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Nonlinear delay integrodifferential systems with Caputo fractional derivative in infinite-dimensional spaces

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Abstract. This paper is mainly concerned with existence of mild solutions and optimal controls for nonlinear delay integrodifferential systems with Caputo fractional derivative in infinite-dimensional spaces. We do not assume that the relevant strongly continuous semigroup is compact.

1. Introduction. In this paper, we consider the following delay integrodifferential systems with Caputo fractional derivative:

$$
\begin{cases} {}^{C}D_t^q x(t) = Ax(t) + f\Big(t, x_t, \int_0^t g(t,s,x_s)\,ds\Big) \\ \qquad\qquad + B(t)u(t),\ 0 \le t \le T, \\ x(t) = \varphi(t), \quad -r \le t \le 0, \end{cases} \tag{1.1}
$$

where ${}^{C}D_t^q$ denotes the Caputo fractional derivative of order $q \in (0,1)$, A is the generator of a strongly continuous semigroup $\{S(t), t \ge 0\}$ on a Banach space X, f and g are X-valued functions specified later, u takes values in another separable reflexive Banach space Y, $B(t)$ is a linear operator from Y into X, and $x_t : [-r, 0] \to X$, $t \ge 0$, which is defined by setting $x_t = \{x(t+s) \mid s \in [-r, 0]\}$, represents the history of the state from time $t - r$ up to the present time t.

Fractional differential equations have recently been proved to be valuable tools in the modelling of many phenomena in various fields of engineering, physics and economics. There has been significant progress in the theory of such equations (see the monographs of Kilbas et al. [KST], Miller and Ross [MR], Podlubny [P], Lakshmikantham et al. [LLD] and the surveys of Agarwal et al. [ABH], [ABB]).

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Recently, some papers appeared on fractional delay differential equations or inclusions in Banach spaces [AZH], [BHNO], [HO], [HRS], [M], [RQS], [ZJL]. To study the theory of abstract fractional differential equations involving the Caputo derivative in Banach spaces, the crux of the problem is a concept of mild solution. A pioneering work has been reported in El-Borai [E1], [E2]. Hernández et al. [HRB] pointed out that some recent works [ABB], [BB], [CKA], [HRS], [JOM], [RQS] on abstract fractional differential equations in Banach spaces were incorrect, and used another approach based on the well developed theory of resolvent operators for integral equations. Moreover, Zhou et al. [WZ], [ZJ1], [ZJ2] also introduced a suitable definition of mild solutions based on the well known theory of Laplace transform and probability density functions.

A pioneering work on the existence of mild solutions for system (1.1) has been reported in Hu et al. [HRS]. However, the definition of mild solution in [HRS] is not appropriate for such problems although it has been utilized by several authors. In the present paper, we revisit this interesting problem and establish some new existence principles for solutions to the system (1.1). Firstly, we use a more appropriate definition (Definition 3.1) for mild solutions based on earlier work [WZ], [ZJ1], [ZJ2]. Secondly, we prove the existence and uniqueness of mild solutions for system (1.1). The main techniques used here are fractional calculus, the singular Gronwall inequality with the $\mathcal{B}$-norm and the contraction mapping principle. We also discuss the continuous dependence of mild solutions. Thirdly, we consider a Lagrange problem (P), and a result on existence of optimal controls is presented.

The rest of this paper is organized as follows. In Section 2, some notation and preparation results are given. In Section 3, results on the existence and uniqueness of mild solutions for system (1.1) are given. Moreover, continuous dependence of mild solutions is discussed. In Section 4, the Lagrange problem (P) for system (1.1) is formulated and a result on existence of optimal controls is presented. Finally, an example is given to demonstrate the applicability of our result.

2. Preliminaries. Throughout this paper, let X and Y be two Banach spaces, and $L_b(X,Y)$ the space of bounded linear operators from X to Y. Suppose $r > 0$, $T > 0$, let $J = [0,T]$. Denote $M = \sup_{t\in J} \|S(t)\|_{L_b(X,X)}$, which is a finite number. Let $C([-r,a],X)$, $a \geq 0$, be the Banach space of continuous functions from $[-r,a]$ to X with the usual supremum norm. For brevity, we denote $C([-r,a],X)$ simply by $C_{-r,a}$ and its norm by $\|\cdot\|_{-r,a}$. For any $x \in C_{-r,T}$ and $t \in J$, we define $x_t(s) = x(t+s)$ for $-r \leq s \leq 0$; then $x_t \in C_{-r,0}$. We will also use $\|f\|_{L^p(J,\mathbb{R}^+)}$ to denote the $L^p(J,\mathbb{R}^+)$ norm of $f \in L^p(J,\mathbb{R}^+)$ for $1 < p < \infty$.

Let us recall the following known definitions. For more details, see [KST].

DEFINITION 2.1. The *fractional integral* of order γ for a function $f : [0,\infty) \to \mathbb{R}$ is defined as

$$I^{\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{f(s)}{(t-s)^{1-\gamma}}\, ds, \qquad t > 0,\ \gamma > 0,$$

provided the right side is pointwise defined on $[0,\infty)$, where $\Gamma(\cdot)$ is the gamma function.

