if it intersects γ_4 then it forms a crossed V with γ_3 and γ_4, in either case contradicting Corollary 7.2. Similarly we can show that the shortest geodesic join from p_4 to p_5 intersects none of those already considered. It is now obvious that the shortest geodesic joins from p_3 to p_2 and p_4 intersect none of the earlier ones.

If we now drop the assumption that J intersects γ_3, we are left with the possibilities that J intersects γ_5 or that it does not. In the former case we repeat the above discussion. In the latter, we consider the similar three possibilities for the shortest geodesic join from p_3 to p_5. It is easy to see that in all cases there is just one pair of intersecting shortest geodesic joins.

We now map the one puncture that is not involved in either intersecting pair to infinity, and relabel the punctures so that p_2, p_3, p_4, p_5 still appear in anticlockwise order relative to the point p_1 at infinity. In summary, the ten shortest geodesic joins form an *envelope* pattern, with a distinguished quadrilateral whose interior angles are all less than $1/2$, its diagonals being the shortest geodesic joins joining opposite vertices.

Let the angles at p_1 between the joins, in order, be h_1, h_2, h_3, and h, summing to 1. We are still at liberty to choose the labelling in such a way that the inequalities (7.7) and (7.8) hold. Assume this has been done, and that h_4, h_5 and h_6 are as in Figure 1.

As simple exercises in pseudo-trigonometry, we can readily obtain the following expressions for the interior angles at p_2, p_3, p_4, p_5 respectively:

$$\text{angle at } p_2 = \frac{h_1(h_4+h_5)}{h_6(h_4+h_5+h_6+h_1)+h_1(h_4+h_5)}, \tag{7.11}$$

$$\text{angle at } p_3 = \frac{h_3(h_5+h_6)}{h_4(h_1+h_2)+h_3(h_5+h_6)}, \tag{7.12}$$

$$\text{(7.13)} \qquad \text{angle at } p_4 = \frac{h_1(h_4+h_5)}{h_6(h_2+h_3)+h_1(h_4+h_5)},$$

$$\text{(7.14)} \qquad \text{angle at } p_5 = \frac{h_3(h_5+h_6)}{h_4(h_3+h_4+h_5+h_6)+h_3(h_5+h_6)}.$$

Since the angles at p_3 and p_4 are less than $1/2$, the stated inequalities (7.9) and (7.10) follow from (7.12) and (7.13) respectively. The other necessary conditions, that the angles at p_2 and p_5 are also less than $1/2$, follow from the above formulas, which, with (7.7) and (7.8), imply that these angles are smaller than those at p_4 and p_3 respectively.

It remains to prove the last statement of the theorem. The cases of equality in (7.7) and (7.8) are ones which allow rotations of the distinguished quadrilateral. If there is a coordinate set $[h_i]$ for which equality holds only in (7.7), then there is another set $[h_i']$ for a conformally equivalent Ω_n for which equality holds only in (7.8). Equality holds in both (7.7) and (7.8) if and only if the distinguished quadrilateral is one whose vertices are those of a Euclidean parallelogram.

The cases of equality in (7.9) and (7.10) are ones which allow replacement of the distinguished quadrilateral. If equality holds for the set $[h_i]$ only in (7.9) then there is another set $[h_i']$ for a conformally equivalent Ω_n for which equality holds *again* only in (7.9) *or* in (7.10). To see this, note that the distinguished quadrilateral has the angle at p_3 equal to $1/2$, so can be replaced by the distinguished quadrilateral which excludes p_5. We obtain a configuration equivalent to the one considered by relabelling p_5, p_4, p_3, p_2, p_1 as p_1, p_2, p_3, p_4, p_5 *or* p_5, p_1, p_4, p_3, p_2 as p_1, p_2, p_3, p_4, p_5. The case of equality in (7.10) is discussed in a similar fashion. Equality holds in both (7.9) and (7.10) if and only if the punctures are concyclic.

Thus the theorem is proved. ■

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