DEFINITION 2.2. The *Riemann–Liouville* derivative of order γ for $f : [0,\infty) \to \mathbb{R}$ is defined as

$${}^{L}D^{\gamma} f(t) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\gamma+1-n}}\, ds, \qquad t > 0,\ n-1 < \gamma < n.$$

DEFINITION 2.3. The *Caputo derivative* of order γ for $f : [0,\infty) \to \mathbb{R}$ is defined as

$${}^{C}D^{\gamma} f(t) = {}^{L}D^{\gamma}\Big(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0)\Big), \qquad t > 0,\ n-1 < \gamma < n.$$

REMARK. (i) If $f \in C^n[0,\infty)$, then

$${}^{C}D^{\gamma} f(t) = \frac{1}{\Gamma(n-\gamma)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\gamma+1-n}}\, ds = I^{n-\gamma} f^{(n)}(t),\ t > 0,\ n-1 < \gamma < n.$$

(ii) The Caputo derivative of a constant is equal to zero.

(iii) If f is an abstract function with values in X, then the integrals which appear in Definitions 2.1 and 2.2 are taken in Bochner's sense.

The following results will be used throughout this paper.

LEMMA 2.4. *A measurable function $V\colon J \to X$ is Bochner integrable if $\|V\|$ is Lebesgue integrable.*

LEMMA 2.5 ([XK, Lemma 1.2]). *Suppose that $x \in C_{-r,T}$ satisfies*

$$\begin{cases} \|x(t)\| \le a + b\int_0^t (t-s)^{q-1} \|x_s\|_{-r,0}\, ds + c\int_0^t (t-s)^{q-1} \|x(s)\|\, ds, & t \in J, \\ x(t) = \varphi(t), \quad -r \le t \le 0, \end{cases}$$

where $\varphi \in C_{-r,0}$ and $a,b,c \ge 0$ are constants. Then there exists a constant $M^* > 0$ independent of a and φ such that

$$\|x(t)\| \le M^*(a + \|\varphi\|_{-r,0}) \qquad \text{for all } t \in J.$$

Define $\mathcal{B} = \{x \in C_{-r,T} : x(t) = \varphi(t) \text{ for } -r \le t \le 0\}$ and $\|x_t\|_{\mathcal{B}} = \sup_{-r\le\theta\le t} \|x(\theta)\|$ for $x \in \mathcal{B}$, $0 \le t \le T$. It is easy to see that $\|x_t\|_{\mathcal{B}}$ is continuous and increasing on J, and $\|x_t\|_{-r,0} \le \|x_t\|_{\mathcal{B}}$. Then we can obtain the following result immediately.

LEMMA 2.6. *Suppose that $x \in C_{-r,T}$ satisfies*

$$\begin{cases} \|x_t\|_{\mathcal{B}} \le (a + \|\varphi\|_{-r,0}) + b\int_0^t (t-s)^{q-1}\|x_s\|_{\mathcal{B}}\,ds, & t \in J, \\ x(t) = \varphi(t), \quad -r \le t \le 0, \end{cases}$$

where $\varphi \in C_{-r,0}$ and $a, b \ge 0$ are constants. Then there exists a constant $M^* > 0$ independent of T, a and φ such that

$$\|x_t\|_{\mathcal{B}} \le M^*(a + \|\varphi\|_{-r,0}) \quad \text{for all } t \in J.$$

Thus, $\|x(t)\| \le \|x_t\|_{\mathcal{B}} \le M^*(a + \|\varphi\|_{-r,0})$ for all $t \in J$.

REMARK. From Lemma 2.6, it is obvious that there exists a constant $\rho = \max\{M^*(a + \|\varphi\|_{-r,0}), \|\varphi\|_{-r,0}\} > 0$ such that $\|x\|_{-r,T} \le \rho$.

3. System analysis. We set $D := \{(t,s) \in J \times J \mid 0 \le s \le t\}$ and make the following assumptions.

ASSUMPTION [HF]. $f : J \times C_{-r,0} \times X \to X$ satisfies:

(i) f is measurable for $t \in J$.
(ii) For every $\rho > 0$ there exists a constant $L_f(\rho) > 0$ such that for all $\xi_1, \xi_2 \in C_{-r,0}$ and $\eta_1, \eta_2 \in X$ satisfying $\|\xi_1\|_{-r,0}, \|\xi_2\|_{-r,0}, \|\eta_1\|, \|\eta_2\| \le \rho$, and all $t \in J$,
$$\|f(t,\xi_1,\eta_1) - f(t,\xi_2,\eta_2)\| \le L_f(\rho)(\|\xi_1 - \xi_2\|_{-r,0} + \|\eta_1 - \eta_2\|).$$
(iii) There exists a constant $a_f > 0$ such that for all $\xi \in C_{-r,0}$, $\eta \in X$ and $t \in J$,
$$\|f(t,\xi,\eta)\| \le a_f(1 + \|\xi\|_{-r,0} + \|\eta\|).$$

ASSUMPTION [HG]. $g : D \times C_{-r,0} \to X$ satisfies:

(i) g is continuous for $(t,s) \in D$.
(ii) For every $\rho > 0$ there exists a constant $L_g(\rho) > 0$ such that for all $(t,s) \in D$ and ξ_1, $\xi_2 \in C_{-r,0}$ satisfying $\|\xi_1\|_{-r,0}, \|\xi_2\|_{-r,0} \le \rho$,
$$\|g(t,s,\xi_1) - g(t,s,\xi_2)\| \le L_g(\rho)\|\xi_1 - \xi_2\|_{-r,0}.$$
(iii) There exists a constant $M_g > 0$ such that for all $(t,s) \in D$ and $\xi \in C_{-r,0}$,
$$\|g(t,s,\xi)\| \le M_g(1 + \|\xi\|_{-r,0}).$$

ASSUMPTION [HB]. Let Y be a separable reflexive Banach space in which the controls u take their values. Then we have $B \in L^\infty(J, L(Y,X))$, i.e., $\operatorname{ess\,sup}_{t\in J}\|B(t)\|_{L(Y,X)} < \infty$. Let $\|B\|_\infty$ stand for the norm of $B \in L^\infty(J, L(Y,X))$.

ASSUMPTION [HU]. Multivalued maps $U(\cdot) : J \to 2^Y\backslash\{\emptyset\}$ have closed, convex and bounded values. $U(\cdot)$ is graph measurable and $U(\cdot) \subseteq \Omega$ where Ω is a bounded set of Y.

Define the admissible set

$$U_{\mathrm{ad}} = \{v(\cdot) \mid J \to Y \text{ strongly measurable}, v(t) \in U(t) \text{ a.e.}\}.$$

Obviously, $U_{\mathrm{ad}} \neq \emptyset$ [Z, Theorem 2.1] and $U_{\mathrm{ad}} \subset L^p(J,Y)$ $(1 < p < \infty)$ is bounded, closed and convex. It is obvious that $Bu \in L^p(J,X)$ for all $u \in U_{\mathrm{ad}}$.

Based on Lemma 3.1 and Definition 3.1 of our previous work [WZ], we introduce the following definition.

DEFINITION 3.1. For every $u \in U_{\mathrm{ad}}$, if there exists a $T = T(u) > 0$ and $x \in C_{-r,T}$ such that

$$x(t) = \begin{cases} \mathscr{T}(t)\varphi(0) \\ \quad + \int_0^t (t-s)^{q-1}\mathscr{S}(t-s)f\Big(s, x_s, \int_0^s g(s,\tau,x_\tau)\,d\tau\Big)\,ds \\ \quad + \int_0^t (t-s)^{q-1}\mathscr{S}(t-s)B(s)u(s)\,ds, \quad 0 \le t \le T, \\ \varphi(t), \quad -r \le t \le 0, \end{cases} \tag{3.1}$$

then system (1.1) is called *mildly solvable with respect to u* on $[-r,T]$, where

$$\mathscr{T}(t) = \int_0^\infty \xi_q(\theta)S(t^q\theta)\,d\theta, \qquad \mathscr{S}(t) = q\int_0^\infty \theta\xi_q(\theta)S(t^q\theta)\,d\theta,$$

$$\xi_q(\theta) = \frac{1}{q}\theta^{-1-1/q}\varpi_q(\theta^{-1/q}) \ge 0,$$

$$\varpi_q(\theta) = \frac{1}{\pi}\sum_{n=1}^\infty (-1)^{n-1}\theta^{-qn-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q), \quad \theta \in (0,\infty),$$

and ξ_q is a probability density function defined on $(0,\infty)$, that is,

$$\xi_q(\theta) \ge 0, \quad \theta \in (0,\infty), \qquad \int_0^\infty \xi_q(\theta)\,d\theta = 1.$$

LEMMA 3.2 ([ZJ1, Lemmas 3.2–3.3]). *The operators $\mathscr{T}$ and $\mathscr{S}$ have the following properties:*

(i) *For any fixed $t \ge 0$, $\mathscr{T}(t)$ and $\mathscr{S}(t)$ are bounded linear operators, viz., for any $x \in X$,*

$$\|\mathscr{T}(t)x\| \le M\|x\| \quad \text{and} \quad \|\mathscr{S}(t)x\| \le \frac{qM}{\Gamma(1+q)}\|x\|.$$

(ii) *$\{\mathscr{T}(t), t \ge 0\}$ and $\{\mathscr{S}(t), t \ge 0\}$ are strongly continuous.*

In order to discuss the existence of mild solutions of system (1.1), we need the following important lemma.

LEMMA 3.3. *Let Assumptions* [HF] *and* [HG] *hold. Suppose system* (1.1) *is mildly solvable on* $[-r,T]$ *with respect to* u. *Then there exists a constant* $M^{**}>0$ *such that*

$$\|x(t)\| \le M^{**} \quad \text{for all } t \in J.$$

Proof. If x is a mild solution of system (1.1) with respect to u on $[-r,T]$, then x satisfies (3.1). Direct calculation gives

$$\begin{aligned}\|x(t)\| &\le \|\mathscr{T}(t)\varphi(0)\| + \int_0^t (t-s)^{q-1}\|\mathscr{S}(t-s)B(s)u(s)\|\,ds \\ &\quad + \int_0^t (t-s)^{q-1}\Big\|\mathscr{S}(t-s)f\Big(s,x_s,\int_0^s g(s,\tau,x_\tau)d\tau\Big)\Big\|\,ds \\ &\le a + \frac{a_f qM(1+M_gT)}{\Gamma(1+q)}\int_0^t (t-s)^{q-1}\|x_s\|_{\mathcal{B}}\,ds,\end{aligned}$$

which yields

$$\begin{aligned}\|x_t\|_{\mathcal{B}} &\le \|\varphi\|_{-r,0} + \sup_{0\le\theta\le t}\|x(\theta)\| \\ &\le a + \|\varphi\|_{-r,0} + \frac{a_f qM(1+M_gT)}{\Gamma(1+q)}\int_0^t (t-s)^{q-1}\|x_s\|_{\mathcal{B}}\,ds,\end{aligned}$$

where

$$\begin{aligned} a &= M\|\varphi(0)\| \\ &\quad + \frac{a_f(1+M_gT)MT^q}{\Gamma(1+q)} + \frac{qM\|B\|_\infty}{\Gamma(1+q)}\left(\frac{p-1}{pq-1}\right)^{(p-1)/p} T^{q-1/p}\|u\|_{L^p(J,Y)}.\end{aligned}$$

By Lemma 2.6, there exists a constant $M^*>0$ such that

$$\|x(t)\| \le \|x_t\|_{\mathcal{B}} \le M^*(a+\|\varphi\|_{-r,0}) \quad \text{for all } t\in J,$$

Let $M^{**} = \max\{M^*(a+\|\varphi\|_{-r,0}), \|\varphi\|_{-r,0}\} > 0$. Thus $\|x(t)\| \le M^{**}$ for $t\in J$. ∎

THEOREM 3.4. *Assume that* [HF], [HG], [HB] *and* [HU] *are satisfied. Then for each* $u\in U_{\rm ad}$ *and for some* $p\in(1,\infty)$ *with* $pq>1$, *system* (1.1) *is mildly solvable on* $[-r,T]$ *with respect to* u, *and the mild solution is unique.*

Proof. Let $C_{-r,T_1} = C([-r,T_1],X)$ with the usual supremum norm and $T_1>0$ to be specified later, and set

$$\mathcal{S}(1,T_1)=\Big\{h\in C_{-r,T_1}\ \Big|\ \max_{s\in[0,T_1]}\|h(s)-\varphi(0)\|\le 1,$$
$$h(s)=\varphi(s)\text{ for }-r\le s\le 0\Big\}.$$

Then $\mathcal{S}(1,T_1)$ is a closed convex subset of C_{-r,T_1}. According to [HF](i) and [HG](i), it is easy to deduce that $f(s,h_s,\int_0^s g(s,\tau,h_\tau)\,d\tau)$ is a measurable function on $[0,T_1]$. Let $h\in\mathcal{S}(1,T_1)$. Then the constant $\rho^*=\max\{\|\varphi(0)\|+1,\|\varphi\|_{-r,0}\}>0$ is such that $\|h\|_{-r,T_1}\le\rho^*$. Using [HF](iii) and [HG](iii), we have

$$\Big\|f\Big(s,h_s,\int_0^s g(s,\tau,h_\tau)\,d\tau\Big)\Big\|\le a_f(1+\rho^*+M_gT(1+\rho^*))=:K.$$

In light of Lemma 3.2(i), we obtain

$$\int_0^t(t-s)^{q-1}\Big\|\mathscr{S}(t-s)f\Big(s,h_s,\int_0^s g(s,\tau,h_\tau)\,d\tau\Big)\Big\|\,ds\le\frac{MKT_1^q}{\Gamma(1+q)}.$$

Thus, $(t-s)^{q-1}\mathscr{S}(t-s)f(s,h_s,\int_0^s g(s,\tau,h_\tau)\,d\tau)$ is Bochner integrable with respect to $s\in[0,t]$ for all $t\in[0,T_1]$ due to Lemma 2.4.

On the other hand, by Lemma 3.2(i), [HB], [HU] and $pq>1$, we have

$$\begin{aligned}(3.2)\qquad&\int_0^t(t-s)^{q-1}\|\mathscr{S}(t-s)B(s)u(s)\|\,ds\\&\le\frac{qM\|B\|_\infty}{\Gamma(1+q)}\Big(\frac{p-1}{pq-1}\Big)^{(p-1)/p}T^{q-1/p}\|u\|_{L^p(J,Y)}.\end{aligned}$$

Thus, $(t-s)^{q-1}\mathscr{S}(t-s)B(s)u(s)$ is also Bochner integrable with respect to $s\in[0,t]$ for all $t\in[0,T_1]$.

Now we can define P: $\mathcal{S}(1,T_1)\to C_{-r,T_1}$ by

$$(3.3)\qquad (Ph)(t)=\begin{cases}\mathscr{T}(t)\varphi(0)\\ \quad+\int_0^t(t-s)^{q-1}\mathscr{S}(t-s)f\Big(s,h_s,\int_0^s g(s,\tau,h_\tau)\,d\tau\Big)\,ds\\ \quad+\int_0^t(t-s)^{q-1}\mathscr{S}(t-s)B(s)u(s)\,ds,\quad 0<t\le T_1,\\ \varphi(t),\quad -r\le t\le 0,\end{cases}$$

By using the properties of $\mathscr{T}$ and $\mathscr{S}$ and our assumptions, we will verify that P is a contraction map on $\mathcal{S}(1,T_1)$ with a suitable $T_1>0$.

For $t \in [0, T_1]$, it is not difficult to obtain the inequality

$$\|(Ph)(t) - \varphi(0)\| \le \|\mathscr{T}(t)\varphi(0) - \varphi(0)\| + \frac{MK}{\Gamma(1+q)} t^q + \frac{qM\|B\|_\infty \|u\|_{L^p(J,Y)}}{\Gamma(1+q)} \left(\frac{p-1}{pq-1}\right)^{(p-1)/p} t^{q-1/p}. \tag{3.4}$$

By Lemma 3.2(ii), we can choose $\varepsilon = 1/2$ such that

$$\|\mathscr{T}(t)\varphi(0) - \varphi(0)\| \le 1/2 \quad \text{for } \varphi(0) \in X. \tag{3.5}$$

Let

$$T_{11} = \min\left\{\frac{1}{2}, \left[\frac{\Gamma(1+q)}{2M(K + q\|B\|_\infty \|u\|_{L^p(J,Y)})\left(\frac{p-1}{pq-1}\right)^{(p-1)/p}}\right]^{p/(pq-1)}\right\}.$$

Then for all $t \le T_{11}$, it follows from (3.4) and (3.5) that

$$\|(Ph)(t) - \varphi(0)\| \le 1.$$

On the other hand, for $-r \le t \le 0$, $(Ph)(t) = \varphi(t)$.

Hence $P(\mathcal{S}(1, T_1)) \subseteq \mathcal{S}(1, T_1)$.

Let $h_1, h_2 \in \mathcal{S}(1, T_1)$, so $\|h_1\|_{-r,T_1}, \|h_2\|_{-r,T_1} \le \rho^*$. For $t \in [0, T_1]$, using Lemma 3.2(i), [HF](ii) and [HG](ii), we also obtain

$$\begin{aligned}
&\|(Ph_1)(t) - (Ph_2)(t)\| \\
&\quad \le \frac{qML_f(\rho^*)}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} \|(h_1)_s - (h_2)_s\|_{-r,0}\, ds \\
&\qquad + \frac{qML_f(\rho^*)}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} \Big(\int_0^s \|g(s,\tau,(h_1)_\tau) - g(s,\tau,(h_2)_\tau)\|\, d\tau\Big)\, ds \\
&\quad \le \frac{qML_f(\rho^*)}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} \|(h_1)_s - (h_2)_s\|_{-r,0}\, ds \\
&\qquad + \frac{qML_f(\rho^*)L_g(\rho^*)T}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} \|(h_1)_\tau - (h_2)_\tau\|_{-r,0}\, ds \\
&\quad \le \frac{qML_f(\rho^*)(1 + L_g(\rho^*)T)}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} \|(h_1)_s - (h_2)_s\|_{\mathcal{B}}\, ds,
\end{aligned}$$

which implies that

$$\|(Ph_1)(t) - (Ph_2)(t)\| \le \frac{ML_f(\rho^*)(1 + L_g(\rho^*)T)}{\Gamma(1+q)} t^q \|h_1 - h_2\|_{-r,T_1}.$$

Note that for $t \in [-r, 0]$,

$$\|(Ph_1)(t) - (Ph_2)(t)\| = 0.$$

Thus,

$$\|Ph_1 - Ph_2\|_{-r,T_1} \le \frac{ML_f(\rho^*)(1 + L_g(\rho^*)T)}{\Gamma(1+q)} t^q \|h_1 - h_2\|_{-r,T_1}.$$

Let

$$T_{12} = \frac{1}{2}\bigg[\frac{\Gamma(1+q)}{ML_f(\rho^*)(1 + L_g(\rho^*)T)}\bigg]^{1/q}, \quad T_1 = \min\{T_{11}, T_{12}\}.$$

Then P is a contraction map on $\mathcal{S}(1, T_1)$. It follows from the contraction mapping principle that P has a unique fixed point $h \in \mathcal{S}(1, T_1)$, and h is the unique mild solution of system (1.1) with respect to u on $[-r, T_1]$.

Let $T_{21} = T_1 + T_{11}$, $T_{22} = T_1 + T_{12}$, and $\Delta T = \min\{T_{21} - T_1, T_{12}\} > 0$. Similarly, one can verify that (1.1) has a unique mild solution on $[-r, \Delta T]$. Repeating the above procedure in each interval $[\Delta T, 2\Delta T]$, $[2\Delta T, 3\Delta T], \ldots,$ we immediately obtain the global existence of mild solutions for system (1.1). ∎

To end this section, we prove a result on the continuous dependence of mild solutions for system (1.1).

Theorem 3.5. *Suppose $\varphi^1(0), \varphi^2(0) \in \Pi$ where Π is a bounded subset of X, $\varphi^1, \varphi^2 \in C_{-r,0}$ and $u, v \in U_{\mathrm{ad}}$. Let $x^1(t)$ (respectively, $x^2(t)$) be the mild solution of system (1.1) corresponding to (φ^1, u) (respectively, (φ^2, v)). Then the constant*

$$C^* = \max\bigg\{M^*M, M^*, \frac{qM^*M\|B\|_\infty}{\Gamma(1+q)}\bigg(\frac{p-1}{pq-1}\bigg)^{(p-1)/p} T^{q-1/p}\bigg\} > 0 \tag{3.6}$$

satisfies

$$\|x^1(t) - x^2(t)\| \le C^*(\|\varphi^1(0) - \varphi^2(0)\| + \|\varphi^1 - \varphi^2\|_{-r,0} + \|u - v\|_{L^p(J,Y)})$$

for $t \in J$; moreover

$$\|x^1(t) - x^2(t)\| = \|\varphi^1(t) - \varphi^2(t)\|$$

for $-r \le t \le 0$, where M^* depends on the domain of each solution.

Proof. By Lemma 3.3 and [HG](iii), one can check that there exists a constant $\rho > 0$ such that $\|x^1_s\|_{-r,0} \le \rho$, $\|x^2_s\|_{-r,0} \le \rho$, $\|\int_0^s g(s, \tau, x^1_\tau)\, d\tau\| \le \rho$ and $\|\int_0^s g(s, \tau, x^2_\tau)\, d\tau\| \le \rho$. For $t \in J$, by Lemma 3.2(i), [HF](ii), [HG](ii), [HB], [HU] and Hölder's inequality, we have

$$\begin{aligned}
&\|x^1(t)-x^2(t)\|\\
&\quad\le M\|\varphi^1(0)-\varphi^2(0)\|\\
&\qquad+\frac{qML_f(\rho)}{\Gamma(1+q)}\int_0^t (t-s)^{q-1}\|x_s^1-x_s^2\|_{-r,0}\,ds\\
&\qquad+\frac{qML_f(\rho)L_g(\rho)T}{\Gamma(1+q)}\int_0^t (t-s)^{q-1}\|(x^1)_\tau-(x^2)_\tau\|_{-r,0}\,ds\\
&\qquad+\frac{qM\|B\|_\infty}{\Gamma(1+q)}\int_0^t (t-s)^{q-1}\|u(s)-v(s)\|_Y\,ds\\
&\quad\le M\|\varphi^1(0)-\varphi^2(0)\|\\
&\qquad+\frac{qM\|B\|_\infty}{\Gamma(1+q)}\Big(\int_0^t (t-s)^{\frac{p}{p-1}(q-1)}\,ds\Big)^{(p-1)/p}\Big(\int_0^t \|u(s)-v(s)\|_Y^p\,ds\Big)^{1/p}\\
&\qquad+\frac{qML_f(\rho)(1+L_g(\rho)T)}{\Gamma(1+q)}\int_0^t (t-s)^{q-1}\|x_s^1-x_s^2\|_{\mathcal B}\,ds\\
&\quad\le M\|\varphi^1(0)-\varphi^2(0)\|+\frac{qM\|B\|_\infty}{\Gamma(1+q)}\Big(\frac{p-1}{pq-1}\Big)^{(p-1)/p}T^{q-1/p}\|u-v\|_{L^p(J,Y)}\\
&\qquad+\frac{qML_f(\rho)(1+L_g(\rho)T)}{\Gamma(1+q)}\int_0^t (t-s)^{q-1}\|x_s^1-x_s^2\|_{\mathcal B}\,ds,
\end{aligned}$$

which implies that

$$\begin{aligned}
\|x_t^1-x_t^2\|_{\mathcal B}\le{}& M\|\varphi^1(0)-\varphi^2(0)\|+\|\varphi^1-\varphi^2\|_{-r,0}\\
&+\frac{qM\|B\|_\infty}{\Gamma(1+q)}\Big(\frac{p-1}{pq-1}\Big)^{(p-1)/p}T^{q-1/p}\|u-v\|_{L^p(J,Y)}\\
&+\frac{qML_f(\rho)(1+L_g(\rho)T)}{\Gamma(1+q)}\int_0^t (t-s)^{q-1}\|x_s^1-x_s^2\|_{\mathcal B}\,ds.
\end{aligned}$$

Invoking Lemma 2.6 again, we obtain

$$\|x^1(t)-x^2(t)\|\le C^*(\|\varphi^1(0)-\varphi^2(0)\|+\|\varphi^1-\varphi^2\|_{-r,0}+\|u-v\|_{L^p(J,Y)})$$

for $t\in J$, where C^* is given by (3.6).

Finally, note that

$$\|x^1(t)-x^2(t)\|\le\|\varphi^1(t)-\varphi^2(t)\|\quad\text{for } -r\le t\le 0. \ \blacksquare$$

4. Optimal control problem. In the following, we consider the following Lagrange problem:

(P) Find a control $u^0 \in U_{\rm ad}$ such that
$$\mathcal{J}(u^0) \le \mathcal{J}(u) \quad \text{for all } u \in U_{\rm ad},$$
where
$$\mathcal{J}(u) = \int_0^T \mathcal{L}(t, x_t^u, x^u(t), u(t))\, dt$$
and x^u denotes the mild solution of system (1.1) corresponding to the control $u \in U_{\rm ad}$.

For the existence of solution for problem (P), we shall introduce the following assumption:

Assumption [HL].

(i) The functional $\mathcal{L} : J \times C_{-r,0} \times X \times Y \to \mathbb{R} \cup \{\infty\}$ is Borel measurable.
(ii) $\mathcal{L}(t, \cdot, \cdot, \cdot)$ is sequentially lower semicontinuous on $C_{-r,0} \times X \times Y$ for almost all $t \in J$.
(iii) $\mathcal{L}(t, x, y, \cdot)$ is convex on Y for each $x \in C_{-r,0}$, $y \in X$ and almost all $t \in J$.
(iv) There exist constants $d, e \ge 0$, $j > 0$, and $\mu \in L^1(J, \mathbb{R})$ nonnegative such that
$$\mathcal{L}(t, x, y, u) \ge \mu(t) + d\|x\|_{-r,0} + e\|y\| + j\|u\|_Y^p.$$

Now we can give the following result on existence of optimal controls for problem (P).

Theorem 4.1. *Let the assumptions of Theorem* 3.4 *and* [HL] *hold. If B is a strongly continuous operator, then the Lagrange problem* (P) *admits at least one optimal pair, that is, there exists an admissible control $u^0 \in U_{\rm ad}$ such that*
$$\mathcal{J}(u^0) = \int_0^T \mathcal{L}(t, x_t^0, x^0(t), u^0(t))\, dt \le \mathcal{J}(u) \quad \textit{for all } u \in U_{\rm ad}.$$

Proof. If $\inf\{\mathcal{J}(u) \mid u \in U_{\rm ad}\} = \infty$, there is nothing to prove. So we assume that $\inf\{\mathcal{J}(u) \mid u \in U_{\rm ad}\} = \epsilon < \infty$. Using [HL], we have $\epsilon > -\infty$. By the definition of infimum there exists a minimizing sequence $\{(x^m, u^m)\} \subset \mathcal{A}_{ad} := \{(x, u) \mid x$ is a mild solution of system (1.1) corresponding to $u \in U_{\rm ad}\}$ such that $\mathcal{J}(x^m, u^m) \to \epsilon$ as $m \to \infty$. Since $\{u^m\} \subseteq U_{\rm ad}$, $\{u^m\}$ is a bounded subset of the separable reflexive Banach space $L^p(J, Y)$, and there exists a subsequence (not relabeled) and $u^0 \in L^p(J, Y)$ such that $u^m \xrightarrow{w} u^0$ in $L^p(J, Y)$. Since $U_{\rm ad}$ is closed and convex, $u^0 \in U_{\rm ad}$ due to the Mazur Lemma.

Let $\{x^m\} \subset C_{-r,T}$ be the mild solutions of (1.1) corresponding to $\{u^m\}$. By Lemma 3.3 again, there exists a $\rho > 0$ such that $\|x^m\|_{-r,T} \le \rho$ for

$m = 0, 1, 2, \ldots$. Hence,

$$\begin{aligned}
&\|x_t^m - x_t^0\|_{\mathcal{B}} \\
&\quad \le \frac{qML_f(\rho)}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} \|x_s^m - x_s^0\|_{-r,0}\, ds \\
&\qquad + \frac{qML_f(\rho)L_g(\rho)T}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} \|x_\tau^m - x_\tau^0\|_{-r,0}\, ds \\
&\qquad + \frac{qM}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} \|B(s)u^m(s) - B(s)u^0(s)\|\, ds \\
&\quad \le \frac{qM}{\Gamma(1+q)} \left(\frac{p-1}{pq-1}\right)^{(p-1)/p} t^{q-1/p} \left(\int_0^t \|B(s)u^m(s) - B(s)u^0(s)\|^p\, ds\right)^{1/p} \\
&\qquad + \frac{qML_f(\rho)(1+L_g(\rho)T)}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} \|x_s^m - x_s^0\|_{\mathcal{B}}\, ds.
\end{aligned}$$

Note that, for $-r \le t \le 0$,

$$x^m(s) - x^0(s) = 0. \tag{4.1}$$

Again applying Lemma 2.6, we obtain

$$\|x^m(t) - x^0(t)\| \le \|x_t^m - x_t^0\|_{\mathcal{B}} \le M^* \|Bu^m - Bu^0\|_{L^p(J,Y)} \tag{4.2}$$

for $t \in J$, where M^* is a constant independent of u, m, t.

Since B is strongly continuous, we have

$$\|Bu^m - Bu^0\|_{L^p(J,Y)} \xrightarrow{s} 0 \quad \text{as } m \to \infty. \tag{4.3}$$

It follows from (4.1)–(4.3) that

$$\|x^m - x^0\|_{-r,T} \xrightarrow{s} 0 \quad \text{as } m \to \infty,$$

which yields

$$x^m \xrightarrow{s} x^0 \quad \text{in } C_{-r,T} \text{ as } m \to \infty.$$

Note that our assumption [HL] implies the assumptions of Balder (see [B, Theorem 2.1]). Hence, by Balder's theorem we conclude that $(x_t \times x, u) \mapsto \int_0^T \mathcal{L}(t, x_t, x(t), u(t))\, dt$ is sequentially lower semicontinuous in the weak topology of $L^p(J,Y) \subset L^1(J,Y)$, and strong topology of $L^1(J, C_{-r,0} \times X)$. Hence, $\mathcal{J}$ is weakly lower semicontinuous on $L^p(J,Y)$, and since by [HL](iv), $\mathcal{J} > -\infty$, $\mathcal{J}$ attains its infimum at $u^0 \in U_{\mathrm{ad}}$, that is,

$$\begin{aligned}\epsilon &= \lim_{m\to\infty} \int_0^T \mathcal{L}(t, x_t^m, x^m(t), u^m(t))\,dt \\ &\ge \int_0^T \mathcal{L}(t, x_t^0, x^0(t), u^0(t))\,dt = \mathcal{J}(u^0) \ge \epsilon. \qquad \blacksquare\end{aligned}$$

5. Example. As an application we consider the following problem:

$$(5.1)\qquad \begin{cases} {}^C D_t^{2/3} x(t,y) = \Delta x(t,y) + x(t+s,y) + \int_{-r}^{t} h(t-s)x(s,y)\,ds \\ \qquad\qquad + \int_{\Omega} \mathcal{K}(y,s)u(s,t)\,ds, \quad y\in\Omega,\ -r\le s\le t,\ 0<t\le T, \\ x(t,y) = \varphi(t,y), \quad y\in\overline{\Omega},\ -r\le t\le 0, \\ x(t,y) = 0, \quad y\in\partial\Omega,\ t\in J, \end{cases}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with $\partial\Omega \in C^3$, Δ is the Laplace operator, $\varphi \in C^{2,1}([-r,0],\overline{\Omega})$, $u \in L^2(J\times\Omega)$, $h \in L^1([-r,T+r],\mathbb{R})$ and $\mathcal{K} : \overline{\Omega}\times\overline{\Omega} \to \mathbb{R}$ is continuous.

Define $X = Y = L^2(J\times\Omega)$, $D(A) = H^2(\Omega)\cap H_0^1(\Omega)$,

$$Ax = -\left(\frac{\partial^2 x}{\partial y_1^2} + \frac{\partial^2 x}{\partial y_2^2} + \frac{\partial^2 x}{\partial y_3^2}\right) \quad \text{for } x\in D(A).$$

Then A is the generator of a strongly continuous semigroup $\{T(t), t\ge 0\}$ on X. The controls are functions $u : Sx(\Omega) \to \mathbb{R}$ such that $u \in L^2(Sx(\Omega))$. We claim that $t \mapsto u(\cdot,t)$ from J into Y is measurable. Set $U(t) = \{u \in Y \mid \|u\|_Y \le \nu\}$, where $\nu \in L^2(J,\mathbb{R}^+)$. We restrict the admissible controls $U_{\rm ad}$ to be all $u \in L^2(Tx(\Omega))$ such that $\|u(\cdot,t)\|_{L^2(\Omega)} \le \nu(t)$ a.e.

Define

$$x(t)(y) = x(t,y), \quad x_t(y) = x(t+s,y), \quad B(t)u(t)(y) = \int_{\Omega} \mathcal{K}(y,s)u(s,t)\,ds$$

and

$$f\Big(t, x_t, \int_0^t g(t,s,x_s)\,ds\Big)(y) = x_t(y) + \int_{-r}^{t} h(t-s)x(s,y)\,ds.$$

Thus problem (5.1) can be rewritten as

$$(5.2)\qquad \begin{cases} {}^C D_t^{q} x(t) = Ax(t) + f\Big(t, x_t, \int_0^t g(t,s,x_s)\,ds\Big) \\ \qquad\qquad + B(t)u(t), \quad q = 2/3,\ t\in J, \\ x(t) = \varphi(t), \quad t\in[-r,0]. \end{cases}$$

We consider the following cost function:

$$\mathcal{J}(u)=\int_0^T \mathcal{L}(t,x_t^u,x^u(t),u(t))\,dt,$$

where $\mathcal{L}: J\times C^{1,0}([-r,0]\times\overline{\Omega})\times L^2(J\times\Omega)\to\mathbb{R}\cup\{+\infty\}$ for $x\in C^{1,0}([-r,T]\times\overline{\Omega})$ and $u\in L^2(\Omega\times J)$,

$$\begin{aligned}\mathcal{L}(t,x_t^u,x^u(t),u(t))(y)&=\int_\Omega\int_{-r}^0 |x^u(t+s,y)|^2\,ds\,dy\\&\quad+\int_\Omega |x^u(t,y)|^2\,dy+\int_\Omega |u(y,t)|^2\,dy.\end{aligned}$$

Then it satisfies all the assumptions given in Theorem 4.1. Therefore, the problem (5.1) has at least one optimal pair.

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